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Stationary response of multi-degree-of-freedom vibro-impact systems under white noise excitations

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Abstract

A multi-degree-of-freedom vibro-impact system under white noise excitations is formulated as a stochastically excited and dissipated Hamiltonian system. The constraints are modelled as non-linear springs according to the Hertz contact law. The exact stationary solution of the system is derived under certain conditions. The approximate stationary solutions of the system are also obtained by using the stochastic averaging methods for quasi-Hamiltonian systems. It is shown that the stochastic averaging method for quasi-non-integrable-Hamiltonian systems is applicable if the non-linear forces according to the Hertz contact law take an important role in the response of the system while the stochastic averaging method for quasi-integrable-Hamiltonian systems is applicable if the non-linear forces can be neglected. An example for stochastically excited two-degree-of-freedom vibro-impact system is given to illustrate the application of the procedures in detail.

1. Introduction

There has been sustained interest in the response of vibro-impact systems for many years [1]. In recent years the phenomena of bifurcation and chaos in vibro-impact systems under harmonic excitation have been studied extensively [1–7] and the complicated dynamical behaviors of the systems have been found. The exact and approximate stationary solutions of single-degree-of-freedom (SDOF) vibro-impact systems under Gaussian white noise excitation were obtained by Nayak and Jing [8–10] using the Hertz contact law [11], where the contact force between two elastic bodies is assumed to be proportional to the $\frac{3}{2}$ power of the relative displacement between them. It is shown in Ref. [10] that the contact phenomena are

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almost negligible when the clearance is about twice the root mean square displacement of the corresponding linear system. Hess et al. [12,13] studied the normal vibration and friction at a Hertzian contact under random excitation and the result were verified by experiment. Lin and Bapat [14] proposed an approach for estimating the clearance of a stochastically excited vibro-impact system. Dimentberg [15,16] obtained the exact and approximate solutions for stochastic excited SDOF vibro-impact system with rigid barriers. To the authors' knowledge, so far there is no exact or approximate solutions for stochastically excited multi-degree-of-freedom (MDOF) vibro-impact systems.

On the other hand, the theory of exact stationary solutions and stochastic averaging methods for stochastically excited and dissipated Hamiltonian systems have been proposed recently [17–19]. They have been applied to predict response, stochastic stability and stochastic bifurcation successfully [20,21]. The stochastically excited MDOF vibro-impact systems are a subclass of stochastically excited and dissipated Hamiltonian systems. In the present paper, the exact stationary solutions of MDOF vibro-impact systems will be obtained by using the theory of the exact stationary solution for stochastically excited and dissipated Hamilton systems, and the approximate stationary solutions of the systems will be obtained by using the stochastic averaging method for quasi-non-integrable-Hamiltonian systems and for quasi-integrable-Hamiltonian systems. An example of stochastically excited 2DOF vibro-impact system is given to illustrate the application of the proposed procedures.

2. MDOF vibro-impact system

Consider a MDOF vibro-impact system under additive Gaussian white noise excitations as shown is Fig. 1. The equations of motion of the system are of the form

$$\begin{aligned} &m_i\ddot{X}_i + c_{ij}(\mathbf{X},\dot{\mathbf{X}})\dot{X}_j + k_i(X_i - X_{i-1}) + k_{i+1}(X_i - X_{i+1}) + g_i(X_i) = f_{il}W_l(t) \\ &X_0 = X_{n+1} = 0, \quad k_{n+1} = 0, \quad i,j = 1,2,\ldots,n, \quad l = 1,2,\ldots,m, \end{aligned} \tag{1}$$

where X_i is the displacement of mass m_i, k_i, k_{i+1} are the stiffnesses of spring at two sides of mass $m_i, c_{ij}(\mathbf{X},\dot{\mathbf{X}})$ are damping coefficients and f_{il} are amplitudes of stochastic excitations. $W_l(t)$ are Gaussian white noises in the sense of Stratonovich [22] with correlation functions $E[W_k(t)W_l(t+\tau)] = 2D_{kl}\delta(\tau)$. Each mass and its constraints are regarded as elastic bodies and the relationship between the contact force and displacement is assumed to be governed by Hertz contact law. Thus

$$g_i(X_i) = \begin{cases} B_{ir}(X_i - \delta_{ir})^{3/2}, & X_i > \delta_{ir}, \\ 0, & -\delta_{il} \leqslant X_i \leqslant \delta_{ir}, \\ -B_{il}(-X_i - \delta_{il})^{3/2}, & X_i < -\delta_{il}, \end{cases} \tag{2}$$

where B_{ir} and B_{il} represent contact stiffnesses of mass m_i, which are functions of geometries and material properties of mass m_i and its both side constraints. A detailed description of the contact stiffness for several cases can be found in Ref. [12].

System (1) can be rewritten as a stochastically excited and dissipated Hamiltonian system. Let

$$Q_i = X_i, \quad P_i = m_i\dot{X}_i. \tag{3}$$

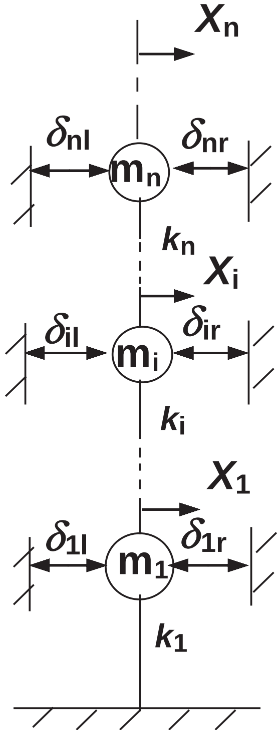

Fig. 1. A sketch of nDOF vibro-impact system.

