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A note on delay-independent stability of a predator–prey model

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Abstract

This paper analyzes the delay-independent stability of a predator–prey model, with the assistance of the delay-independent stability criteria for a class of retarded dynamical systems. An interesting result is obtained that, the delay-independent stability condition is equivalent to the zero-delay stability condition in this model. The physical meaning of this result is also given.

1. Introduction

Stability of retarded dynamical systems, a challenging problem, has caught the researchers' attention since several decades ago [1,2]. This problem widely appears in bioecology, biomechanics, robotics and machine tool vibrations [3]. Although the stability analysis of systems with three or more delays remains hard, things become better for that of systems with two delays, especially for the delay-independent stability (i.e., the asymptotical stability for all delays) analysis. For example, literature [4] presents a systematic approach to the delay-independent stability analysis of a class of dynamical systems with two delays. And research [5] has developed delay-independent stability criteria for a class of retarded dynamical systems, which extends the criteria and applications in Ref. [4].

As shown in Refs. [4,5], the delay-independent stability region of a delayed system is generally smaller than the zero-delay stability (i.e., stability when the delays are zeros) region, which is also seen from the stability charts of a predator–prey model (refer to Fig. 4.1 for system (4.16) in

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Ref. [3]) and a machine tool vibration system (refer Fig. 3 for system (2.8) in Ref. [6]).[2] So an interesting problem is: is there a system whose delay-independent stability region is equal to the zero-delay stability one? One possible system with this quality, to be checked in this paper, is the two-delay linearized predator–prey model [3,7]

$$\begin{aligned} \dot{x}_1(t) &= -a_{11}x_1(t) - a_{12}x_2(t) + b_{11}x_1(t-\tau_1), \\ \dot{x}_2(t) &= -a_{21}x_1(t) - a_{22}x_2(t) + b_{22}x_2(t-\tau_2), \end{aligned} \quad (1)$$

where $\dot{x}_1(t)$ and $\dot{x}_2(t)$ stand for $\mathrm{d}x_1/\mathrm{d}t$ and $\mathrm{d}x_2/\mathrm{d}t$, respectively; $x_1(t)$ and $x_2(t)$ are the population (relative to the equilibrium of the corresponding non-linear model of Eq. (1) [3,7]) of the predator and prey, respectively. All the constant parameters are positive except the time delays τ_1 and τ_2 are non-negative. The parameters a_{12} and a_{21} can be assumed to be the competition strengths; a_{11} and a_{22} the intrinsic death rates; b_{11} and b_{22} the intrinsic birth rates.

This paper is arranged as follows. Section 2 develops the theoretical background for the stability, and the stability of system (1) is analyzed in Section 3, which is followed by Section 4 giving conclusive remarks.

2. Theoretical background for the stability

In this and next section, the time delays τ_1 and τ_2 may be independent (they are two independent variables; the cases $\tau_1 = \tau \geqslant 0$, $\tau_2 = 0$ and $\tau_1 = 0$, $\tau_2 = \tau \geqslant 0$ may also be included in this case [5, Section 3]) or dependent ($\tau_1 = h_1\tau$, $\tau_2 = h_2\tau$, where h_1, h_2 are two given positive integers and $\tau \geqslant 0$ is the time delay), unless specified. The main result is Theorem 2.1, which is a sufficient stability criterion.

Lemma 2.1 (Wu and Ren [5, Section 2]). *Consider the linear retarded system*

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau_1) + A_2x(t-\tau_2), \quad (2)$$

where $t \in [0, +\infty) \triangleq \bar{\mathbb{R}}^+$; $A_0, A_1, A_2 \in \mathbb{R}^{n\times n}$, $\mathbb{R} = (-\infty, +\infty)$, $n \geqslant 1$; $\tau_1, \tau_2 \in \bar{\mathbb{R}}^+$ *are two times delays*; $x(t), x(t-\tau_1), x(t-\tau_2) \in \mathbb{R}^{n\times 1}$; *rank*$(A_1) =$ *rank*$(A_2) = 1$. *The characteristic equation of system* (2) *is*

