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Journal of Sound and Vibration 275 (2004) 267–281

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Sound dispersion in a deformable tube with polymeric liquid and elastic central rod

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Received 28 October 2002; accepted 23 June 2003

Abstract

Sound waves in deformable tubes containing a compressible polymeric liquid are investigated. The central part of the tube is occupied by a coaxial elastic rod. A dispersion equation is derived for this system which accounts for viscoelastic effects in the fluid and elastic deformations of the tube and the rod. The equation is valid in the low frequency range where the sound wavelength is greater than the tube radius. Its analysis has shown that rheological properties of the liquid essentially influence the speed and attenuation of sound in the waveguide. This influence depends on the gap width and elastic properties of both the tube wall and the internal rod.

1. Introduction

Propagation of acoustic waves in liquid or gas filled tubes has been thoroughly investigated since the classic Kirchhoff study [1]. The solution of the problem, in terms of non-dimensional parameters, was obtained and studied by Tijdeman [2]. His results for air-filled tubes were extended in Ref. [3] for a wider range of dimensionless parameter variation. One important feature of sound attenuation in tubes is its frequency dependence. Numerical analysis of the attenuation coefficient has been done in Refs. [3,4]; in Ref. [5] it was shown that friction losses at pressure wave propagation in a liquid-filled tube are highly dependent on the initial signal form or, in other words, on its frequency spectrum. The speed of pressure waves is also frequency-dependent due to the inertia of the tube wall and viscosity of the liquid [6,7]. Experimental data on sound dispersion and attenuation in water-filled tubes were reported in Ref. [8]. A general

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.06.021

overview of different models of tube acoustics, including acousto-elastic interactions, can be found in Ref. [9].

The mathematical model of the process becomes much more complex in the case where the fluid is non-Newtonian. It occurs, for instance, for polymeric solutions and melts having a ramified molecular structure. Such a fluid is characterized by a spectrum of relaxation times associated with different structure elements, and its hereditary properties can highly influence the dynamic behaviour of the system. The features of viscoelasticity at sound wave propagation in a thin-walled tube containing polymeric solution were investigated in Refs. [10,11]. This analysis was generalized later in Ref. [12] for the case where the central part of the tube is occupied by a rigid rod which represents an additional source of dissipation in view of the non-slip boundary condition on its surface. It was shown particularly that the presence of an internal rod can additionally illuminate non-Newtonian properties of the liquid in the region of viscoelastic transition.

The current paper is devoted to the theoretical analysis of a more general and realistic case where the rod is elastic. The analysis is aimed at describing the dispersion and attenuation of acoustic waves in the system while taking into account both the external shell and internal rod deformations and hereditary properties of the polymeric liquid.

2. Formulation of the problem

The acoustics of the system is studied below in a conjugated quasi-one-dimensional formulation. The wall of the tube is considered as a thin elastic cylindrical shell; a polymeric solution, filling the gap, is treated as a hereditary compressible liquid; the central circular rod is supposed to be pure elastic. Dynamic equations for each region are formulated separately; they are coupled through appropriate boundary conditions. The basic assumption of the model developed here is $R_1 \ll l \ll L$, where l is the wavelength.

2.1. Dynamic problem for a thin-walled tube

Axisymmetric dynamics of a thin-walled cylindrical circular shell can be described by Kirchhoff–Love equations [13] which, in terms of displacements, have the form:

$$\frac{E_s}{1-\nu_s^2}\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\nu_s}{R_1}\frac{\partial u_r}{\partial x}\right)=\rho_s\frac{\partial^2 u_x}{\partial t^2}, \tag{1}$$

$$\frac{E_s h}{(1-\nu_s^2)R_1}\left(\nu_s\frac{\partial u_x}{\partial x}+\frac{u_r}{R_1}\right)+\frac{E_s h^3}{3(1-\nu_s^2)}\frac{\partial^4 u_r}{\partial x^4}-\frac{1}{2}\Delta p=-\rho_s h\frac{\partial^2 u_r}{\partial t^2}. \tag{2}$$

In Eqs. (1) and (2) $x,\ r$ are the co-ordinates of the cylindrical co-ordinate system with the origin on the tube axis and Δp is the contact pressure equal to normal stress in the liquid at the pipe wall. It is supposed that the assumption $\varepsilon_1 = h/R_1 \ll 1$ is valid here.

The dimensionless variables are:

$$\{u_1,u_2,\xi,\zeta\}=R_1^{-1}\{u_r,u_x,r,x\},\quad p_c=\Delta p/p_0,\quad \bar{E}_s=E_s/p_0, \tag{3}$$

where p_0 is the equilibrium pressure within the tube. The solution of Eqs. (1) and (2) for longitudinal waves is sought in the form

$$\{u_1, u_2, p_c\} = \{\hat{u}_1, \hat{u}_2, \hat{p}_c\}\exp[\mathrm{i}(\omega\tau - k\zeta)],$$

$$\omega = \Omega t_0, \quad \tau = t/t_0, \quad t_0 = R_1(\rho_s/p_0)^{1/2}. \tag{4}$$

Here Ω is the frequency and k is the dimensionless wave number. It leads to the following equations for the complex amplitudes:

$$\mathrm{i}\nu_s k\hat{u}_1 + (k^2 - \bar{E}_s^{-1}\omega^2(1-\nu_s^2))\hat{u}_2 = 0, \tag{5}$$

$$\left[1 + \tfrac{1}{3}\varepsilon_1^2 k^4 - \bar{E}_s^{-1}(1-\nu_s^2)\omega^2\right]\hat{u}_1 - \mathrm{i}k\nu_s\hat{u}_2 - [(1-\nu_s^2)/(2\varepsilon_1\bar{E}_s)]\hat{p}_c = 0. \tag{6}$$

The contact pressure amplitude $\hat{p}_c$ in Eq. (6) must be found from boundary conditions, formulated at the liquid–shell interface for $r = R_1 - h \approx R_1$. They have the form

$$v_r = \frac{\partial u_r}{\partial t}, \quad v_x = \frac{\partial u_x}{\partial t}, \quad \Delta p = \Delta p_f - \tau_{rr}, \quad \Delta p_f = p_f - p_0. \tag{7}$$

In Eqs. (7) τ_{rr} is the normal component of deviatoric stress in the liquid at the interface.

