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Strongly non-linear oscillators with slowly varying parameters

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Abstract

A multiple scales method, which gives the approximate solution in terms of elliptic functions, is used for the study of strongly non-linear oscillators with slowly varying parameters. As an application, quadratic and cubic non-linear oscillators are studied in detail. Two examples are considered: a generalized Van der Pol oscillator and a pendulum with variable length. Comparisons are also made with numerical results to show the efficiency of the present method.

1. Introduction

This paper is to study the following strongly non-linear oscillator of the form

$$\frac{d^2y}{dt^2} + g(y, \tilde{t}) = \varepsilon h\left(y, \frac{dy}{dt}, \tilde{t}\right), \tag{1}$$

where $\tilde{t} = \varepsilon t$ is the slow scale. We assume that functions g and h are arbitrary non-linear functions of their arguments and Eq. (1) has periodic solutions when $\varepsilon = 0$. The special cases of system (1) have been studied by many authors. For the case of cubic polynomial with respect to y in $g(y, \tilde{t})$, Yuste extended the KB method by using the Jacobian elliptic functions [1] and Cveticanin applied adiabatic invariants and elliptic KB method to find the asymptotic solutions [2]. For the case of linear damping in $h(y, dy/dt, \tilde{t})$, Kuzmak proposed a multiple scales method to obtain the conditions of periodicity and asymptotic solutions of first order [3] and Luke extended Kuzmak's method to higher order [4]. Kevorkian and Li reviewed and compared the Kuzmak–Luke method and that of near-identity averaging transformations [5,6]. The equations of motion of electrons in a free electron laser (FEL) [7] are also of the special form of Eq. (1). In this paper, we will follow

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Kuzmak–Luke's procedure to Eq. (1) and will discuss applications to quadratic and cubic non-linear oscillators. For illustration, a generalized Van der Pol oscillator and a pendulum with slowly varying length are studied in detail. Comparisons of asymptotic and numerical solutions are also made to show the efficiency of the present method.

2. General theory

We assume that the solution of (1) can be developed in the form

$$y(t,\varepsilon) = y_0(t^+,\tilde{t}\,) + \varepsilon y_1(t^+,\tilde{t}\,) + \varepsilon^2 y_2(t^+,\tilde{t}\,) + \cdots, \tag{2}$$

where $\tilde{t} = \varepsilon t$ is the slow scale. The fast scale t^+, following Kuzmak [3], is defined as

$$\frac{\mathrm{d}t^+}{\mathrm{d}t} = \omega(\tilde{t}\,)$$

with an unknown $\omega(\tilde{t}\,)$ to be determined by the periodicity of the solution of (1). $y_0, y_1, \ldots$ must be periodic functions of t^+, otherwise the expansions cannot be asymptotic.

Substituting (2) into Eq. (1), expanding $h(y, \mathrm{d}y/\mathrm{d}t, \tilde{t}\,)$ in power series of ε and equating coefficients of like power of ε yield the following equations:

$$\omega^2(\tilde{t}\,)\frac{\partial^2 y_0}{\partial t^{+2}} + g(y_0,\tilde{t}\,) = 0, \tag{3}$$

$$\omega^2(\tilde{t}\,)\frac{\partial^2 y_n}{\partial t^{+2}} + g'_y(y_0,\tilde{t}\,)y_n = F_n(y_0, y_1, \ldots, y_{n-1}, \tilde{t}\,), \tag{4}$$

where $n = 1, 2, \ldots,$ F_1 can be worked out in the form

$$F_1 = -2\omega\frac{\partial^2 y_0}{\partial t^+ \partial\tilde{t}} - \frac{\mathrm{d}\omega}{\mathrm{d}t}\frac{\partial y_0}{\partial t^+} + h\left(y_0, \omega\frac{\partial y_0}{\partial t^+}, \tilde{t}\right). \tag{5}$$

Note that there is a periodic solution to the homogeneous equation (4) in the form

$$y_{\mathrm{I}} = \frac{\partial y_0}{\partial\varphi}, \quad \varphi = t^+ + \varphi_0(\tilde{t}\,). \tag{6}$$