$$D(\lambda, \tau_1, \tau_2) = P_{12}(\lambda)\mathrm{e}^{-\lambda(\tau_1+\tau_2)} + P_1(\lambda)\mathrm{e}^{-\lambda\tau_1} + P_2(\lambda)\mathrm{e}^{-\lambda\tau_2} + P_0(\lambda) = 0, \quad (3)$$

where $P_0(\lambda), P_1(\lambda), P_2(\lambda)$ *and* $P_{12}(\lambda)$ *are real coefficient polynomials of the complex number* λ; *the leading coefficient of* $P_0(\lambda)$ *is assumed to be* 1; *deg*$[P_0(\lambda)] >$ *deg*$[P_1(\lambda)] >$ *deg*$[P_{12}(\lambda)]$, *deg*$[P_{12}(\lambda)] <$ *deg*$[P_2(\lambda)] <$ *deg*$[P_0(\lambda)]$.

Lemma 2.2 (Wu and Ren [5, Lemmas 3.1 and 3.2]). *Linear retarded dynamical system* (2) *with characteristic function* (3) *is asymptotically stable if and only if*: (i) *the function* $D(\lambda, 0, 0) = P_{12}(\lambda) + P_1(\lambda) + P_2(\lambda) + P_0(\lambda)$ *is Hurwitz stable*, *and* (ii) *the equation* $D(\lambda, \tau_1, \tau_2) = 0$ *has no non-zero root* λ *on the imaginary axis for any given delays* τ_1 *and* τ_2.

[2] Note the delay-independent region for each of the two systems can be obtained by applying Theorem 2.1 in this paper or Corollary 3.2 in Ref. [5].

Denote

$$\begin{aligned}
&P_{12}(\mathrm{i}\omega) = P_{12R}(\omega) + \mathrm{i}P_{12I}(\omega), \quad P_1(\mathrm{i}\omega) = P_{1R}(\omega) + \mathrm{i}P_{1I}(\omega), \qquad \omega \in \mathbb{R},\\
&P_2(\mathrm{i}\omega) = P_{2R}(\omega) + \mathrm{i}P_{2I}(\omega), \quad P_0(\mathrm{i}\omega) = P_{0R}(\omega) + \mathrm{i}P_{0I}(\omega), \qquad \mathrm{i} = \sqrt{-1},\\
&a(\omega) = P_{12R}(\omega) + P_{0R}(\omega), \quad b(\omega) = P_{12I}(\omega) - P_{0I}(\omega),\\
&c(\omega) = P_{12I}(\omega) + P_{0I}(\omega), \quad d(\omega) = P_{0R}(\omega) - P_{12R}(\omega),\\
&e(\omega) = P_{1R}(\omega) + P_{2R}(\omega), \quad f(\omega) = P_{1I}(\omega) - P_{2I}(\omega),\\
&g(\omega) = P_{1I}(\omega) + P_{2I}(\omega), \quad h(\omega) = -P_{1R}(\omega) + P_{2R}(\omega).
\end{aligned}$$

Lemma 2.3. *Characteristic equation* (3) *has no non-zero root λ on the imaginary axis for any given delays τ_1 and τ_2 if the equation $\bar{A}_L(\omega) = 0$ has no non-zero real root ω, where*

$$\begin{aligned}
\bar{A}_L(\omega) = &- [h(\omega)a(\omega) - f(\omega)c(\omega)]^2 - [e(\omega)c(\omega) - g(\omega)a(\omega)]^2\\
&- [h(\omega)b(\omega) - f(\omega)d(\omega)]^2 - [e(\omega)d(\omega) - g(\omega)b(\omega)]^2\\
&+ [b(\omega)c(\omega) - a(\omega)d(\omega)]^2 + [e(\omega)h(\omega) - f(\omega)g(\omega)]^2.
\end{aligned} \tag{4}$$