Eq. (1) can be rewritten as

$$\begin{aligned} \dot{Q}_i &= \frac{\partial H}{\partial P_i}, \\ \dot{P}_i &= -\frac{\partial H}{\partial Q_i} - c_{ij}\frac{\partial H}{\partial P_j} + f_{il}W_l(t), \quad i,j = 1,2,\ldots,n; \;\; l = 1,2,\ldots,m, \end{aligned} \tag{4}$$

where

$$\begin{aligned} H &= \frac{1}{2}\sum_{i=1}^{n}\frac{P_i^2}{m_i} + \frac{1}{2}\sum_{i=1}^{n}k_i(Q_i - Q_{i-1})^2 + \sum_{i=1}^{n}U_i(Q_i), \\ U_i(Q_i) &= \int_0^{Q_i} g_i(X_i)\,\mathrm{d}X_i = \begin{cases} \frac{2}{5}B_{ir}(Q_i - \delta_{ir})^{5/2}, & Q_i > \delta_{ir}, \\ 0, & -\delta_{il} \leqslant Q_i \leqslant \delta_{ir}, \\ \frac{2}{5}B_{il}(-Q_i - \delta_{il})^{5/2}, & Q_i < -\delta_{il}. \end{cases} \end{aligned} \tag{5}$$

Eq. (4) can be further converted into Itô differential equations [22]

$$
\begin{aligned}
\mathrm{d}Q_i &= \frac{\partial H}{\partial P_i}\,\mathrm{d}t \\
\mathrm{d}P_i &= \left[-\frac{\partial H}{\partial Q_i} - c_{ij}(\mathbf{Q},\mathbf{P})\frac{\partial H}{\partial P_j}\right]\mathrm{d}t + f_{il}\,\mathrm{d}B_l(t), \quad i,j = 1,2,\ldots,n; \;\; l = 1,2,\ldots,m,
\end{aligned} \tag{6}
$$

where $B_l(t)$ are standard Wiener processes.

3. Exact stationary solution

The Hamiltonian system associated with Eq. (6) without dampings and stochastic excitations is generally non-integrable. Based on the theory of exact stationary solution of stochastically excited and dissipated Hamiltonian systems [17], the exact stationary solution of Eq. (6) is of the form

$$
p(\mathbf{Q},\mathbf{P}) = C\exp\left[-\int_0^H h(u)\,\mathrm{d}u\right]\Bigg|_{H=H(\mathbf{Q},\mathbf{P})}, \tag{7}
$$

where C is a normalization constant and

$$
h(H) = \frac{2c_{ij}\frac{\partial H}{\partial p_j},}{b_{ij}\frac{\partial H}{\partial p_j}} \quad i,j = 1,2,\ldots,n; \;\; k,l = 1,2,\ldots,m \tag{8}
$$

and $b_{ij} = 2D_{kl} f_{ik} f_{jl}$.

4. Approximate stationary solution by using the stochastic averaging method for quasi-non-integrable-Hamiltonian systems

It can be seen from Eq. (8) that the exact stationary solution exists only in very limited cases. Thus, only the approximate stationary solution of system (6) can be obtained in most cases. The stochastic averaging method for quasi-Hamiltonian systems is a powerful method to predict the response of stochastically excited and dissipated quasi-Hamiltonian systems. Based on the stochastic averaging method for quasi-non-integrable-Hamiltonian systems [18], the following averaged Itô equation can be obtained from Eq. (6)

$$
\mathrm{d}H = s(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{d}B(t), \tag{9}
$$

where $B(t)$ is standard Wiener process,

$$s(H) = \frac{1}{T(H)}\int_{\Omega}\left[\left(-c_{ij}\frac{\partial H}{\partial P_i}\frac{\partial H}{\partial P_j} + D_{kl}f_{ik}f_{jl}\right)\Big/\frac{\partial H}{\partial P_1}\right]\mathrm{d}Q_1\cdots\mathrm{d}Q_n\,\mathrm{d}P_2\cdots\mathrm{d}P_n,$$

$$\sigma^2(H) = \frac{1}{T(H)}\int_{\Omega}\left[2D_{kl}f_{ik}f_{jl}\Big/\frac{\partial H}{\partial P_1}\right]\mathrm{d}Q_1\cdots\mathrm{d}Q_n\,\mathrm{d}P_2\cdots\mathrm{d}P_n,$$

$$T(H) = \int_{\Omega}\left(1\Big/\frac{\partial H}{\partial P_1}\right)\mathrm{d}Q_1\cdots\mathrm{d}Q_n\,\mathrm{d}P_2\cdots\mathrm{d}P_n,$$

$$\Omega = \{(Q_1,\ldots,Q_n,P_2,\ldots,P_n)|H(Q_1,\ldots,Q_n,0,P_2,\ldots,P_n)\leqslant H\}. \tag{10}$$

The exact stationary solution of the Fokker–Planck–Kolmogorov (FPK) equation associated with averaged Itô equation (9) is of the form

$$p(H) = \frac{C}{\sigma^2(H)}\exp\left[-\int_0^H \frac{2s(X)}{\sigma^2(X)}\mathrm{d}X\right], \tag{11}$$

where C is a normalization constant. The joint stationary probability density of generalized displacements q_i and momenta p_i is then obtained by using relationship $p(\mathbf{q},\mathbf{p}) = p(q_1,\ldots,q_n;p_2,\ldots,p_n\,|\,H)p(H)|\partial H/\partial p_1|$ and $p(q_1,\ldots,q_n;p_2,\ldots,p_n|H) = 1/(T(H)|\partial H/\partial p_1|)$ as $p(\mathbf{q},\mathbf{p}) = [p(H)/T(H)]|_{H=H(\mathbf{q},\mathbf{p})}$ [18].

5. Approximate stationary solution by using the stochastic averaging method for quasi-integrable-Hamiltonian systems

For stochastically excited SDOF vibro-impact system, i.e., $n = 1$ in Eq. (1), the non-linear term in Eq. (1) can be neglected if $\min(\delta_{1l},\delta_{1r}) > 2\bar{\sigma}(X_1)$, where $\bar{\sigma}(X_1)$ is the root mean square displacement of oscillator [9]. Similar conclusion can be drawn for MDOF vibro-impact systems. Suppose that all the non-linear terms in Eq. (1) can be neglected, then Eq. (1) describes a stochastically excited and dissipated integrable-Hamiltonian system. The equations of motion in this case are

$$m_i\ddot{X}_i + c_{ij}(\mathbf{X},\dot{\mathbf{X}})\dot{X}_j + k_i(X_i - X_{i-1}) + k_{i+1}(X_i - X_{i+1}) = f_{il}W_l(t)$$
$$X_0 = X_{n+1} = 0,\quad k_{n+1} = 0,\quad i = 1,2,\ldots,n,\ \ l = 1,2,\ldots,m \tag{12}$$

or

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F}\mathbf{W}, \tag{13}$$

where

$$\mathbf{X} = \{X_1, X_2, \ldots, X_n\}^{\mathrm{T}}, \quad \mathbf{W} = \{W_1, W_2, \ldots, W_m\}^{\mathrm{T}}, \quad \mathbf{M} = \mathrm{diag}\{m_1, m_2, \ldots, m_n\},$$

$$[\mathbf{C}]_{ij} = c_{ij}(\mathbf{X}, \dot{\mathbf{X}}), \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 \ \cdots & & & -k_n & k_n \end{bmatrix}, \quad [\mathbf{F}]_{il} = f_{il},$$
$$i,j = 1,2,\ldots,n, \quad l = 1,2,\ldots,m.$$

