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Sensitivity analysis of non-conservative eigensystems

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Abstract

An expression for the derivatives of eigenvalues and eigenvectors of non-conservative systems is presented. Contrary to previous methods that use state space form ($2N$-space) to consider damping, proposed method solves the eigenpair derivatives of damped system explicitly. The computation size of N-order is maintained and the eigenpair derivatives are obtained simultaneously from one equation so that it is efficient in CPU time and storage capacity. Moreover, this method can be extended to asymmetric non-conservative damped systems. Although additional problems are generated contrary to the eigenpair sensitivity methods of symmetric systems, in asymmetric case, an algebraic method for the eigenpair derivatives can be obtained through similar procedure. The proposed expression is derived by combining the differentiations of the eigenvalue problem and normalization condition into one linear algebraic equation. The numerical stability is proved by showing non-singularity of the proposed equation, and the efficiency of the derived expression is illustrated by considering a cantilever beam with lumped dampers and a whirling beam.

1. Introduction

Natural frequencies and mode shapes of systems are essential to understand dynamic behavior of structure. However, design parameters can be varied with damage, deterioration, corrosion, etc. and this causes variation in natural frequency and mode shape. The variation of eigenpair brings about variation of dynamic behavior of systems and this affects the stability of structure

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directly. Therefore, eigen-sensitivity analysis has played a central part in structural stability analysis and has emerged as an important area of research. And eigenpair sensitivity is used in many areas, the optimization of structure subject to natural frequency, system identification, finite modelling updating, structural control, etc.

A number of methods for eigenpair sensitivity of undamped system have been developed. Fox and Kapoor [1] finded the eigenpair derivatives with the term of system matrix and eigenpair. Nelson [2] represented eigenvector derivative by sum of the homogeneous solution and the particular solution, and Ojalvo [3] and Dailey [4] extended Nelson's method to the multiple eigenvalue problem. Modal method [5,6] and its modified one [7,8] approximated the mode shape derivatives by the linear combination of mode shapes, and Lee and Jung [9,10] presented the algebraic methods for eigenpair derivatives of systems having the distinct and multiple eigenvalue.

A number of the prescribed methods can be applied to the damped systems. However, almost eigen-sensitivity methods have to use state space equation based on $2N$-space to solve the problems induced by damping. These methods are at a disadvantage in CPU time and storage capacity because of double computation size. In order to overcome these drawbacks, Zimoch [11] presented direct method for the eigenpair derivatives of damped systems without use of state space equation. However, this method is restricted to mechanical systems because the available design parameter is limited to the component of the system matrices. Sodipon Adhikari [12] proposed eigen-sensitivity method based on N-space, too. However, it did not give exact solution and only is applicable to small sized damped systems. On the other hand, Lee et al. [13,14] developed analytical method that give exact solutions while it maintain 'N-space', but it finds eigenvalue derivative from classical method as before.

Many eigenpair sensitivity methods are restricted to systems whose characteristic matrices are symmetric. However, a number of real systems have asymmetric mass, damping, and stiffness matrices, for example, the behavior of structure in fluid, moving vehicles on roads, the study of aircraft flutter and gyroscopic systems. It is difficult to solve the eigenpair sensitivity of asymmetric systems by using the previous methods because of additional problems due to asymmetric characteristic matrices. And this difficulty is possibly motivation for authors that have tried to solve the eigenpair sensitivity of asymmetric systems.

Fox and Kapoor [1] presented exact expression for eigenpair derivatives of symmetric undamped systems in the earliest time and many authors [15–17] have extended his method [1] to asymmetric systems. Rudisill [18] solved the eigenvector derivatives of general matrices analytically and Murthy and Haftka [19] have written an excellent review on calculating the eigenpair derivatives of general matrices.

However, above methods don't explicitly consider the damping of systems. Brandon [20] presented the modal method for asymmetric damped systems. This method solved the problems due to asymmetric matrices by using the left eigenvector. However, it has disadvantages in CPU time and storage capacity because it uses state space form to consider damping of systems and requires many of eigenpair information to find eigenpair sensitivity.

In this paper, an efficient algebraic method for the eigenpair sensitivities of damped systems is presented. Contrary to previous methods the proposed method finds the eigenvalue and eigenvector sensitivities simultaneously from one equation. And the proposed method does not use state space equation ($2N$-space), instead of it, the method maintain 'N-space' because singularity problem is solved by using only a side condition. The proposed method gives exact solutions because it is the analytical method. And it only requires the corresponding eigenpair information differently from modal methods.

Moreover, an efficient algebraic method for the eigenpair sensitivity of asymmetric damped systems is derived through similar procedure of symmetric case. The problems due to asymmetric system matrices are solved by finding the derivatives of eigenvalue and eigenvector simultaneously. It does not require the left eigenvector and state space form contrary to previous methods. And the solutions are also exact and numerically stable.

2. Eigenpair sensitivity

2.1. Eigenpair sensitivity in symmetric damped systems

Consider the eigenvalue problem of damped systems with N degrees of freedom described as

$$(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{u}_j = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively, and $n \times n$ symmetric matrices. $\mathbf{M}$ is positive definite, $\mathbf{K}$ is positive definite or semi-positive definite and λ_j is the jth eigenvalue and $\mathbf{u}_j$ is the jth eigenvector of systems.

In order to determine the eigenvalue derivatives, the differentiation of Eq. (1) is used. Eq. (1) is differentiated with respect to a design parameter α, then

$$(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{u}_{j,\alpha} = -(2\lambda_j\mathbf{M} + \mathbf{C})\mathbf{u}_j\lambda_{j,\alpha} - (\lambda_j^2\mathbf{M}_{,\alpha} + \lambda_j\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j, \tag{2}$$

where $(\bullet)_{,\alpha}$ represents the derivative of $(\bullet)$ with respect to design parameter α.