The other solution linearly independent of can be found by the reduction of order

$$y_{\mathrm{II}} = y_{\mathrm{I}}\int^{\varphi}\frac{1}{y_{\mathrm{I}}^2}\,\mathrm{d}\psi. \tag{7}$$

Unfortunately, the solution y_{II} is no longer periodic to general non-linear system. Using variation of parameters, we obtain the general solution of the inhomogeneous equation (4) in the form

$$\begin{aligned} y_n &= C_n(\tilde{t}\,)y_{\mathrm{I}} + D_n(\tilde{t}\,)y_{\mathrm{II}} - \frac{y_{\mathrm{I}}}{\omega^2}\int^{\varphi} F_n y_{\mathrm{II}}\,\mathrm{d}\psi + \frac{y_{\mathrm{II}}}{\omega^2}\int^{\varphi} F_n y_{\mathrm{I}}\,\mathrm{d}\psi \\ &= y_{\mathrm{I}}\left[C_n(\tilde{t}\,) + \int^{\varphi}\frac{\mathrm{d}\psi}{y_{\mathrm{I}}^2}\left(D_n(\tilde{t}\,) + \frac{1}{\omega^2}\int^{\psi} F_n y_{\mathrm{I}}\,\mathrm{d}\gamma\right)\right], \end{aligned} \tag{8}$$

where coefficients C_n and D_n can be determined by the periodicity of higher order solutions. To have y_n periodic in φ, the inner integral and the outer integral in Eq. (8) must be periodic in ψ and

φ, respectively. We thus have, with the periodic normalized to be T,

$$\int_0^T F_n y_{\mathrm{I}}\, \mathrm{d}\varphi = 0, \tag{9}$$

$$\int_0^T \frac{\mathrm{d}\varphi}{y_{\mathrm{I}}^2}\left(D_n(\tilde{t}) + \frac{1}{\omega^2}\int_0^\varphi F_n y_{\mathrm{I}}\, \mathrm{d}\psi\right) = 0. \tag{10}$$

This paper just concerns applications of leading order approximations. More details of higher order solution, readers can refer to [7].

The leading order solution has two parameters $\omega(\tilde{t})$ and $\varphi_0(\tilde{t})$ to be determined. As shown in Ref. [7], φ_0 is constant and is determined by initial conditions. Substituting Eqs. (5) and (6) into (9) with $n = 1$ yields

$$\int_0^T \left(2\omega f_{\varphi\tilde{t}} f_\varphi + \frac{\mathrm{d}\omega}{\mathrm{d}\tilde{t}} f_\varphi^2 - h(f, \omega f_\varphi, \tilde{t}) f_\varphi\right) \mathrm{d}\varphi = 0. \tag{11}$$

Then we obtain the following equation to determine $\omega(\tilde{t})$:

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\left(\omega \int_0^T f_\varphi^2\, \mathrm{d}\varphi\right) - \int_0^T h(f, \omega f_\varphi, \tilde{t}) f_\varphi\, \mathrm{d}\varphi = 0. \tag{12}$$

In above two equations, notation $y_0 = f(\varphi, \tilde{t})$ has been used. When the damping is linear, i.e., $h(y, \mathrm{d}y/\mathrm{d}t, \tilde{t}) = k(y, \tilde{t})\, \mathrm{d}y/\mathrm{d}t$, Eq. (12) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\left(\omega \int_0^T f_\varphi^2\, \mathrm{d}\varphi\right) - \omega \int_0^T k(f, \tilde{t}) f_\varphi^2\, \mathrm{d}\varphi = 0. \tag{13}$$