Let

$$\mathbf{X} = \mathbf{TQ}, \tag{14}$$

where $\mathbf{T}$ is the modal matrix and $\mathbf{Q} = \{Q_1, \ldots, Q_n\}^{\mathrm{T}}$ is a vector of modal displacements. Substituting Eq. (14) into Eq. (13) and multiplying $\mathbf{T}^{\mathrm{T}}$ on both sides of Eq. (13) lead to

$$M_i^* \ddot{Q}_i + c_{ij}^* \dot{Q}_{ij} + K_i^* Q_i = f_{il}^* W_l(t), \quad i,j = 1,2,\ldots,n, \tag{15}$$

where M_i^* and K_i^* are modal mass and stiffness, respectively, $c_{ij}^* = \sum_{r=1}^{n} T_{ri} c_{rs} T_{sj}$, $f_{il}^* = \sum_{r=1}^{n} T_{ri} f_{rl}$. Eq. (15) can be rewritten as

$$\begin{aligned} \dot{Q}_i &= \frac{\partial H}{\partial P_i}, \\ \dot{P}_i &= -\frac{\partial H}{\partial Q_i} - c_{ij}^* \frac{\partial H}{\partial P_j} + f_{il}^* \,\mathrm{d}B_l(t), \quad i,j = 1,2,\ldots,n; \ \ l = 1,2,\ldots,m, \end{aligned} \tag{16}$$

where $B_l(t)$ are standard Wiener processes,

$$H = \sum_{i=1}^{n} \left(\frac{1}{2} \frac{P_i^2}{M_i^*} + \frac{1}{2} K_i^* Q_i^2 \right) = \sum_{i=1}^{n} H_i, \quad P_i = M_i^* \dot{Q}_i.$$

Eq. (16) without dampings and stochastic excitations is an integrable-Hamiltonian system. Suppose that there is no internal resonant relations in the system, i.e., $l_i \omega_i \neq O(\varepsilon)$, where ω_i are the natural frequencies of the system and l_i are integers and ε is a small positive parameter. The averaged Itô equations for $H_1, H_2, \ldots, H_n$ can be derived by using stochastic averaging method for quasi-integrable-Hamiltonian systems [19], i.e.,

$$\mathrm{d}H_i = s_i(\mathbf{H})\,\mathrm{d}t + \sigma_{il}\,\mathrm{d}B_l(t), \quad i = 1,2,\ldots,n; \ \ l = 1,2,\ldots,m, \tag{17}$$

where

$$
\begin{aligned}
s_i(\mathbf{H}) &= \left\langle -c_{ij}^* \frac{P_i}{M_j^*} P_j + \frac{D_{kl}}{M_i^*} f_{ik}^* f_{il}^* \right\rangle,\\
b_{ij}(\mathbf{H}) &= [\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}]_{ij} = \left\langle 2D_{kl} f_{ik}^* f_{jl}^* \frac{P_i}{M_i^*} \frac{P_j}{M_j^*} \right\rangle,\\
\langle \cdot \rangle &= \frac{1}{T_1 \cdots T_n} \int_0^{2\pi} \cdots \int_0^{2\pi} \langle \cdot \rangle \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n,\\
T_i &= \frac{2\pi}{\omega_i} = 2\pi \Big/ \sqrt{\frac{K_i^*}{M_i^*}},\\
Q_i &= \sqrt{2H_i/K_i^*} \cos\theta_i,\\
P_i &= -\sqrt{2H_i M_i^*} \sin\theta_i. \qquad (18)
\end{aligned}
$$

The FPK equation associated with averaged Itô equation (17) is of the form

$$
\frac{\partial p}{\partial t} = -\frac{\partial(s_i p)}{\partial H_i} + \frac{1}{2}\frac{\partial^2(b_{ij}p)}{\partial H_i\, \partial H_j}. \qquad (19)
$$

The stationary probability density $p(\mathbf{H})$ can be obtained from Eq. (19) if the diffusion and drift coefficients satisfy the compatibility conditions [19]. The joint probability density of modal displacements and momenta is then of the form

$$
p(\mathbf{Q}, \mathbf{P}) = \frac{p(\mathbf{H})}{T_1 \cdots T_n}\bigg|_{H_i = H_i(Q_i, P_i)} \qquad (20)
$$

and joint probability density of displacements and velocities, $p(\mathbf{X}, \dot{\mathbf{X}})$, can be derived by using transformation (14).

6. Example

Consider a 2DOF vibro-impact system under Gaussian white noise excitations as shown in Fig. 2. The equations of motion of the system are of the form

$$
\begin{aligned}
&m_1\ddot{X}_1 + c_1\dot{X}_1 + k_1X_1 + k_2(X_1 - X_2) = W_1(t),\\
&m_2\ddot{X}_2 + c_2\dot{X}_2 + k_2(X_2 - X_1) + g_2(X_2) = W_2(t), \qquad (21)
\end{aligned}
$$

where the notations of $c_i, k_i, g_2(X_2)$ are the same as those in Eq. (1) and $W_i(t)$ are independent Gaussian white noise in the senses of Stratonovich with intensity $2D_i$.

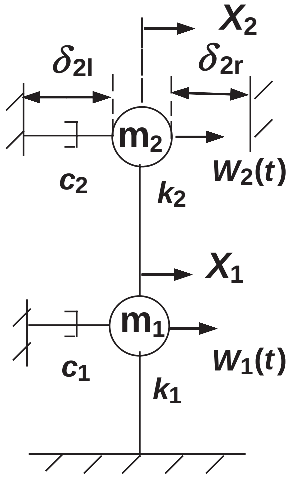

Fig. 2. A sketch of 2DOF vibro-impact system.