Pre-multiplying at each side of Eq. (2) by $\mathbf{u}_j^{\mathrm{T}}$ gives

$$\mathbf{u}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{u}_{j,\alpha} = -\mathbf{u}_j^{\mathrm{T}}(2\lambda_j\mathbf{M} + \mathbf{C})\mathbf{u}_j\lambda_{j,\alpha} - \mathbf{u}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M}_{,\alpha} + \lambda_j\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j. \tag{3}$$

Eq. (3) can be transposed because it is scalar, and its transposition enables one to eliminate the left side of it due to symmetry of $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$. As a result, the eigenvalue derivative is obtained as follows:

$$\lambda_{j,\alpha} = -\mathbf{u}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M}_{,\alpha} + \lambda_j\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j. \tag{4}$$

However, eigenvector derivative is not solved from Eq. (2) directly, because the coefficient matrix of left side of Eq. (2) is singular. Hence, a number of studies have focused on the eigenvector sensitivity generally in the case of symmetric damped systems.

2.1.1. Zimoch's method

Zimoch [11] presented method for eigenpair sensitivity subject to the components of system matrices as follows:

$$\partial\mathbf{\Lambda}/\partial m_{ij} = \mathbf{diag}[-6\lambda_k^2\mathbf{u}_{ik}\mathbf{u}_{jk}],$$
$$\partial\mathbf{\Lambda}/\partial m_{ii} = \mathbf{diag}[-3\lambda_k^2\mathbf{u}_{ik}^2] \quad (k = 1, 2, \ldots, 2n),$$

$$\begin{aligned}
\partial\mathbf{\Phi}/\partial m_{ii} &= \Phi[\lambda_l \mathbf{u}_{ik}\mathbf{u}_{il}(\lambda_l + 2\lambda_k)/(\lambda_k - \lambda_l)] \quad (l \neq k),\\
\partial\mathbf{\Phi}/\partial m_{ii} &= \Phi[\lambda_k \mathbf{u}_{ik}^2] \quad (l = k),\\
\partial\mathbf{\Phi}/\partial m_{ij} &= \Phi[(2\lambda_k\lambda_l + \lambda_l^2)(\mathbf{u}_{il}\mathbf{u}_{jk} + \mathbf{u}_{ik}\mathbf{u}_{jl})/(\lambda_k - \lambda_l)] \quad (l \neq k),\\
\partial\mathbf{\Phi}/\partial m_{ij} &= \Phi[-2\lambda_k \mathbf{u}_{ik}\mathbf{u}_{jk}] \quad (l = k) \;\; (l, k = 1, 2, \ldots, 2n).
\end{aligned} \tag{5}$$

This equation uses the component of mass matrix, m_{ij}, as design parameter and similar methods about damping matrix and stiffness matrix are also presented. It is efficient in CPU time because it requires only corresponding eigenpair to obtain eigenpair derivatives. However, it cannot use general design parameters such as area, length, thickness, moment of inertia, etc. In other words, it is the limited method that is only applicable to discrete systems.

2.1.2. Zeng's method

Zeng [22] (see Ref. [13]) used the classical modal method for eigenpair derivatives of damped systems:

$$\begin{aligned}
\mathbf{u}_{j,\alpha} = -\Bigg\{ & (\mathbf{B} + \beta\mathbf{A})^{-1} \sum_{m=0}^{M_a - 1} [-(\lambda_j - \beta)\mathbf{A}(\mathbf{B} + \beta\mathbf{A})^{-1}]^m \\
& + \sum_{k=1, k\neq j}^{N} \left[\left(\frac{\lambda_j - \beta}{\lambda_k - \beta}\right)^{M_a} \frac{(\mathbf{u}_k\mathbf{u}_k^{\mathrm{T}})}{\lambda_j - \lambda_k} + \left(\frac{\lambda_j - \beta}{\lambda_k^* - \beta}\right)^{M_a} \frac{(\mathbf{u}_k\mathbf{u}_k^{\mathrm{T}})^*}{\lambda_j - \lambda_k^*} \right] \\
& + \left(\frac{\lambda_j - \beta}{\lambda_j^* - \beta}\right)^{M_a} \frac{(\phi_j\phi_j^{\mathrm{T}})^*}{\lambda_j - \lambda_j^*} \Bigg\} (\lambda_j'\mathbf{A} + \lambda_j\mathbf{A}_{,\alpha} + \mathbf{B}_{,\alpha})\mathbf{u}_j - (\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{A}_{,\alpha}\mathbf{u}_j)/2.
\end{aligned} \tag{6}$$

Although it is the improved modal method that uses the accelerated and shifted poles, it has disadvantages in CPU time and storage capacity because it requires many of eigenpair information for one eigenpair derivative. And state space equation ($2N$-space) is introduced to extend damped system, and it gives approximated solutions when truncated modes is used.

2.1.3. Nelson's method [2]

In this method, the eigenvector derivative is expressed as sum of particular solution and homogeneous solution as follows:

$$\mathbf{u}_{j,\alpha} = v_{j\alpha} + c_{j\alpha}\mathbf{u}_j, \tag{7}$$

where $c_{j\alpha} = -\mathbf{u}_j^{\mathrm{T}}\mathbf{M}v_{j\alpha} - 0.5\mathbf{u}_j^{\mathrm{T}}\mathbf{M}_{,\alpha}\mathbf{u}_j$: homogeneous solution.

The method is known as the most efficient one among previous eigen-sensitivity methods for undamped systems. Not only its algorithm is simple but also it gives exact solution and only needs a corresponding eigenpair for eigenpair derivatives. However, it did not consider damping explicitly, hence it requires state space form to solve damped systems as ever.