Integrating (13) gives

$$\omega(\tilde{t}) = \frac{c}{\int_0^T f_\varphi^2\, \mathrm{d}\varphi} \exp\left(\int_0^{\tilde{t}} \frac{\int_0^T k(f, \tau) f_\varphi^2\, \mathrm{d}\varphi}{\int_0^T f_\varphi^2\, \mathrm{d}\varphi}\, \mathrm{d}\tau\right), \tag{14}$$

where c is a constant. If the damping k depends on y, the calculation of $\omega(\tilde{t})$ will be rather involved. An approach of average damping is proposed in Ref. [7]. Instead of k, we use the leading term of its Taylor series expansion around $f = y_r$, the resonance center, i.e., we assume

$$k(y, \tilde{t}) = k(y_r, \tilde{t}) + k_y(y_r, \tilde{t})(y - y_r) + \tfrac{1}{2} k_{yy}(y_r, \tilde{t})(y - y_r)^2 + \cdots, \tag{15}$$

where y_r is the resonance center of system (1) and is determined by $g(y_r, \tilde{t}) = 0$. Because the system oscillates around the center y_r, the second term of expansion (15) vanishes on average. Therefore, substitution of $k(y, \tilde{t}) \approx k(y_r, \tilde{t})$ into (14) should give a good approximation for $\omega(\tilde{t})$. The result is

$$\omega(\tilde{t}) = \frac{c}{\int_0^T f_\varphi^2\, \mathrm{d}\varphi} \exp\left(\int_0^{\tilde{t}} k(y_r, \tau)\, \mathrm{d}\tau\right). \tag{16}$$

Numerical examples in Section 4 shows that the results are quite satisfactory.

3. Applications to quadratic and cubic non-linear oscillators

3.1. Quadratic non-linear oscillator

We now apply the results summarized in previous section to the following quadratic non-linear system:

$$\frac{d^2y}{dt^2} + a(\tilde{t})y + b(\tilde{t})y^2 = \varepsilon k(y, \tilde{t})\frac{dy}{dt}. \tag{17}$$

Suppose that the solution of (17) can be developed in the form of asymptotic expression (2). The leading order equation corresponding to (3) has the form

$$\omega^2(\tilde{t})\frac{\partial^2 y_0}{\partial t^{+2}} + a(\tilde{t})y_0 + b(\tilde{t})y_0^2 = 0. \tag{18}$$

Its energy integral is

$$\frac{\omega^2(\tilde{t})}{2}\left(\frac{\partial y_0}{\partial t^+}\right)^2 + V(y_0, a, b) = E_0(\tilde{t}), \tag{19}$$

where

$$V(y_0, a, b) = \tfrac{1}{2}a(\tilde{t})y_0^2 + \tfrac{1}{3}b(\tilde{t})y_0^3 \tag{20}$$

is the potential, and $E_0(\tilde{t})$ is the slowly varying energy of the system. It can be seen from (20) that V has a minimum at $y_0 = 0$ for $a(\tilde{t}) > 0$. So Eq. (17) has periodic solutions around $y_0 = 0$ and the resonance center is at $y_r = 0$. V has a minimum at $y_0 = -a(\tilde{t})/b(\tilde{t})$ for $a(\tilde{t}) < 0$. Eq. (17) has periodic solutions around $y_0 = -a(\tilde{t})/b(\tilde{t})$ and the resonance center is at $y_r = -a(\tilde{t})/b(\tilde{t})$. This paper just concerns the symmetrical oscillations, i.e., the case of $a(\tilde{t}) > 0$. The calculations for the case of $a(\tilde{t}) < 0$ are essentially similar to that of $a(\tilde{t}) > 0$.

By integrating (19), we can obtain y_0 in terms of elliptic function of t^+. However, here is an alternative. We first assume that the solution is in the form of elliptic function, and then determine its amplitude and modulus via (18). Suppose that we have

$$y_0 = A_0(\tilde{t})cn^2[K(v)\varphi, v(\tilde{t})] + B_0(\tilde{t}), \tag{21}$$

where $\varphi = t^+ + \varphi_0$ and $K(v)$ is the complete elliptic integral of the first kind associated with the modulus $\sqrt{v}$. Substituting (21) into (18) yields

$$\begin{aligned} &2\omega^2K^2A_0(1-v) + aB_0 + B_0^2 + A_0[4\omega^2K^2(2v-1) + a + 2bB_0]cn^2(u,v) \\ &\quad + A_0(bA_0 - 6\omega^2K^2v)cn^4(u,v) = 0, \end{aligned} \tag{22}$$

where $u = K(v)\varphi$ and the equation

$$\frac{\partial^2}{\partial u^2}[cn^2(u,v)] = 2(1-v) + 4(2v-1)cn^2(u,v) - 6vcn^4(u,v)$$

has been used. From (22) we obtain algebraic equations:

$$2\omega^2K^2A_0(1-v)+aB_0+B_0^2=0,$$
$$A_0[4\omega^2K^2(2v-1)+a+2bB_0]=0,$$
$$A_0(bA_0-6\omega^2K^2v)=0.$$