6.1. Exact stationary solution

By using the procedure described in Section 3, the exact stationary solution of system (21) is obtained as follows:

$$p(\mathbf{Q},\mathbf{P}) = C\exp\left[-\frac{c_1}{D_1}H\right]\Bigg|_{H=\frac{P_1^2}{2m_1}+\frac{P_2^2}{2m_2}+\frac{1}{2}k_1Q_1^2+\frac{1}{2}k_2(Q_2-Q_1)^2+U_2(Q_2)} \tag{22}$$

provided that $c_1/D_1 = c_2/D_2 = h$. The stationary probability density of displacement q_2 is obtained from (22) as follows:

$$p(q_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(q_1, q_2, p_1, p_2)\,\mathrm{d}q_1\,\mathrm{d}p_1\,\mathrm{d}p_2. \tag{23}$$

The exact stationary probability density $p(q_2)$ of system (21) with both side constraints are shown in Fig. 3 for different parameter values while that for the system with right hand side constraint only are shown in Fig. 4. It can be seen from Figs. 3 and 4 that the property of the stationary response of the system is determined by the contact stiffness, the clearances between the masses and constraints, and the ratios of damping coefficients to intensities of stochastic excitations. The response of the system is non-Gaussian when the contact stiffness B is bigger than 0.1, the clearances δ is less than 2, and the ratio h is less than 10. The contact force takes an important role in the response of the system for these parameter values. On the other hand, the response of the system are almost Gaussian and the non-linear term $g(X_2)$ in Eq. (21) can be neglected if the contact stiffness B is less than 0.1, or the clearance δ is bigger than 2, or the ratio h is bigger than 10. So, the approximate stationary solution of system (21) must be obtained by using the stochastic averaging method for quasi-non-integrable-Hamiltonian systems and for quasi-integrable-Hamiltonian systems, respectively.

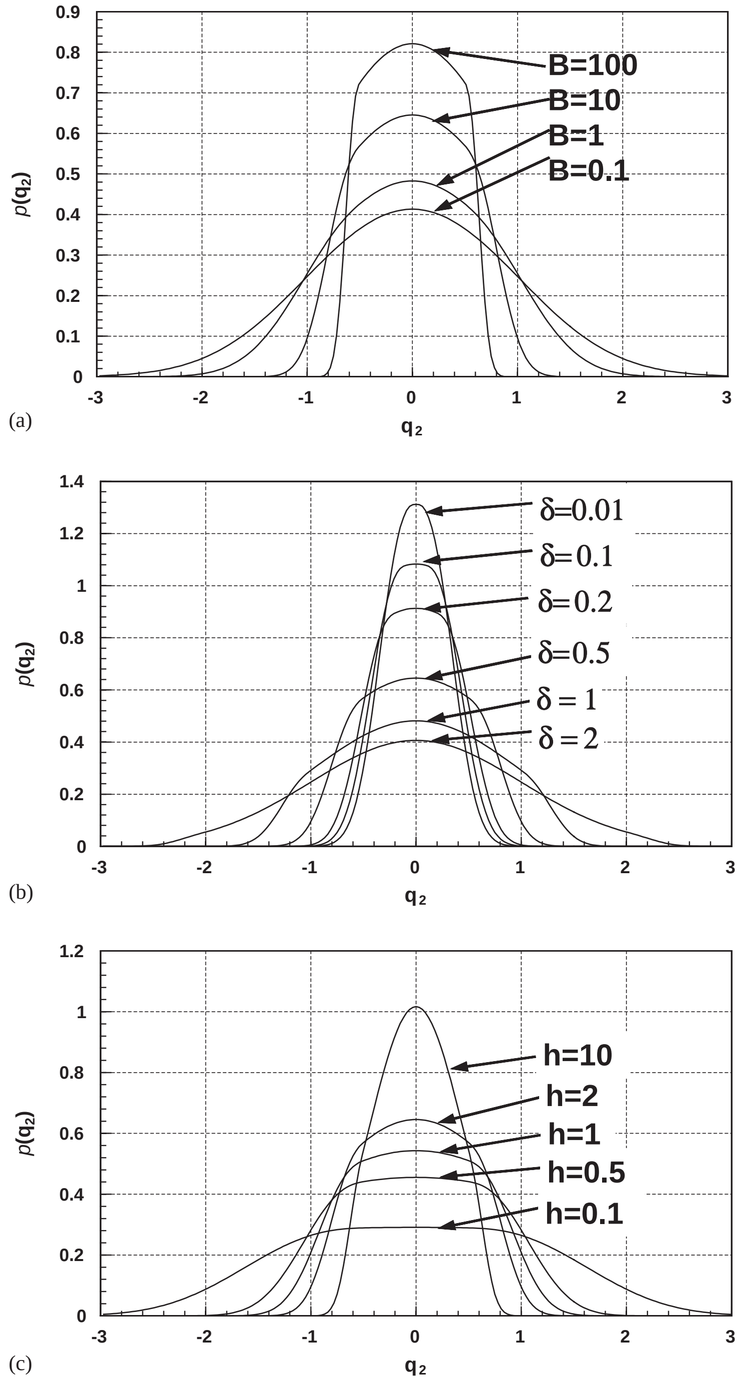

Fig. 3. Exact stationary probability density of displacement q_2 of system (21) with both side constraints. $m_1=m_2=1$, $k_1=k_2=1$. (a) $h=\frac{c_1}{D_1}=\frac{c_2}{D_2}=2.0, \delta=\delta_{2l}=\delta_{2r}=0.5, B=B_{2l}=B_{2r}$. (b) $h=\frac{c_1}{D_1}=\frac{c_2}{D_2}=2.0, B=B_{2l}=B_{2r}=10, \delta=\delta_{2l}=\delta_{2r}$. (c) $\delta=\delta_{2l}=\delta_{2r}=0.5, B=B_{2l}=B_{2r}=10, h=\frac{c_1}{D_1}=\frac{c_2}{D_2}$.