2.1.4. Sondipon Adhikari's method [12]

This is the modal method that deals with damped systems with approximated eigenpair. In this method, the eigenpair of damped systems is approximated using the eigenpair of undamped systems. It maintains N-space, however, it does not give exact solution, and it is only applicable to small sized damped systems

$$\mathbf{u}_{j,\alpha} \approx -0.5(\mathbf{u}_j^{\mathrm{T}}\mathbf{M}_{,\alpha}\mathbf{u}_j)\varphi_j + \sum_{k \neq j}^{N} \frac{1}{2\omega_k}\left[\frac{(1-\gamma_{kj})\mathbf{u}_k^{\mathrm{T}}\tilde{F}_{i,\alpha}\mathbf{u}_j}{\lambda_j - \lambda_k} - \frac{(1-\bar{\gamma}_{kj})\mathbf{u}_k^{*\mathrm{T}}\tilde{F}_{i,\alpha}\mathbf{u}_j}{\lambda_j + \lambda_k^*}\right](\varphi_k - \alpha_k^{(i)}\varphi_j), \tag{8}$$

where

$$\lambda_j \approx \omega_j + 0.5\varphi_j^{\mathrm{T}} C\varphi_j \cdot i,$$
$$\mathbf{u}_j \approx \varphi_j + \sum_{k=1}^{N} \frac{i\lambda_j(\varphi_k^{\mathrm{T}} C\varphi_j)\varphi_k}{(\lambda_j - \lambda_k)(\lambda_j - \lambda_k^*)}.$$

2.2. Eigenpair sensitivity in asymmetric damped systems

Generalized eigenvalue problem for asymmetric damped systems is same as one for symmetric damped systems:

$$(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{u}_j = \mathbf{0}, \tag{9}$$

where λ_j is the jth eigenvalue, $\mathbf{u}_j$ is jth eigenvector, $\mathbf{M}$ is mass matrix, $\mathbf{C}$ is damping matrix and $\mathbf{K}$ is stiffness. In this case, one have to pay attention to that $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ is asymmetric system matrices.

By differentiating Eq. (9) with respect to design parameter, Eq. (10) is obtained,

$$(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{u}_{j,\alpha} = -(2\lambda_j\mathbf{M} + \mathbf{C})\mathbf{u}_j\lambda_{j,\alpha} - (\lambda_j^2\mathbf{M}_{,\alpha} + \lambda_j\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j, \tag{10}$$

where $(\bullet)_{,\alpha}$ represents the derivative of $(\bullet)$ with respect to design parameter α.

Pre-multiplying at each side of Eq. (10) by $\mathbf{u}_j^{\mathrm{T}}$, we have

$$\lambda_{j,\alpha} = -\mathbf{u}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{u}_{j,\alpha} - \mathbf{u}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M}_{,\alpha} + \lambda_j\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j. \tag{11}$$

Eq. (11) can be transposed due to its scalar and transposing Eq. (11) enables one to eliminate the first term of right side of Eq. (11) in the case that system matrix $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are symmetric, because the first term of right side contains eigenvalue problem as Eq. (9). Hence the eigenvalue derivative is obtained clearly. However, the first term of right side is remained in the case of asymmetric systems such as Eq. (12) because of $\mathbf{M}^{\mathrm{T}} \neq \mathbf{M}$, $\mathbf{C}^{\mathrm{T}} \neq \mathbf{C}$ and $\mathbf{K}^{\mathrm{T}} \neq \mathbf{K}$:

$$\lambda_{j,\alpha} = -\mathbf{u}_{j,\alpha}^{\mathrm{T}}(\lambda_j^2\mathbf{M}^{\mathrm{T}} + \lambda_j\mathbf{C}^{\mathrm{T}} + \mathbf{K}^{\mathrm{T}})\mathbf{u}_j - \mathbf{u}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M}_{,\alpha}^{\mathrm{T}} + \lambda_j\mathbf{C}_{,\alpha}^{\mathrm{T}} + \mathbf{K}_{,\alpha}^{\mathrm{T}})\mathbf{u}_j. \tag{12}$$

Therefore, it is difficult to find the eigenvalue derivative by previous approach that is used in symmetric systems.

A number of authors have used the left eigenvector to solve above problems due to asymmetric characteristics. The left eigenvector satisfies the next condition in damped systems,

$$\mathbf{v}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K}) = \mathbf{0}. \tag{13}$$

For eigenvalue and eigenvector derivatives, pre-multiplying at each side of Eq. (10) by $\mathbf{v}_j^{\mathrm{T}}$, we can obtain a new equation,

$$\lambda_{j,\alpha} = -\mathbf{v}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{u}_{j,\alpha} - \mathbf{v}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M}_{,\alpha} + \lambda_j\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j. \tag{14}$$

The first term of right side of Eq. (14) is eliminated because of Eq. (13). Therefore, the derivative of eigenvalue for asymmetric damped systems is obtained such as

$$\lambda_{j,\alpha} = -\mathbf{v}_j^{\mathrm{T}}(\lambda_j^2\mathbf{M}_{,\alpha} + \lambda_j\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j. \tag{15}$$

The left eigenvector serves as the clue that led to solution of the eigenvector derivative for asymmetric systems in many studies [16,20].

2.2.1. *Brandon's method [20]*

Brandon [20] presented the modal method that finds the eigenvector derivative of asymmetric damped system as sum of several modes:

$$\mathbf{u}_{j,\alpha} = \sum_{i=1, i\neq j}^{n} a_{ij}\mathbf{u}_i, \tag{16}$$

where

$$a_{ij} = \frac{-\mathbf{v}_i^{\mathrm{T}}(\mathbf{A}_{,\alpha} + \lambda_j\mathbf{B}_{,\alpha})\mathbf{u}_j}{(\lambda_j - \lambda_i)\mathbf{v}_i^{\mathrm{T}}\mathbf{B}\mathbf{u}_i}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}.$$

In the coefficient part of Eq. (16) this method uses state space form. Therefore, significant numerical effort is required as the size of the problem doubles. And the method has disadvantages of the modal method that many eigenpairs are used for one eigenpair derivative and error arises when truncated in order to reduce the available modes.