Proof. Characteristic equation (3) has no non-zero root λ on the imaginary axis for any given delays τ_1 and τ_2 if and only if the equation

$$\begin{bmatrix} P(\omega,\tau_2) & q(\omega,\tau_2)\\ r(\omega,\tau_2) & s(\omega,\tau_2)\end{bmatrix}\begin{bmatrix}\cos\frac{\omega\tau_1}{2}\\ \sin\frac{\omega\tau_1}{2}\end{bmatrix} = 0 \tag{5}$$

has no non-zero real root ω for any given delays τ_1 and τ_2, where

$$\begin{aligned}
p(\omega,\tau_2) &= [a(\omega) + e(\omega)]\cos\frac{\omega\tau_2}{2} + [b(\omega) - f(\omega)]\sin\frac{\omega\tau_2}{2},\\
q(\omega,\tau_2) &= [b(\omega) + f(\omega)]\cos\frac{\omega\tau_2}{2} + [e(\omega) - a(\omega)]\sin\frac{\omega\tau_2}{2},\\
\tau(\omega,\tau_2) &= [c(\omega) + g(\omega)]\cos\frac{\omega\tau_2}{2} + [d(\omega) - h(\omega)]\sin\frac{\omega\tau_2}{2},\\
s(\omega,\tau_2) &= [d(\omega) + h(\omega)]\cos\frac{\omega\tau_2}{2} + [g(\omega) - c(\omega)]\sin\frac{\omega\tau_2}{2}.
\end{aligned}$$

The determinant of the left part of Eq. (5) is

$$det(\omega,\tau_2) = \frac{m_2(\omega) + m_0(\omega)}{2} + \frac{-m_2(\omega) + m_1(\omega)}{2}\cos\omega\tau_2 + \frac{m_1(\omega)}{2}\sin\omega\tau_2,$$

where

$$\begin{aligned}
m_2(\omega) &= [b(\omega) - f(\omega)][g(\omega) - c(\omega)] - [d(\omega) - h(\omega)][e(\omega) - a(\omega)],\\
m_1(\omega) &= 2[-f(\omega)d(\omega) + b(\omega)h(\omega) - e(\omega)c(\omega) + a(\omega)g(\omega)],\\
m_0(\omega) &= [a(\omega) + e(\omega)][d(\omega) + h(\omega)] - [b(\omega) + f(\omega)][c(\omega) + g(\omega)].
\end{aligned}$$

If $\forall\omega\in\mathbb{R}\backslash\{0\}\triangleq\mathbb{R}^*$, $\bar{A}_L(\omega)\neq 0$, or equivalently $m_1^2(\omega) - 4m_2(\omega)m_0(\omega) < 0$, then $\forall\omega\in\mathbb{R}^*$, $det(\omega,\tau_2)\neq 0$. This means Eq. (5) has no non-zero root ω for any given delays τ_1 and τ_2; thus, this lemma is proved. □

Theorem 2.1. *Linear retarded dynamical system* (2) *with characteristic equation* (3) *is delay-independently stable if*: (i) *the function* $D(\lambda,0,0) = P_{12}(\lambda) + P_1(\lambda) + P_2(\lambda) + P_0(\lambda)$ *is Hurwitz stable, and* (ii) *the equation* $\bar{A}_L(\omega) = 0$ *has no non-zero real root* ω.

Proof. This theorem, can be proved by applying Lemmas 2.2 and 2.3. □

3. Delay-independent stability analysis

Denote $b_{11} = u,\ b_{22} = v,\ a_{11} = u + x,\ a_{22} = v + y,\ a_{12}a_{21} = t$, then the characteristic equation of system (1) is

$$\begin{aligned} D(\lambda,\tau_1,\tau_2) = {} & uv\mathrm{e}^{-\lambda(\tau_1+\tau_2)} - u(\lambda + v + y)\mathrm{e}^{-\lambda\tau_1} - v(\lambda + u + x)\mathrm{e}^{-\lambda\tau_2} \\ & + \lambda^2 + (u + v + x + y)\lambda + (u + x)(v + y) - t. \end{aligned}$$