Then, we have

$$A_0=\frac{3av}{2b\sqrt{v^2-v+1}},\quad B_0=-\frac{a}{2b}\left(\frac{2v-1}{\sqrt{v^2-v+1}}+1\right), \tag{23, 24}$$

$$\omega^4=\frac{a^2}{16K^4(v^2-v+1)}. \tag{25}$$

Substituting (21) into (16), we get another form of $\omega(\tilde{t})$

$$\omega^5=\frac{cb^2}{144K^5v^2J(v)}\exp\left(\int_0^{\tilde{t}}k(0,\tau)\,\mathrm{d}\tau\right), \tag{26}$$

where

$$J(v)=\int_0^K sn^2(u,v)cn^2(u,v)\,\mathrm{d}n^2(u,v)\,\mathrm{d}u$$
$$=\frac{1}{15v^2}[(1-v)(v-2)K(v)+2(v^2-v+1)E(v)].$$

From (25) and (26), we have an equation for v

$$\frac{v^2J(v)}{(v^2-v+1)^{5/4}}=\frac{2cb^2}{9a^{5/2}}\exp\left(\int_0^{\tilde{t}}k(0,\tau)\,\mathrm{d}\tau\right). \tag{27}$$

3.2. Cubic non-linear oscillator

Consider the following cubic non-linear oscillator:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+a_1(\tilde{t})y+b_1(\tilde{t})y^3=\varepsilon k_1(y,\tilde{t})\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{28}$$

Here, we only consider the case of the resonance center at origin, i.e., the case of $a(\tilde{t})>0$. Similar to the quadratic non-linear oscillator, we can get leading order approximate solution

$$y_0=\sqrt{\frac{-2a_1v}{b_1(1+v)}}\,sn(K\varphi,v) \tag{29}$$

and

$$\omega(\tilde{t})=\frac{-c_1b_1(1+v)}{2a_1vK(v)L(v)}\exp\left(\int_0^{\tilde{t}}k_1(0,\tau)\,\mathrm{d}\tau\right), \tag{30}$$

where

$$L(v) = \int_0^K cn^2(u,v)\,dn^2(u,v)\,du = \frac{1}{3v}[(1+v)E(v) - (1-v)K(v)].$$

The equation governed v becomes

$$\frac{v^2L^2(v)}{(1+v)^3} = \frac{c_1^2 b_1^2}{4a_1^3}\exp\left(2\int_0^{\tilde{t}} k_1(0,\tau)\,d\tau\right), \tag{31}$$

where constant c_1 can be determined by the initial values of the system.

4. Examples

Example 1. Consider the following generalized Van der Pol oscillator:

$$\frac{d^2y}{dt^2} + (1+\varepsilon t)^2 y - (1+\varepsilon t)^{5/2} y^2 = \varepsilon\left(\frac{1}{1+\varepsilon t} - y^2\right)\frac{dy}{dt}, \tag{32}$$

$$y(0) = 0.5, \quad \dot{y}(0) = 0. \tag{33}$$

From initial conditions we can obtain $\varphi_0 = 1$ and $c = 0.27312$. The comparison of numerical solution and asymptotic solution obtained by (21)–(27) with $\varepsilon = 0.01$ is shown in Fig. 1. In this paper the symbolic language *Mathematica* is used to implement the asymptotic and numerical solutions.