3. Proposed method

3.1. *Eigenpair sensitivity in symmetric damped systems*

3.1.1. *Algorithm*

The algorithm of the proposed method is simple. The problems due to singularity and damping are solved simultaneously from the proposed equation. The method uses the derivatives of eigenvalue problem and the side condition as fundamental equations.

By rearranging Eq. (2) which is the differentiation of eigenvalue problem, one gets

$$(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{u}_{j,\alpha} + (2\lambda_j\mathbf{M} + \mathbf{C})\mathbf{u}_j\lambda_{j,\alpha} = -(\lambda_j^2\mathbf{M}_{,\alpha} + \lambda_j\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j. \tag{17}$$

The eigenvectors of damped system are normalized as follows:

$$\mathbf{z}_j^{\mathrm{T}}\mathbf{B}\mathbf{z}_j = \begin{Bmatrix} \mathbf{u}_j \\ \lambda_i\mathbf{u}_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_j \\ \lambda_j\mathbf{u}_j \end{Bmatrix} = \mathbf{u}_j^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\mathbf{u}_j = 1. \tag{18}$$

The derivative of normalization condition is used as side condition. Differentiating Eq. (18) subject to design variable (α) gives

$$\mathbf{u}_j^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{u}_{j,\alpha}+\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j\lambda_{j,\alpha}=-0.5\mathbf{u}_j^{\mathrm{T}}(2\lambda_j\mathbf{M}_{,\alpha}+\mathbf{C}_{,\alpha})\mathbf{u}_j. \tag{19}$$

Because the unknown or interested values are $\mathbf{u}_{j,\alpha}$ and $\lambda_{j,\alpha}$, Eqs. (17) and (19) can be combined into single matrix form as follows:

$$\begin{bmatrix} \lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K} & (2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{u}_j \\ \mathbf{u}_j^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C}) & \mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \end{bmatrix}\begin{Bmatrix} \mathbf{u}_{j,\alpha} \\ \lambda_{j,\alpha} \end{Bmatrix}=-\begin{Bmatrix} (\lambda_j^2\mathbf{M}_{,\alpha}+\lambda_j\mathbf{C}_{,\alpha}+\mathbf{K}_{,\alpha})\mathbf{u}_j \\ 0.5\mathbf{u}_j^{\mathrm{T}}(2\lambda_j\mathbf{M}_{,\alpha}+\mathbf{C}_{,\alpha})\mathbf{u}_j \end{Bmatrix}. \tag{20}$$

This is the key of the proposed method. Contrary to previous method, the sensitivities of the eigenvalue and eigenvector are obtained simultaneously from one equation. It is a matter of course that the method is efficient in CPU time and storage capacity because it maintains N-space without use of state space equation and finds the eigenpair derivatives simultaneously. The proposed method requires only corresponding eigenpair information differently from modal methods, and gives exact solution and guarantees numerical stability.

3.1.2. Numerical stability

Numerical stability is guaranteed by proving non-singularity of the coefficient matrix $\mathbf{A}^{\#}$ of proposed Eq. (20). To prove that the coefficient matrix $\mathbf{A}^{\#}$ is non-singular, introduce the determinant property such as

$$\det(\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{\#}\mathbf{Y})=\det(\mathbf{Y}^{\mathrm{T}})\det(\mathbf{A}^{\#})\det(\mathbf{Y}). \tag{21}$$

If $\det(\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{\#}\mathbf{Y})\neq 0$ is proved with the arbitrary non-singular matrix $\mathbf{Y}$, $\det(\mathbf{A}^{\#})\neq 0$ is proved. Assuming the arbitrary non-singular matrix $\mathbf{Y}$ such as

$$\mathbf{Y}=\begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{22}$$

where $\mathbf{\Psi}=[\psi_1,\psi_2\cdots\psi_{n-1}\mathbf{u}_j]$, $\mathbf{u}_j$ is the jth eigenvector of systems and $\boldsymbol{\psi}$'s are arbitrary vectors to be independent of $\mathbf{u}_j$. $\mathbf{\Psi}$ is a $n\times n$ matrix and Y is a $(n+1)\times(n+1)$ matrix.

Pre- and post-multiplying $\mathbf{A}^{\#}$ by $\mathbf{Y}^{\mathrm{T}}$ and $\mathbf{Y}$ yields

$$\begin{aligned}\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{\#}\mathbf{Y}&=\begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K} & (2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{u}_j \\ \mathbf{u}_j^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C}) & \mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \end{bmatrix}\begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\\ &=\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{\Psi} & \mathbf{\Psi}^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{u}_j \\ \mathbf{u}_j^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{\Psi} & \mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \end{bmatrix},\end{aligned} \tag{23}$$

The last column and row of the matrix $\mathbf{\Psi}^{\mathrm{T}}(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{\Psi}$ is zero because the last column of $\mathbf{\Psi}$ is the eigenvector $\mathbf{u}_j$. That is

$$\mathbf{\Psi}^{\mathrm{T}}(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{\Psi}=\begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \tag{24}$$

where $\mathbf{\Psi}^{\mathrm{T}}(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{\Psi}$ is $n \times n$ matrix, and this matrix has the rank of $n-1$ because λ_j is distinct eigenvalue. Therefore the matrix $\tilde{\mathbf{A}}$ of order $(n-1)$ is non-singular.