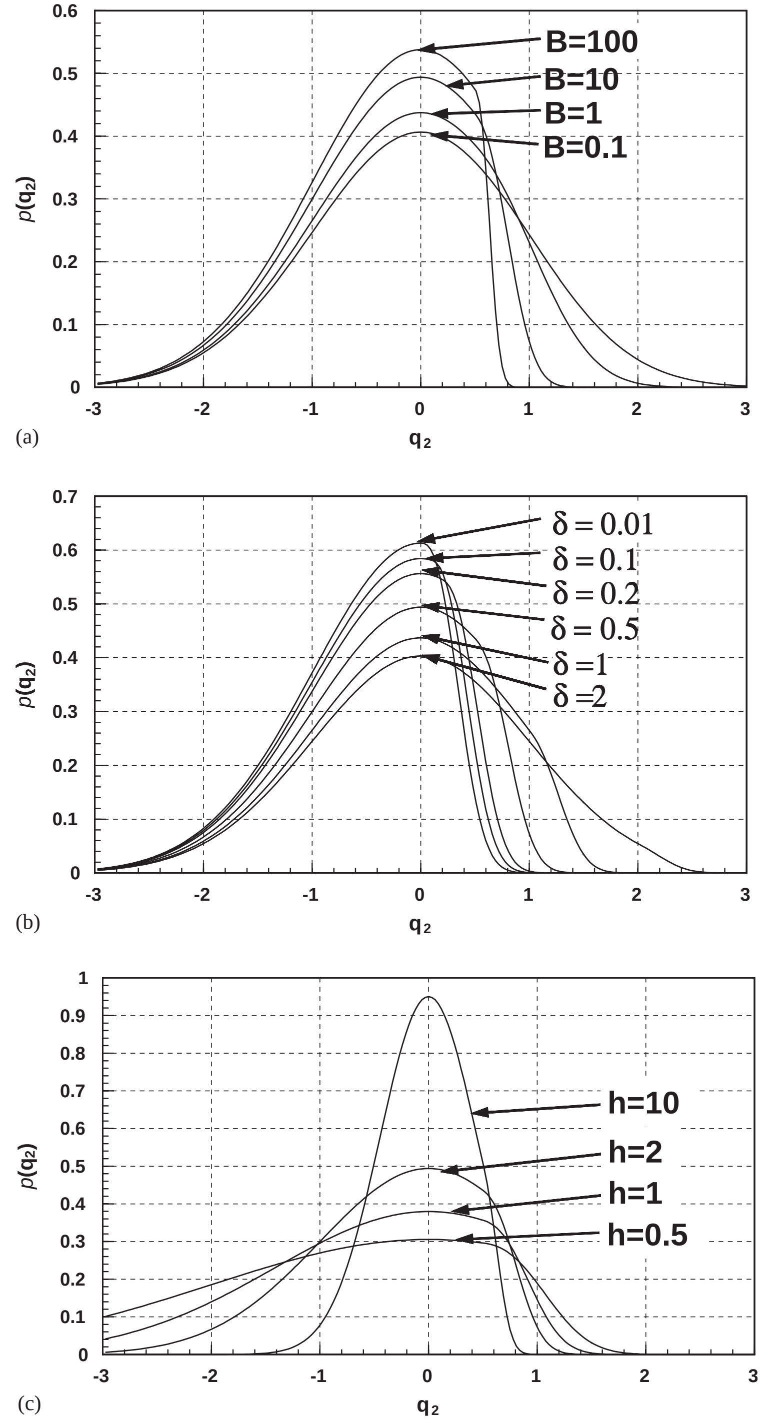

Fig. 4. Exact stationary probability density of displacement q_2 of system (21) with right hand side constraint only. $m_1 = m_2 = 1, k_1 = k_2 = 1$. (a) $h = \frac{c_1}{D_1} = \frac{c_2}{D_2} = 2.0, \delta = \delta_{2r} = 0.5, B = B_{2r}$. (b) $h = \frac{c_1}{D_1} = \frac{c_2}{D_2} = 2.0, B = B_{2r} = 10, \delta = \delta_{2r}$. (c) $\delta = \delta_{2r} = 0.5, B = B_{2r} = 10, h = \frac{c_1}{D_1} = \frac{c_2}{D_2}$.

6.2. Approximate stationary solution by using stochastic averaging method for quasi-non-integrable Hamiltonian systems

If $c_1/D_1 \neq c_2/D_2$ and the non-linear term $g(X_2)$ in Eq. (21) cannot be neglected, the approximate stationary solution of system (21) can be obtained by using the procedure described in Section 4. The averaged Itô equation for Hamiltonian H is of the form

$$\mathrm{d}H = s(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{d}B(t), \tag{24}$$

where

$$\begin{aligned}
s(H) &= \frac{1}{T(H)}\int_{\Omega'}[B_1(H-V)+B_2]\,\mathrm{d}Q_1\,\mathrm{d}Q_2,\\
\sigma^2(H) &= \frac{2}{T(H)}\int_{\Omega'} B_2(H-V)\,\mathrm{d}Q_1\,\mathrm{d}Q_2,\\
T(H) &= \int_{\Omega'}\pi\,\mathrm{d}Q_1\,\mathrm{d}Q_2,\\
B_1 &= -\pi(c_1+c_2),\\
B_2 &= \pi(D_1+D_2),\\
V(Q_1,Q_2) &= \tfrac{1}{2}k_1Q_1^2 + \tfrac{1}{2}k_2(Q_2-Q_1)^2 + U_2(Q_2),\\
\Omega' &= \{(Q_1,Q_2)\,|\,H(Q_1,Q_2,0,0)\leqslant H\}.
\end{aligned} \tag{25}$$

The integral region Ω' is shown in Fig. 5 where

$$\begin{aligned}
Q_1^{\pm} &= (k_2Q_2 \pm Z)/(k_1+k_2),\\
Z = Z(Q_2,H) &= \sqrt{k_2^2Q_2^2 - 2(k_1+k_2)\left[\tfrac{1}{2}k_2Q_2^2 + G(Q_2) - H\right]}
\end{aligned} \tag{26}$$

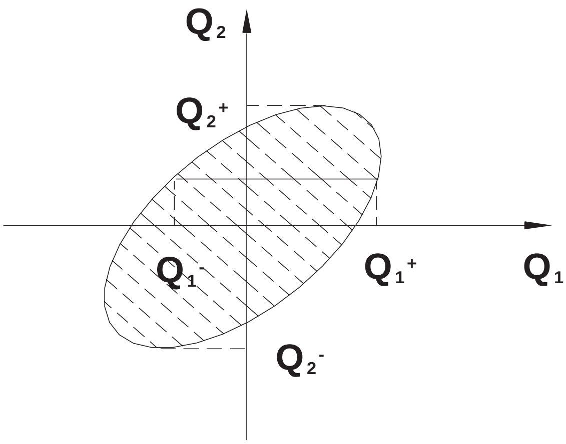

Fig. 5. Approximate integral domain Ω' in Eq. (25).