According to Theorem 2.1, system (1) is delay-independently stable if: (i) The function $D(\lambda,0,0) = \lambda^2 + (x + y)\lambda + xy - t$ is Hurwitz stable, i.e., $x > 0$, $y > 0$, $xy - t > 0$, and (ii) the equation $\bar{A}_L(\omega) = 0$ has no non-zero real root ω, where

$$\bar{A}_L(\omega) = \omega^8 + c_6\omega^6 + c_4\omega^4 + c_2\omega^2 + c_0,$$

in which

$$c_6 = 2y^2 + 4ux + 4vy + 4t + 2x^2,$$

$$\begin{aligned} c_4 = {} & 4x^2y^2 + (4v^2y^2 - 4txv + x^4) + (4u^2x^2 - 4tyu + y^4) + 6t^2 + 4ux^3 \\ & + 4uv(4xy - t) + 8xy(uy + vx) + 8t(ux + vy) + 4(x^2 + y^2 - xy)t + 4vy^3, \end{aligned}$$

$$c_2 = c_{23}t^3 + c_{22}t^2 + c_{21}t + c_{20},$$

$$c_0 = (xy - t)[x(y + 2v) - t][y(x + 2u) - t][(x + 2u)(y + 2v) - t],$$

with

$$c_{23} = 4,$$

$$c_{22} = 2x^2 + 2y^2 + 4ux + 4vy - 8(u + x)(v + y),$$

$$c_{21} = -4(x + u)(y + v)(x^2 + 2ux + y^2 + 2vy) + 4xy(x + 2u)(y + 2v),$$

$$c_{20} = 2xy(x + 2u)(y + 2v)(x^2 + 2ux + y^2 + 2vy).$$

Now it is to be proved that condition (i) contains (ii). Suppose condition (i) is true. It is easy to see $c_6 > 0$, $c_4 > 0$ and $c_0 > 0$. And there exists $c_2 > 0$ because (let $t = kxy,\ 0<k<1$)

$$c_2|_{t=0} = c_{20} > 0,$$

$$c_2|_{t=xy} = (x + y)^2 + 2ux + 2vy > 0,$$

$$\begin{aligned}\frac{\mathrm{d}(c_2)}{\mathrm{d}t} &< -2(1-k)xy[x^2-2(1-3k)xy+y^2]-(12-8k)xy(ux+vy)\\ &\quad -2xy[4u^2-4(1-2k)uy+(1-k)y^2]-8xy[u^2+(k-1)uv+v^2]\\ &\quad -2xy[4v^2-4(1-2k)vx+(1-k)x^2]\\ &< 0.\end{aligned}$$

So $\forall\omega\in\mathbb{R}^*$, $\bar{A}_L(\omega)>0$. This means that condition (i) contains (ii).

To summarize, system (1) is delay-independently stable if the zero-delay stability condition is held true.

4. Conclusive remarks

In Ref. [7], Freedman and Rao proved that system (1) is asymptotically stable if

$$(a_{11}-b_{11})(a_{22}-b_{22})>a_{12}a_{21},$$

$$a_{11}-b_{11}+a_{22}-b_{22}>(\tau_1+\tau_2)(b_{11}a_{22}+a_{11}b_{22}+b_{11}b_{22}). \tag{6}$$

In Ref. [3, pp. 95–98], Stépán obtained a less restrictive result that system (1) is asymptotically stable if

$$(a_{11}-b_{11})(a_{22}-b_{22})>a_{12}a_{21},$$

$$a_{11}+a_{22}-b_{11}-b_{22}>b_{11}(b_{22}+0.22a_{22})\tau_1+b_{22}(b_{11}+0.22a_{11})\tau_2. \tag{7}$$