And the last elements of the column vector $\mathbf{\Psi}^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j$ and the row vector $\phi_j^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{\Psi}$ are unity due to its normalization condition:

$$\mathbf{\Psi}^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{u}_j=\begin{Bmatrix}\tilde{\mathbf{b}}\\ 1\end{Bmatrix},\quad \mathbf{u}_j^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{\Psi}=\{\tilde{\mathbf{b}}^{\mathrm{T}}\ \ 1\},\tag{25}$$

where $\tilde{\mathbf{b}}$ is non-zero vector. Substituting Eqs. (24) and (25) into Eq. (23) gives

$$\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{\#}\mathbf{Y}=\begin{bmatrix}\tilde{\mathbf{A}} & \mathbf{0} & \tilde{\mathbf{b}}\\ \mathbf{0} & 0 & 1\\ \tilde{\mathbf{b}}^{\mathrm{T}} & 1 & \mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j\end{bmatrix}.\tag{26}$$

In order to find the determinant of this matrix, apply the determinant property of partitioned matrix such as

$$\det\left(\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\right)=\det\mathbf{A}\times\det(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}).\tag{27}$$

Therefore, the determinant of Eq. (16) is as follows:

$$\det(\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{\#}\mathbf{Y})=\det\begin{bmatrix}0 & 1\\ 1 & \mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j\end{bmatrix}\det\left(\tilde{\mathbf{A}}-[\mathbf{0}\ \tilde{\mathbf{b}}]\begin{bmatrix}0 & 1\\ 1 & \mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\ \tilde{\mathbf{b}}^{\mathrm{T}}\end{bmatrix}\right),\tag{28}$$

where

$$[\mathbf{0}\ \tilde{\mathbf{b}}]\begin{bmatrix}0 & 1\\ 1 & \mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\ \tilde{\mathbf{b}}^{\mathrm{T}}\end{bmatrix}=0\quad\text{and}\quad\det\begin{bmatrix}0 & 1\\ 1 & \mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j\end{bmatrix}=-1.\tag{29}$$

Rearranging Eq. (28) yields

$$\det(\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{\#}\mathbf{Y})=-\det(\tilde{\mathbf{A}})\neq 0.\tag{30}$$

In other words, the matrix $\mathbf{A}^{\#}$ is non-singular.

3.2. Eigenpair sensitivity in asymmetric damped systems

3.2.1. Algorithm

The proposed method does not require the left eigenvector. The problems due to asymmetric system matrix are solved by finding the derivatives of eigenvalue and eigenvector simultaneously.

The algorithm of the proposed method is also simple. By rearranging Eq. (10) which is the differentiation of eigenvalue problem, one gets

$$(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{u}_{j,\alpha}+(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{u}_j\lambda_{j,\alpha}=-(\lambda_j^2\mathbf{M}_{,\alpha}+\lambda_j\mathbf{C}_{,\alpha}+\mathbf{K}_{,\alpha})\mathbf{u}_j.\tag{31}$$

Eigenvector of damped systems is normalized with using state space form such as

$$\begin{Bmatrix} \mathbf{u}_j \\ \lambda_i \mathbf{u}_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_j \\ \lambda_j \mathbf{u}_j \end{Bmatrix} = \mathbf{u}_j^{\mathrm{T}}(2\lambda_i \mathbf{M} + \mathbf{C})\mathbf{u}_j = 1. \tag{32}$$

And we use the differentiation of normalization condition as side condition. Differentiating Eq. (32) with respect to design variable gives

$$\mathbf{u}_j^{\mathrm{T}}(2\lambda_j \mathbf{M} + 2\lambda_j \mathbf{M}^{\mathrm{T}} + \mathbf{C} + \mathbf{C}^{\mathrm{T}})\mathbf{u}_{j,\alpha} + 2\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \lambda_{j,\alpha} = -\mathbf{u}_j^{\mathrm{T}}(2\lambda_j \mathbf{M}_{,\alpha} + \mathbf{C}_{,\alpha})\mathbf{u}_j. \tag{33}$$

Contrary to symmetric systems, the transpositions of mass and damping matrix are appeared in Eq. (33) because of $\mathbf{M}^{\mathrm{T}} \neq \mathbf{M}$, $\mathbf{C}^{\mathrm{T}} \neq \mathbf{C}$, and $\mathbf{K}^{\mathrm{T}} \neq \mathbf{K}$.

Combining Eqs. (31) and (33) into one linear algebraic equation yields

$$\begin{bmatrix} \lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K} & (2\lambda_j \mathbf{M} + \mathbf{C})\mathbf{u}_j \\ \mathbf{u}_j^{\mathrm{T}}(2\lambda_j \mathbf{M} + 2\lambda_j \mathbf{M}^{\mathrm{T}} + \mathbf{C} + \mathbf{C}^{\mathrm{T}}) & 2\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{j,\alpha} \\ \lambda_{j,\alpha} \end{Bmatrix} = -\begin{Bmatrix} (\lambda_j^2 \mathbf{M}_{,\alpha} + \lambda_j \mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{u}_j \\ \mathbf{u}_j^{\mathrm{T}}(2\lambda_j \mathbf{M}_{,\alpha} + \mathbf{C}_{,\alpha})\mathbf{u}_j \end{Bmatrix}. \tag{34}$$

Eq. (34) is the proposed equation. In Eq. (34), the proposed method only requires the corresponding eigenpair information to find the jth eigenpair sensitivity, and the left eigenvectors are not required differently to previous method to solve asymmetric systems. The method has advantages in numerical efforts because it finds eigenpair derivatives simultaneously from one equation and maintains N-space without the use of state space form. And there is no doubt that it gives exact solutions.

3.2.2. Numerical stability

Numerical stability is guaranteed by proving non-singularity of the coefficient matrix $\mathbf{A}^*$ of Eq. (34).