and Q_2^+, Q_2^- are the two roots of non-linear equation

$$\frac{k_1 k_2}{2(k_1+k_2)} Q_2^2 + U_2(Q_2) - H = 0. \tag{27}$$

Eq. (25) can be further simplified as

$$\begin{aligned}
s(H) &= \frac{c_1+c_2}{3(k_1+k_2)} \frac{I_2(H)}{I_1(H)} + D_1 + D_2,\\
\sigma^2(H) &= \frac{2(D_1+D_2)}{3(k_1+k_2)} \frac{I_2(H)}{I_1(H)},\\
I_1(H) &= \int_{Q_2^-}^{Q_2^+} Z(Q_2,H)\,\mathrm{d}Q_2.\\
I_2(H) &= \int_{Q_2^-}^{Q_2^+} Z^3(Q_2,H)\,\mathrm{d}Q_2.
\end{aligned} \tag{28}$$

Noting that $\mathrm{d}I_2/\mathrm{d}H = 3(k_1+k_2)I_1$, the stationary solution of the FPK equation associated with averaged Itô equation (24) can be derived by using Eq. (11) as follows:

$$p(H) = CI_1(H)\exp\left[-\frac{c_1+c_2}{D_1+D_2}H\right], \tag{29}$$

where C is a normalization constant. The joint stationary probability densities of generalized displacement and momenta and the stationary probability density of displacement q_2 are then

$$\begin{aligned}
p(q_1,q_2,p_1,p_2) &= \frac{p(H)}{T(H)}\bigg|_{H=H(\mathbf{q},\mathbf{p})} = \bar{C}\exp\left[-\frac{c_1+c_2}{D_1+D_2}H\right]\bigg|_{H=H(\mathbf{q},\mathbf{p})},\\
p(q_2,p_2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(q_1,q_2,p_1,p_2)\,\mathrm{d}q_1\,\mathrm{d}p_1,\\
p(q_2) &= \int_{-\infty}^{+\infty} p(q_2,p_2)\,\mathrm{d}p_2.
\end{aligned} \tag{30}$$

Note that the joint stationary probability density of generalized displacements and momenta in Eq. (30) is the same as that in Eq. (22) if $c_1/D_1 = c_2/D_2$, i.e., the stationary solution obtained by using the stochastic averaging method for quasi-non-integrable-Hamiltonian systems is the exact stationary solutions of the system in this special case.

The stationary probability density of displacement q_2 of system (21) with both side constraints for different parameter values obtained by using the stochastic averaging method and digital simulation is shown in Fig. 6. It is seen from Fig. 6 that for larger contact stiffness and the intensity of stochastic excitation and smaller clearance between the mass and constraints, the stochastic averaging method yields better results. The joint stationary probability density of displacement q_2 and momentum p_2 of the system with both side constraints is shown in Fig. 7. The

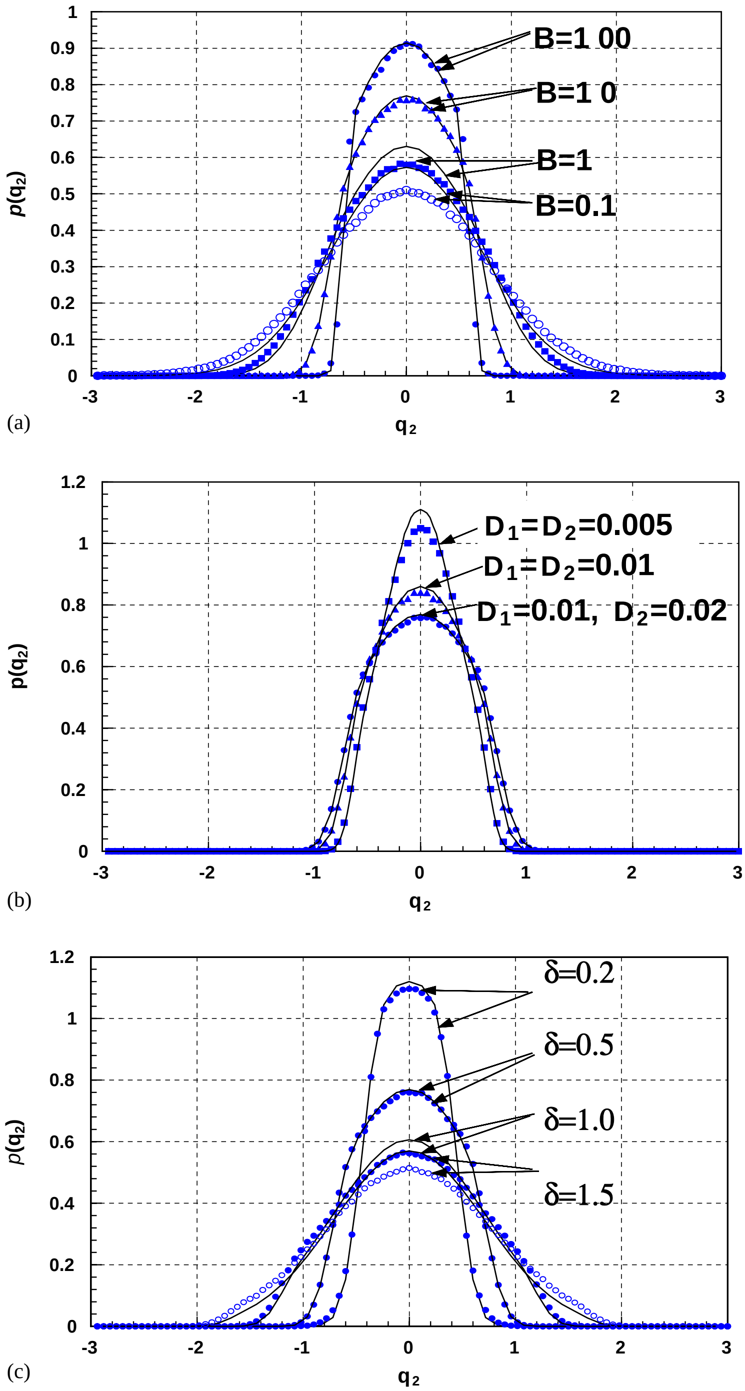

Fig. 6. Approximate stationary probability density of displacement q_2 of system (21) with both side constraints. ——, analytical solution; ●▲■○, from digital simulation. $m_1 = m_2 = 1, k_1 = k_2 = 1$. (a) $D_1 = 0.01, D_2 = 0.02, c_1 = 0.08$, $c_2 = 0.04, \delta = \delta_{2r} = \delta_{2l} = 0.5, B = B_{2l} = B_{2r}$. (b) $B = B_{2l} = B_{2r} = 10, c_1 = 0.08, c_2 = 0.04, \delta = \delta_{2r} = \delta_{2l} = 0.5, D_1, D_2$. (c) $B = B_{2l} = B_{2r} = 10, D_1 = 0.01, D_2 = 0.02, c_1 = 0.08, c_2 = 0.04, \delta = \delta_{2r} = \delta_{2l}$.