In Section 3, this paper has shown that system (1) is asymptotically stable if

$$(a_{11}-b_{11})(a_{22}-b_{22})>a_{12}a_{21},$$

$$a_{11}-b_{11}+a_{22}-b_{22}>0. \tag{8}$$

Condition (8) has largely improved on Eqs. (6) and (7), and it is the best sufficient condition for the delay stability of system (1) because it coincides with the zero-delay stability of this system. A more delicate analysis based on the theory in Ref. [5] has shown that system (1) is delay-independently stable if and only if condition (8) is held true when any one of the three conditions is satisfied: (1) τ_1 and τ_2 are independent, (2) $\tau_1=\tau$, $\tau_2=0$, or $\tau_1=0$, $\tau_2=\tau$, (3) $\tau_1=\tau_2=\tau$. The elaboration of this analysis is so long that it is deferred to be shown in Appendix A.

It should be mentioned that the above discussion remains right for the extreme cases $a_{12}a_{21}=0$ or $b_{11}b_{22}=0$.

To understand the above discussion, consider cats and mice in one area as the predator and prey, respectively. Take the two reasonable assumptions: (1) the death rate of cats and mice are, respectively, greater than their birth rate; (2) the competitive strengths of mice to cats is zero while that of cats to mice is positive. For example, let

$$a_{11}=0.05,\quad a_{12}=0,\quad b_{11}=0.01,\quad a_{21}=1,\quad a_{22}=0.3,\quad b_{22}=0.2.$$

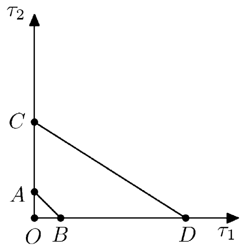

Fig. 1. Contrast condition (11) with Eqs. (9) and (10).

Conditions (6), (7) and (8) are reduced to Eqs. (9), (10) and (11), respectively:

$$\tau_1 + \tau_2 < 28/3, \quad \tau_1 \geqslant 0, \ \tau_2 \geqslant 0. \tag{9}$$

$$133\tau_1 + 210\tau_2 < 7000, \quad \tau_1 \geqslant 0, \ \tau_2 \geqslant 0. \tag{10}$$

$$\tau_1 \geqslant 0, \quad \tau_2 \geqslant 0. \tag{11}$$

These three conditions are plotted in Fig. 1.

In Fig. 1, the inner of triangle AOB together with lines OA and OB except points A and B is corresponding to Eq. (9); the inner of triangle COD together with lines OC and OD except points C and D is corresponding to Eq. (10); the first quadrant of the $\tau_1\tau_2$ plane together with axes τ_1 and τ_7 is corresponding to Eq. (11). It is observed from Fig. 1 that, condition (11) is best of all, and then conditions (10) and (9).

Now we want to give the physical meaning of what we get in Section 3. Define $(a_{11} - b_{11})$ and $(a_{22} - b_{22})$ as the net death rates of the predators and the preys, respectively; then the number of each of the two creatures tends to be stable in spite of the delays in the births, if their net death rates are positive, and if the product of these two net death rates are bigger than that of their competitive strengths.

Appendix A. A more delicate stability analysis

Denote $b_{11} = u > 0$, $b_{22} = v > 0$, $a_{11} = u + x > 0$, $a_{22} = v + y > 0$, $a_{12}a_{21} = t > 0$.

A.1. The case for τ_1 and τ_2 independent

By Theorem 3.1 in Ref. [5], the sufficient condition for model (1) to be delay-independently stable in Section 3 is also the necessary condition.

A.2. The case for $\tau_1 = \tau, \tau_2 = 0$ or $\tau_1 = 0, \tau_2 = \tau$

Only consider the sub-case $\tau_1 = \tau$, $\tau_2 = 0$ (the sub-case $\tau_1 = 0$, $\tau_2 = 0$ is similar). The characteristic equation is

$$D(\lambda, \tau) = -u(\lambda + y)e^{-\lambda\tau} + \lambda^2 + (u + x + y)\lambda + (u + x)y - t.$$

By Corollary 3.2 in Ref. [5], system (1) is delay-independently stable if and only if: (i) $D(\lambda, 0) = \lambda^2 + (x+y)\lambda + xy - t = 0$ is Hurwitz stable, and (ii) the equation $\bar{S}_L(\omega) = -S_L(\omega) = \omega^4 + (x^2 + y^2 + 2ux + 2t)\omega^2 + (2uy + xy - t)(xy - t)$ has no non-zero real root ω. Conditions (i) and (ii) are simplified as $x > 0$, $y > 0$, $xy - t > 0$, i.e., the zero-delay stability condition.