Arbitrary nonsingular matrices $\mathbf{X}$, $\mathbf{Y}$ are introduced to prove that $\det(\mathbf{A}^*) \neq 0$, as follows:

$$\mathbf{X} = \begin{bmatrix} \Gamma & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{35}$$

where $\Gamma = [\varphi_1 \ \varphi_2 \cdots \varphi_{n-1} \ \mathbf{v}_j]$, $\mathbf{\Psi} = [\psi_1 \ \psi_2 \cdots \psi_{n-1} \ \mathbf{u}_j]$ and $\mathbf{v}_j$ is the jth left eigenvector, $\mathbf{u}_j$ is the jth right eigenvector of system which satisfy the following condition:

$$(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K})\mathbf{u}_j = 0, \quad \mathbf{v}_j^{\mathrm{T}}(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) = 0. \tag{36}$$

And each φ_k and ψ_k are the arbitrary vectors to be independent to $\mathbf{v}_j$ and $\mathbf{u}_j$, respectively.

Pre- and post-multiplying the coefficient matrix $\mathbf{A}^*$ by $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{Y}$, one has

$$\begin{aligned} \mathbf{X}^{\mathrm{T}}\mathbf{A}^*\mathbf{Y} &= \begin{bmatrix} \mathbf{\Gamma} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K} & (2\lambda_j \mathbf{M} + \mathbf{C})\mathbf{u}_j \\ \mathbf{u}_j^{\mathrm{T}}(2\lambda_j \mathbf{M} + 2\lambda_j \mathbf{M}^{\mathrm{T}} + \mathbf{C} + \mathbf{C}^{\mathrm{T}}) & 2\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \end{bmatrix} \begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{\Gamma}^{\mathrm{T}}(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K})\mathbf{\Psi} & \mathbf{\Gamma}^{\mathrm{T}}(2\lambda_j \mathbf{M} + \mathbf{C})\mathbf{u}_j \\ \mathbf{u}_j^{\mathrm{T}}(2\lambda_j \mathbf{M} + 2\lambda_j \mathbf{M}^{\mathrm{T}} + \mathbf{C} + \mathbf{C}^{\mathrm{T}})\mathbf{\Psi} & 2\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \end{bmatrix}. \end{aligned} \tag{37}$$

Because the last columns of Γ and $\mathbf{\Psi}$ are the eigenvectors of system, one can rearrange Eq. (37) as

$$\mathbf{X}^{\mathrm{T}}\mathbf{A}^{*}\mathbf{Y}\begin{bmatrix} \tilde{\mathbf{B}} & \mathbf{0} & \tilde{\mathbf{c}} \\ \mathbf{0} & 0 & c \\ \tilde{\mathbf{d}}^{\mathrm{T}} & 2 & 2\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \end{bmatrix}, \tag{38}$$

where $\tilde{\mathbf{B}}$ is $(n-1)\times(n-1)$ non-singular matrix due to distinct eigenvalue λ_j, $c = \mathbf{v}_j^{\mathrm{T}}(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{u}_j$ and $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$ are non-zero matrices.

Using the determinant property of partitioned matrix, the determinant of Eq. (38) is simplified as

$$\begin{aligned}\det(\mathbf{X}^{\mathrm{T}}\mathbf{A}^{*}\mathbf{Y}) &= \det(\tilde{\mathbf{B}}) \times \det\left(\begin{bmatrix} 0 & c \\ 2 & 2\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{d}}^{\mathrm{T}} \end{bmatrix}[\tilde{\mathbf{B}}]^{-1}[\mathbf{0}\ \tilde{\mathbf{c}}]\right) \\ &= -2b\det(\tilde{\mathbf{A}}) \neq 0. \end{aligned} \tag{39}$$

We can see that $\det(\mathbf{X}^{\mathrm{T}}\mathbf{A}^{*}\mathbf{Y})\neq 0$ from Eq. (39). Therefore, $\det(\mathbf{A}^{*})\neq 0$ because the nonsingular matrix $\mathbf{X}$ and $\mathbf{Y}$ are non-singular. In other words, the coefficient matrix $\mathbf{A}^{*}$ is non-singular matrix.

4. Numerical example

4.1. Eigenpair sensitivity in symmetric damped systems

Numerical example is considered to verify the efficiency and feasibility of the proposed method. This example is a cantilever beam to be equipped with lumped dampers (see Fig. 1).

It is FEM model composed of 30 elements and 31 nodes. Each node has two degrees of freedom (vertical displacement, rotation). Both Rayleigh damping ($\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M}$) and lumped damping are considered. The thickness of beam is chosen as design parameter.

Pentium computer with RAM 64M, CPU capacity 266 Hz is used for analysis.

Some eigenpair derivatives to be obtained by the proposed method are represented in Tables 1, 2 and the CPU times of each method are shown in Fig. 2. Zeng's method for damped systems and Nelson's method for undamped systems are most comparable with the proposed method among previous one. However Nelson's method requires state space equation to solve damped systems.

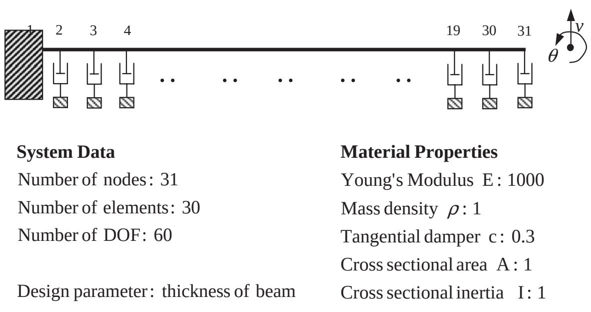

Fig. 1. Cantilever beam with lumped dampers.