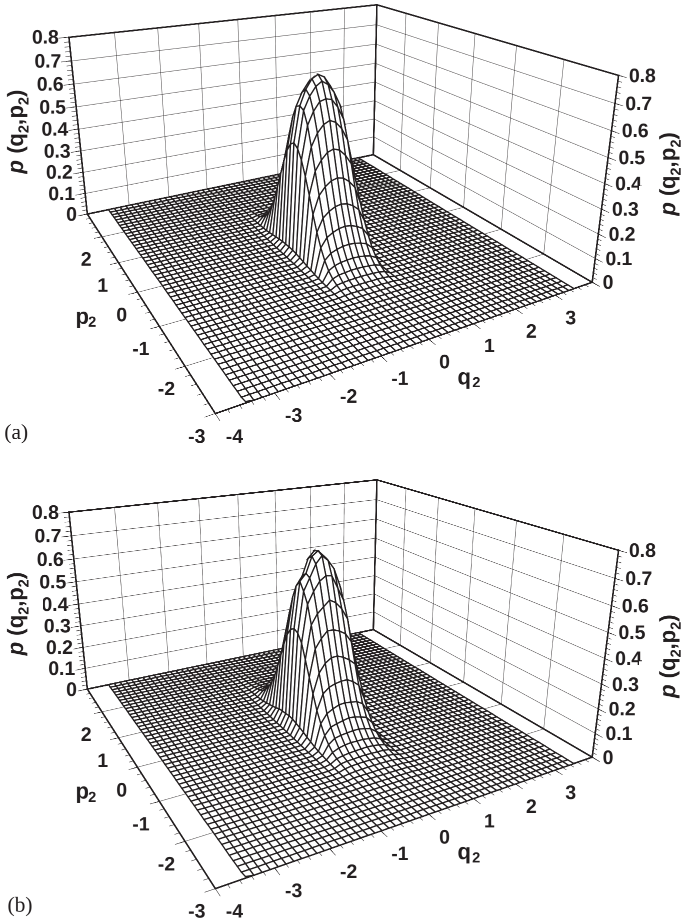

Fig. 7. Approximate joint stationary probability density of displacement and momentum of system (21) with both side constraints. (a) analytical solution; (b) from digital simulation. $m_1 = m_2 = 1, k_1 = k_2 = 1, B = B_{2l} = B_{2r} = 10, D_1 = 0.01, D_2 = 0.02, c_1 = 0.08, c_2 = 0.04, \delta = \delta_{2r} = \delta_{2l} = 0.5$.

results obtained by using the stochastic averaging method agree well with those from digital simulation. The stationary probability density of system (21) with right hand side constraint only for different parameter values can be obtained similarly and the similar conclusion can be drawn.

6.3. Approximate stationary solutions by using stochastic averaging method for quasi-integrable-Hamiltonian systems

It has been pointed out in the last section that the stochastic averaging method for quasi-non-integrable-Hamiltonian system can be employed to predict the response of system (21) if the non-linear terms take an import role in the response of the system. For the case that the non-linear term in Eq. (21) can be negligible, it is better to employ the stochastic averaging method for quasi-integrable-Hamiltonian systems. Eq. (21) in this case is of the form

$$\begin{aligned} m_1\ddot{X}_1 + c_1\dot{X}_1 + k_1X_1 + k_2(X_1 - X_2) &= W_1(t), \\ m_2\ddot{X}_2 + c_2\dot{X}_2 + k_2(X_2 - X_1) &= W_2(t). \end{aligned} \tag{31}$$

By using the procedure described in Section 5, the averaged Itô equations of system (31) are of the form

$$\begin{aligned} \mathrm{d}H_1 &= s_1(\mathbf{H})\,\mathrm{d}t + \sigma_{1j}\,\mathrm{d}B_j(t), \\ \mathrm{d}H_2 &= s_2(\mathbf{H})\,\mathrm{d}t + \sigma_{2j}\,\mathrm{d}B_j(t), \end{aligned} \tag{32}$$

where

$$\begin{aligned} & s_1(\mathbf{H}) = -c_{11}^* H_1, \quad s_2(H) = -c_{22}^* H_2, \\ & H_1 = \frac{P_1^2}{2M_1^*} + \frac{K_1^* Q_1^2}{2}, \quad H_2 = \frac{P_2^2}{2M_2^*} + \frac{K_2^* Q_2^2}{2}, \\ & \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}, \quad P_i = M_i^* \dot{Q}_i, \\ & \begin{bmatrix} M_1^* & 0 \\ 0 & M_2^* \end{bmatrix} = [T]^{\mathrm{T}} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} [T], \quad \begin{bmatrix} K_1^* & 0 \\ 0 & K_2^* \end{bmatrix} = [T]^{\mathrm{T}} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} [T], \\ & b_{11} = [\boldsymbol{\sigma\sigma}^{\mathrm{T}}]_{11} = 2(D_1 T_{11}^2 + D_2 T_{21}^2), \quad b_{22} = [\boldsymbol{\sigma\sigma}^{\mathrm{T}}]_{22} = 2(D_2 T_{12}^2 + D_2 T_{22}^2), \\ & b_{12} = [\boldsymbol{\sigma\sigma}^{\mathrm{T}}]_{12} = b_{21} = 0, \\ & c_{11}^* = c_1 T_{11}^2 + c_2 T_{21}^2, \quad c_{22}^* = c_1 T_{12}^2 + c_2 T_{22}^2, \end{aligned} \tag{33}$$

$[T] = \begin{bmatrix} T_{11} T_{12} \\ T_{21} T_{22} \end{bmatrix}$ is the modal matrix of system (31) without dampings and stochastic excitations.