A.3. The case for $\tau_1 = \tau_2 = \tau$

The characteristic equation is

$$D(\lambda, \tau) = uv\mathrm{e}^{-2\lambda\tau} - [(u+v)\lambda + 2uv + uy + vx]\mathrm{e}^{-\lambda\tau} + \lambda^2 + (u+v+x+y)\lambda + (u+x)(v+y) - t.$$

By Corollary 3.3 in Ref. [5], system (1) is delay-independently stable if and only if: (i) the function $D(\lambda, 0) = \lambda^2 + (x+y)\lambda + xy - t$ is Hurwitz stable, i.e., $x > 0$, $y > 0$, $xy - t > 0$, and (ii) $\forall \omega \in \mathbb{R}^*$, (1)$\bar{E}_L(\omega) = -E_L(\omega) \neq 0$, (2) $E_L(\omega) = 0$ but either $F_L(\omega) < 0$ or $G_L(\omega) < 0$, where

$$F_L(\omega) = \omega^4 + [(u-v)^2 + x^2 + 2ux + y^2 + 2vy + 2t]\omega^2 + [(x+2u)(y+2v) - t](xy - t),$$

$$G_L(\omega) = (uy + vx + xy - t - \omega^2)^2 + \omega^2(2u + 2v + x + y)(x + y),$$

$$\bar{E}_L(\omega) = \omega^8 + c_6\omega^6 + c_4\omega^4 + c_2\omega^2 + c_0(xy - t),$$

in which

$$c_6 = (u-v)^2 + 4(t + ux + vy) + 2(x^2 + y^2),$$

$$\begin{aligned} c_4 = {} & (6t^2 - 4txy + 3x^2y^2) + (12xy - 4t)uv + 2(u-v)^2 t \\ & + [x^2y^2 - 2t(vx + uy) + (vx + uy)^2] + 4(x^2 - xy + y^2)t \\ & + [(x^4 - 2tvx^2y + v^2y^2) + (y^4 - 2tuxy^2 + u^2x^2)] \\ & + 2(vy + ux)(u-v)^2 + 8(ux + vy)t + 2u^2x^2 + 2v^2y^2 \\ & + u^4 + v^4 + u^3x + v^3y + 8(uy^2x + vx^2y) + 4(x^3u + y^3v), \end{aligned}$$

$$c_2 = c_{23}t^3 + c_{22}t^2 + c_{21}t + c_{20},$$

$$\begin{aligned} c_0 = {} & -t^3 + (4uv + 4vx + 3xy + 4uy)t^2 \\ & + (-8uv^2x - 8vx^2y - 8u^2vy - 8uy^2x - 18uvxy)t \\ & + (-5u^2y^2 - 5v^2x^2 - 3x^2y^2)t + 2u^3y^3 + 2x^3v^3 + x^3y^3 \\ & + 4x^2uy^3 + 5x^3v^2y + 4x^3vy^2 + 4y^2u^3v + 5xu^2y^3 + 4x^2uv^3 \\ & + 14uv^2yx^2 + 8u^2v^2xy + 14u^2vxy^2 + 14uy^2vx^2, \end{aligned}$$

with

$$c_{23} = 4,$$

$$c_{22} = 2x^2 + u^2 - 10uv - 8vx + 2y^2 - 8uy + v^2 + 4ux + 4vy - 8xy,$$

$$
\begin{aligned}
c_{21} = & -12uy^2v - 4vx^3 + 4uv^2x - 10uv^2y + 2u^2y^2 \\
& - 12uyx^2 - 12vxy^2 + 4u^2vy - 10u^2vx - 4uy^3 \\
& + 2v^2x^2 - 10v^2xy + 8vx^2y - 10u^2yx - 2u^3y + 16uvxy \\
& + 8uy^2x - 2v^3x - 4x^3y - 4y^3x - 12vx^2u + 4x^2y^2,
\end{aligned}
$$