Table 1
Eigenvalue and its derivatives of system

Mode number	Eigenvalues	First derivatives
1	−2.3427e − 03 − 1.0868e + 00i	6.6237e − 04 − 2.9972e − 01i
2	−2.3427e − 03 + 1.0868e + 00i	6.6231e − 04 + 2.9972e − 01i
3	−1.4162e − 02 − 6.0514e + 00i	4.5231e − 03 − 1.3173e + 00i
4	−1.4162e − 02 + 6.0514e + 00i	4.5266e − 03 + 1.3173e + 00i
5	−3.1855e − 02 − 1.4703e + 01i	8.2032e − 03 − 2.4536e + 00i
6	−3.1855e − 02 + 1.4703e + 01i	8.2040e − 03 + 2.4536e + 00i
7	−5.8513e − 02 − 2.4733e + 01i	1.0219e − 02 − 3.1193e + 00i
8	−5.8513e − 02 + 2.4733e + 01i	1.0245e − 02 + 3.1193e + 00i
9	−9.5243e − 02 − 3.5359e + 01i	1.0631e − 02 − 3.4198e + 00i
10	−9.5243e − 02 + 3.5359e + 01i	1.0656e − 02 + 3.4198e + 00i

Table 2
First eigenvector and its derivative of system

d.o.f. number	First eigenvector	Derivative
1	−5.6474e − 04 − 5.6364e − 04i	1.7040e − 04 + 1.6942e − 04i
2	−3.3629e − 03 − 3.3565e − 03i	1.0142e − 03 + 1.0085e − 03i
3	−2.2249e − 03 − 2.2208e − 03i	6.7061e − 04 + 6.6697e − 04i
4	−6.5726e − 03 − 6.5612e − 03i	1.9789e − 03 + 1.9688e − 03i
5	−4.9294e − 03 − 4.9209e − 03i	1.4842e − 03 + 1.4766e − 03i
⋮	⋮	⋮
57	−2.8334e − 01 − 2.8342e − 01i	8.3339e − 02 + 8.3414e − 02i
58	−4.1107e − 02 − 4.1162e − 02i	1.1937e − 02 + 1.1988e − 02i
59	−2.9705e − 01 − 2.9714e − 01i	8.7318e − 02 + 8.7410e − 02i
60	−4.1111e − 02 − 4.1167e − 02i	1.1937e − 02 + 1.1988e − 02i

As you can see in Fig. 2, the proposed method is more efficient in CPU time than other methods. To calculate 60 eigenpair derivatives of system, 251.4 s for Zeng's method, 9.66 s for Nelson's method and 1.72 s for the proposed method are required. Even if Zeng's method is the improved modal method, state space form is used to solve damped system as ever, and it has a disadvantage of modal method that many eigenpairs are required for one eigenpair derivative as it was. Hence Zeng's method is not efficient in CPU time. However the CPU time of Nelson's method is comparatively good. We can find that Nelson's method is the efficient method although its calculation speed is slower than the proposed method because of its double size for damped systems. Fig. 2 shows the efficiency of the proposed method obviously. This is due attributable to that the proposed method maintains N-space and finds eigenpair derivatives simultaneously.

4.2. Eigenpair sensitivity in asymmetric damped systems

Whirling beam whose system matrices are asymmetric is considered as numerical example. This example is a gyroscopic system rotating with high speed and has a lumped mass in center of beam as Fig. 3.

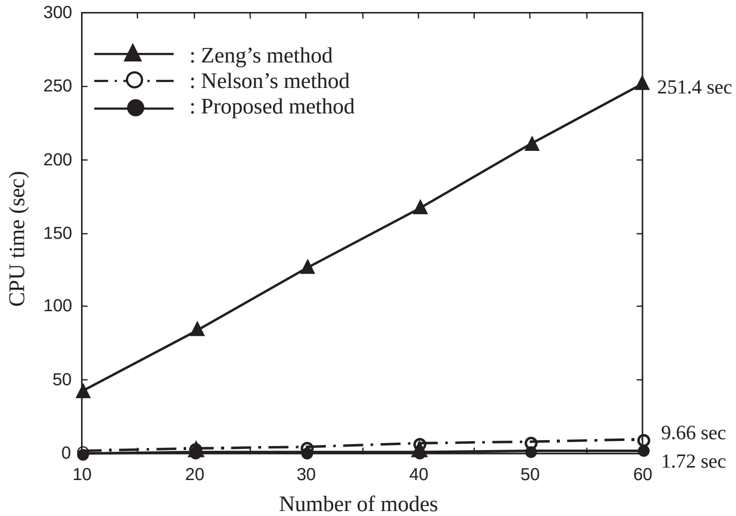

Fig. 2. Comparison of CPU time with previous methods.

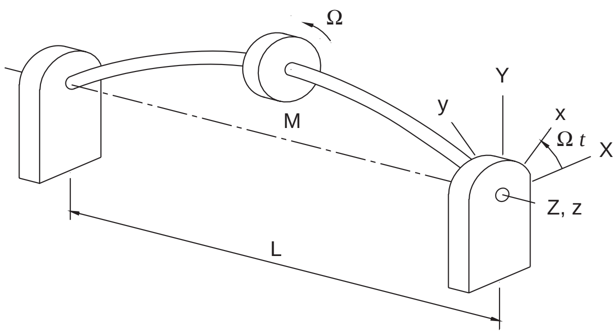

Fig. 3. The whirling beam [21].