Example 2. Consider the following pendulum with slowly varying length:

$$\frac{d}{dt}\left(l^2(\tilde{t})\frac{d\theta}{dt}\right) + gl(\tilde{t})\sin\theta = 0, \tag{34}$$

where θ is the angle of deviation of the pendulum from the vertical, g is the gravitational acceleration, $l(\tilde{t})$ is the slowly varying length, $\tilde{t} = \varepsilon t$ is the slow scale. Such problem was discussed

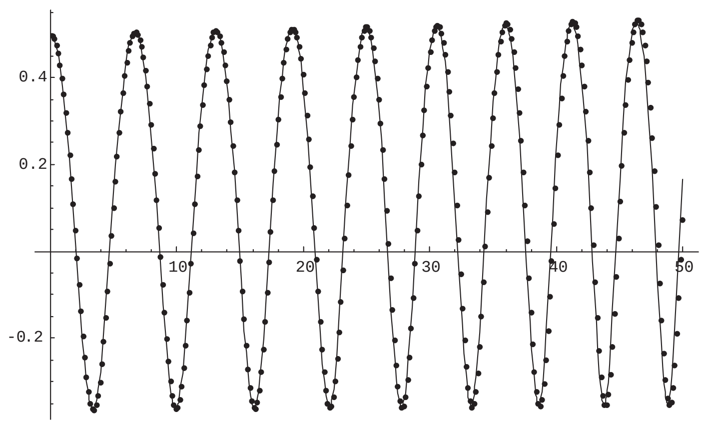

Fig. 1. Solution and approximation of Eq. (32); ——, numerical solution and ⋯⋯, asymptotic solution.

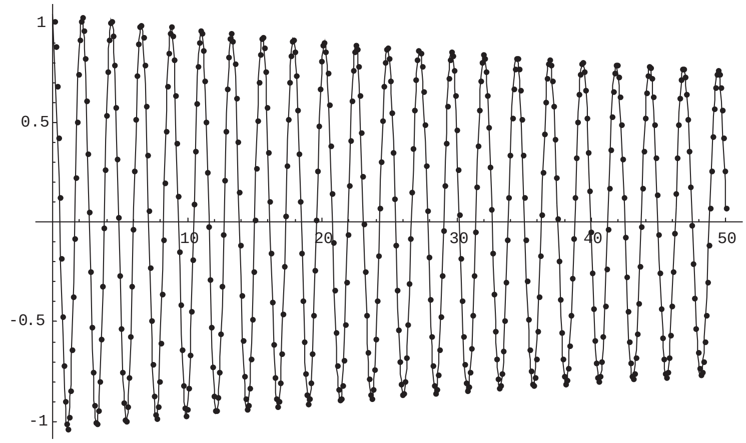

Fig. 2. Solution and approximation of Eq. (36); ——, numerical solution and ⋯⋯, asymptotic solution.

in Ref. [1] but only the amplitude was given there. For not large oscillations, we can approximate $\sin\theta$ by the first two terms of the power series expansion, and then Eq. (34) becomes

$$\frac{d^2\theta}{dt^2} + \varepsilon\frac{2l'(\tilde{t})}{l(\tilde{t})}\frac{d\theta}{dt} + \frac{g}{l(\tilde{t})}\theta - \frac{g}{6l(\tilde{t})}\theta^3 = 0, \tag{35}$$

where $l' = dl/d\tilde{t}$. When $l(\tilde{t}) = 1 + \tilde{t}$ and $g = 9.8$, Eq. (35) becomes

$$\frac{d^2\theta}{dt^2} + \varepsilon\frac{2}{1+\varepsilon t}\frac{d\theta}{dt} + \frac{9.8}{1+\varepsilon t}\theta - \frac{9.8}{6(1+\varepsilon t)}\theta^3 = 0 \tag{36}$$

with initial conditions $\theta(0) = \frac{1}{3}\pi$, $\dot{\theta}(0) = 0$. The comparison of numerical solution and asymptotic solution obtained by (29)–(31) with $\varepsilon = 0.01$ is shown in Fig. 2.

5. Conclusions

1. The Kuzmak–Luke method is used to obtain the conditions of periodicity of certain strongly non-linear oscillators and the asymptotic solutions of quadratic and cubic non-linear oscillators.
2. Two examples are given: the generalized Van der Pol oscillator and the pendulum with slowly varying length. The asymptotic solutions are almost identical with the numerical solutions in Figs. 1 and 2.

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