$$
\begin{aligned}
c_{20} = & 4v^3x^2y + 8uy^2x^3 + 4uy^4x + 4vx^4y + 2uv^2x^3 \\
& + 10u^2y^2x^2 + 2u^2vy^3 + 8vx^2y^3 - 8u^2v^2xy + 4u^3y^2x \\
& + 14vx^3yu + 14uy^3xv + 14uv^2xy^2 + 14u^2vx^2y + 4uv^3xy \\
& + 4u^3vyx - 2uy^2vx^2 - 4uv^2yx^2 - 4u^2vxy^2 \\
& + u^2y^4 + 10v^2x^2y^2 + 2y^4x^2 + 2x^4y^2 + v^2x^4.
\end{aligned}
$$

When the zero-delay stability, i.e., condition (i), is satisfied, there exists $F_L(\omega) > 0$ and $G_L(\omega) > 0$, $\forall \omega \in R^*$. So condition (ii) is reduced to $\forall \omega \in R^*$, $\bar{E}_L(\omega) > 0$ (note that $\bar{E}_L(\omega)$ is a real polynomial of ω). It can be seen that $c_6 > 0$ $c_4 > 0$. And there exists $c_2 > 0$ (let $t = kxy$, $0 < k < 1$) because

$$c_2|_{t=0} > 4uvxy(u-v)^2 \geqslant 0,$$

$$c_2|_{t=xy} > [4uvy(u-v)^2 + uy^2(u-2v)^2] > 0,$$

$$
\begin{aligned}
\frac{\mathrm{d}(c_2)}{\mathrm{d}t} < & -[(4-4k)y^2 + (-12k^2 + 16k - 4)xy + (4-4k)x^2]xy \\
& - [(8-2k)u^2 + (-16+20k)uv + (8-2k)u^2]xy \\
& - (12-8k)ux^2y - (12-8k)vy^2x \\
< & \; 0.
\end{aligned}
$$

Also there exists $c_0 > 0$ because

$$
\begin{aligned}
c_0|_{t=0} = & 2u^3y^3 + 2x^3v^3 + x^3y^3 + 4x^2uy^3 + 5x^3v^2y \\
& + 4x^3vy^2 + 4y^2u^3v + 5xu^2y^3 + 4x^2uv^3 \\
& + 14uv^2yx^2 + 8u^2v^2xy + 14u^2vxy^2 + 14uy^2vx^2 > 0,
\end{aligned}
$$

$$c_0|_{t=xy} = 6uv^2yx^2 + 6u^2vxy^2 + 2u^3y^3 + 2x^3v^3 + 4y^2u^3v + 4x^2uv^3 + 8u^2v^2xy > 0,$$

$$\left.\frac{\mathrm{d}(c_0)}{\mathrm{d}t}\right|_{t=0} = -8uv^2x - 8vx^2y - 8u^2vy - 8uxy^2 - 18uvxy - 5u^2y^2 - 5v^2x^2 - 3x^2y^2 < 0,$$

$$\left.\frac{\mathrm{d}(c_0)}{\mathrm{d}t}\right|_{t=xy} = -10uvxy - 8uv^2x - 8u^2vy - 5u^2y^2 - 5v^2x^2 < 0,$$

$$\frac{\mathrm{d}^2(c_0)}{\mathrm{d}t^2} = 6(xy - t) + 8(uy + vx + uv) > 0.$$

So $\forall \omega \in R^*$, $\bar{E}_L(\omega) > 0$ when the zero-delay stability is held.

To summarize, when $\tau_1 = \tau_2 = \tau$, model (1) is asymptotically stable if and only if the zero-stability condition is held true.

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