The equation of motion of gyroscopic system is as follows:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + (\mathbf{C} + \mathbf{G})\dot{\mathbf{u}}(t) + (\mathbf{K} + \mathbf{H})\mathbf{u}(t) = \mathbf{F}(t), \tag{40}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{F}$ are mass, damping, stiffness and external force matrices, respectively, $\mathbf{G}$ is gyroscopic matrix and $\mathbf{H}$ is circulatory matrix that make system matrix asymmetric,

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{22} \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{22} \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{G}_{12} \\ -\mathbf{G}_{12} & \mathbf{0} \end{bmatrix},$$
$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{22} \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} \mathbf{0} & \mathbf{H}_{12} \\ -\mathbf{H}_{12} & \mathbf{0} \end{bmatrix}. \tag{41}$$

In the case of this system, $\mathbf{G}$ and $\mathbf{H}$ are represented as

$$[\mathbf{G}_{12}]_{ij} = -2\Omega[\mathbf{M}_{11}]_{ij}, \ [\mathbf{H}_{12}]_{ij} = -h\Omega L\delta_{ij}. \tag{42}$$

And material data are as follows:

$$m_0 = 1\ \mathrm{kg/m},\ \mathbf{M} = 1\ \mathrm{kg},\ L = 1\ \mathrm{m},\ \mathbf{K}_1 = \mathbf{K}_2 = L^2/20\ \mathrm{N\ m},\ c = h = 1/4\ \mathrm{N\ s\ m^{-1}},$$

$$\mathbf{EI}_x = 4L^3/5\pi^2\ \mathrm{N\ m^2},\ \mathbf{EI}_y = 9L^3/5\pi^2\ \mathrm{N\ m^2},\ \Omega = \sqrt{21.6\pi}\ \mathrm{rad\ s^{-1}}.$$

Degrees of freedom of system are 20 and the length of beam L is used as design parameter. Computer that has CPU capacity 266 Hz and Ram 64M is used. Brandon's method is comparable to the proposed method and CPU times of each method are compared.

Tables 3 and 4 show some parts of eigenvalue and eigenvector and their derivatives with respect to design variable and the comparison of CPU times for each method is expressed in Fig. 4.

As you can see in Fig. 4, the analysis time to obtain twenty eigenpair derivatives is 4.06 s for Bradon's method and 1.37 s for the proposed method. Because Brandon's method uses state space form to solve damped system and requires many eigenpair to find one eigenpair derivative, but the proposed method maintains N-space by using only a side condition and finds eigenpair derivatives simultaneously from one equation.

5. Conclusion

The exact expression for the eigenpair derivatives of damped system has been derived. In the proposed method, the eigenpair sensitivities of damped systems are obtained simultaneously from one equation. The approach taken here avoids the use of state space equation and considers the damping problem explicitly by introducing a side condition of differentiation of normalization condition. Therefore, computation size of N-order can be maintained and the CPU time of the proposed method can be improved to compare with previous methods.

Moreover, the exact expression for the eigenpair derivatives of asymmetric damped system also has been derived. Traditional restriction of symmetry has not been imposed on the mass, damping and stiffness matrices. The method has solved the problems due to asymmetric system matrices by determining the eigenpair derivatives simultaneously from one equation, contrary to previous methods that use the left eigenvector to solve asymmetric properties.

Table 3
Eigenvalue and its derivatives of system

Mode number	Eigenvalues	Derivatives
1	2.5572e+00	2.6060e+01−4.8587e+01i
2	−2.8132e+00	1.8853e+01+1.7000e+01i
3	−1.5605e−01−8.5386e+00i	6.5060e+00+4.1185e+01i
4	−1.5605e−01+8.5386e+00i	−1.6344e+01−3.8273e+01i
5	1.8030e−01−1.2320e+01i	2.0863e+00−2.7679e+00i
6	1.8030e−01+1.2320e+01i	3.1890e+00−1.9467e+01i
7	−3.4271e−01−1.6881e+01i	−1.2825e−01+2.9982e+00i
8	−3.4271e−01+1.6881e+01i	−7.0710e+01+2.6666e+01i
9	−3.7190e−01−2.9440e+01i	9.2814e+00+1.5815e+01i
10	−3.7190e−01+2.9440e+01i	1.5909e+01−2.8933e+01i

Table 4
First eigenvector and its derivative of system

d.o.f. number	First eigenvector	Derivative
1	1.8853e−02−1.5591e−03i	8.8232e−02+5.0332e−02i
2	−3.3035e−02+7.8354e−02i	1.01178e−01−9.9813e−01i
3	2.6163e−04+1.0292e−02i	−7.5232e−02+6.6204e−02i
4	4.0338e−03−2.3937e−03i	8.9161e−03+8.7611e−03i
5	−1.4006e−03−6.8349e−04i	3.3524e−04−7.2074e−03i
⋮	⋮	⋮
17	−2.3744e−05+4.2393e−04i	−8.3616e−04+1.1098e−03i
18	5.6922e−07+1.6686e−04i	6.4249e−04+3.6554e−04i
19	1.0351e−04−1.3017e−04i	3.4506e−04−1.9786e−04i
20	−1.4742e−05−3.6962e−05i	2.7889e−04−1.8142e−04i

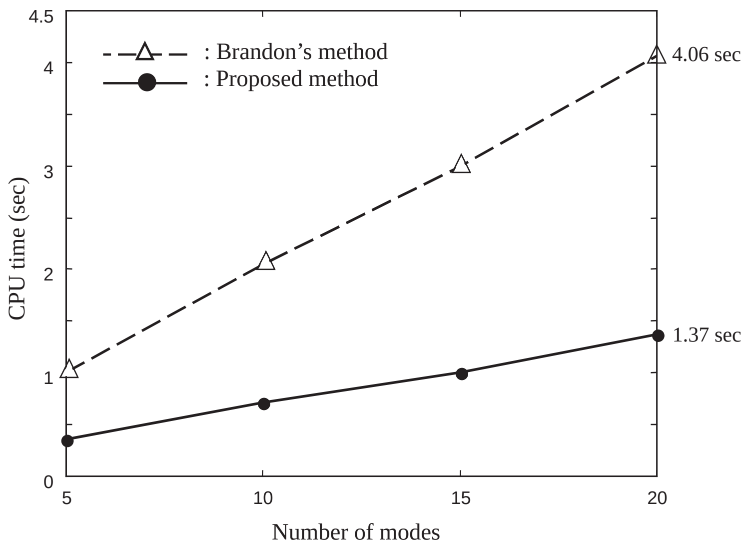

Fig. 4. Comparison of CPU time with Brandon's method.

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