The stationary solution of averaged Itô equation (32) is of the form

$$p(H_1, H_2) = C \exp\left[-\frac{2c_{11}^*}{b_{11}} H_1 - \frac{2c_{22}^*}{b_{22}} H_2\right], \tag{34}$$

where C is a normalization constant. The joint stationary probability density of displacements X_i and velocities $\dot{X}_i$ and the probability density of displacement X_2 can be obtained as follows:

$$\begin{aligned} & p(X_1, X_2, \dot{X}_1, \dot{X}_2) \\ & \quad = \bar{C} \exp\left[-\frac{2c_{11}^*}{b_{11}} H_1 - \frac{2c_{22}^*}{b_{22}} H_2\right]\Bigg|_{\substack{H_1 = \frac{1}{2} K_1^* (T_{11}^{-1} X_1 + T_{12}^{-1} X_2)^2 + \frac{1}{2} M_1^* (T_{11}^{-1} \dot{X}_1 + T_{12}^{-1} \dot{X}_2)^2 \\ H_2 = \frac{1}{2} K_2^* (T_{21}^{-1} X_1 + T_{22}^{-1} X_2)^2 + \frac{1}{2} M_2^* (T_{21}^{-1} \dot{X}_1 + T_{22}^{-1} \dot{X}_2)^2}} \end{aligned} \tag{35}$$

$$p(q_2) = p(X_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(X_1, X_2, \dot{X}_1, \dot{X}_2)\,\mathrm{d}X_1\,\mathrm{d}\dot{X}_1\,\mathrm{d}\dot{X}_2, \tag{36}$$

where $\bar{C}$ is a normalization constant.

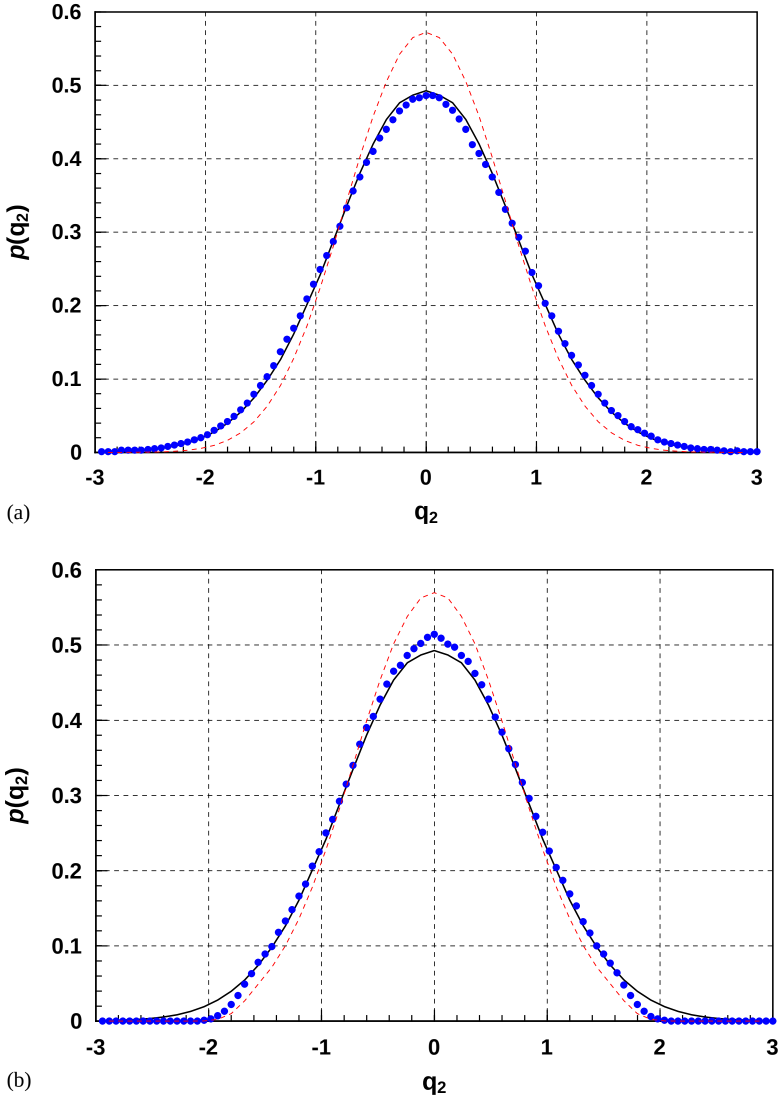

Fig. 8. Stationary probability density of displacement q_2 of system (21) with both side constraints. - - - -, from Eq. (30); ——, from Eq. (36), ●●●●, from digital simulation. (a) $m_1 = m_2 = 1, k_1 = k_2 = 1, B = B_{2l} = B_{2r} = 0.1, D_1 = 0.01, D_2 = 0.02$, $c_1 = 0.08, c_2 = 0.04, \delta = \delta_{2r} = \delta_{2l} = 0.5$. (b) $m_1 = m_2 = 1, k_1 = k_2 = 1, B = B_{2l} = B_{2r} = 10, D_1 = 0.01, D_2 = 0.02$, $c_1 = 0.08, c_2 = 0.04, \delta = \delta_{2r} = \delta_{2l} = 1.5$.

The stationary probability densities of displacement q_2 of system (31) obtained by using Eqs. (30) and (36) are shown in Fig. 8 for the case of both side constraints. It is seen from Fig. 8 that the results obtained by using the stochastic averaging method for quasi-integrable-Hamiltonian systems are better than those by using the stochastic averaging method for quasi-non-integrable-Hamiltonian systems. Further investigation shows that the stochastic averaging method for quasi-integrable-Hamiltonian systems always lead to acceptable results if the contact stiffness is very small or the clearances are larger than twice the root mean square displacement.

7. Concluding remarks

The stochastically excited MDOF vibro-impact systems have been formulated as stochastically excited and dissipated quasi-Hamiltonian system and the exact and approximate stationary solutions of the system have been derived by using the theory of the exact stationary solutions of stochastically excited and dissipated Hamiltonian systems and the stochastic averaging method for quasi-Hamiltonian systems, respectively. It is verified that the stochastic averaging method of quasi-non-integrable-Hamiltonian systems can be employed to predict the response of the system if the non-linear terms in system (1) take an important role in the response of the system. Otherwise, the stochastic averaging method for quasi-integrable-Hamiltonian system are applicable. An example of 2DOF vibro-impact system under stochastic excitations is given to illustrate the application and effectiveness of the proposed procedures. The procedures can be further extended to predict the response of stochastically excited MDOF vibro-impact system with bilinear spring or other contact modelling.

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