2.2. Dynamic problem for an elastic rod

Dynamic equations for a central elastic cylinder of the radius R_2 in the axisymmetric case [14] have the form:

$$(\lambda_r + 2\mu_r)\frac{\partial\Theta}{\partial r} + \mu_r\left(\frac{\partial^2 w_r}{\partial x^2} - \frac{\partial w_x}{\partial x\,\partial r}\right) - \rho_r\frac{\partial^2 w_r}{\partial t^2} = 0, \tag{8}$$

$$(\lambda_r + 2\mu_r)\frac{\partial\Theta}{\partial x} - \frac{\mu_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_r}{\partial x} - r\frac{\partial w_x}{\partial x\,\partial r}\right) - \rho_r\frac{\partial^2 w_x}{\partial t^2} = 0, \tag{9}$$

$$\Theta = \frac{\partial w_r}{\partial r} + \frac{w_r}{r} + \frac{\partial w_x}{\partial x},$$

where ρ_r is the density of the rod's material, and $w_r,\ w_x$ are radial and longitudinal displacements in the rod, respectively. The Lame constants λ_r and μ_r are coupled with Young's and the Poisson modules of the rod's material $E_r,\ \nu_r$ by the relations

$$\mu_r = \frac{E_r}{2(1+\nu_r)}, \quad \lambda_r = \frac{2\mu_r\nu_r}{1-2\nu_r}. \tag{10}$$

The normal component of the stress tensor σ_{rr} in the rod is expressed by the formula

$$\sigma_{rr} = \lambda_r\left(\frac{\partial w_r}{\partial r} + \frac{w_r}{r} + \frac{\partial w_x}{\partial x}\right) + 2\mu_r\frac{\partial w_r}{\partial r}. \tag{11}$$

It is supposed below that $\partial w_r/\partial r \gg \partial w_x/\partial x$, $\partial^2 w_r/\partial r^2 \gg \partial^2 w_r/\partial x^2$, $\partial^2 w_r/\partial r^2 \gg \partial^2 w_x/\partial x\partial r$ (long-wave approximation [6,11]). This allows simplification of the constitutive relation (11) and dynamic equations for the rod. As a result Eqs. (8) and (11) give:

$$(\lambda_r + 2\mu_r)\left(\frac{\partial^2 w_r}{\partial r^2} + \frac{1}{r}\frac{\partial w_r}{\partial r} - \frac{1}{r^2}w_r\right) - \rho_r\frac{\partial^2 w_r}{\partial t^2} = 0, \tag{12}$$

$$\sigma_{rr} \approx (\lambda_r + 2\mu_r)\frac{\partial w_r}{\partial r} + \lambda_r\frac{w_r}{r}. \tag{13}$$

Eqs. (12) and (13) are rewritten in terms of dimensionless variables, using relations

$$\bar{\sigma}_{rr} = p_0^{-1}\sigma_{rr}, \quad w_1 = R_1^{-1}w_r. \tag{14}$$

The solution for w_1 is sought in the form

$$w_1 = \hat{w}_1 \exp[\mathrm{i}(\omega\tau - k\varsigma)]. \tag{15}$$

The Bessel equation for the complex amplitude $\hat{w}_1$ is:

$$\frac{\partial^2 \hat{w}_1}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \hat{w}_1}{\partial \xi} + \left[\beta^2 - \frac{1}{\xi^2}\right]\hat{w}_1 = 0, \tag{16}$$

where $\beta = \omega[\kappa_1(\bar{\lambda}_r + 2\bar{\mu}_r)]^{-1/2}$, $\{\bar{\lambda}_r, \bar{\mu}_r\} = p_0^{-1}\{\lambda_r, \mu_r\}$, $\kappa_1 = \rho_s/\rho_r$.

Its solution, bounded at $\xi \to 0$, is expressed in terms of the Bessel function of the first kind and first order J_1 [15]

$$\hat{w}_1 = A_1 \mathrm{J}_1(\beta\xi). \tag{17}$$

Here A_1 is arbitrary constant. The boundary conditions in the rod on its surface ($r = R_2$) have the form

$$v_x = \frac{\partial w_x}{\partial t}, \quad v_r = \frac{\partial w_r}{\partial t}, \quad \sigma_{rr} = \tau_{rr} - \Delta p_f, \tag{18}$$

2.3. Dynamic problem for polymeric liquid within the gap

It is supposed that the liquid is originally at rest, the tube wall is only slightly deformable, $v_r \ll v_x$ and R_1 can be used instead of the internal radius of the shell. These assumptions allow the linearized equations of momentum and mass balance for the liquid to be written in the form

$$\rho_{f0}\frac{\partial v_x}{\partial t} = -\frac{\partial p_f}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rx}) + \frac{\partial \tau_{xx}}{\partial x}, \tag{19}$$

$$\frac{\partial \rho_f}{\partial t} + \rho_{f0}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_x}{\partial x}\right) = 0. \tag{20}$$

The deviatoric stresses and pressure in the liquid are defined according to linear hereditary model [16,17]:

$$\tau_{ij} = 2\int_{-\infty}^{t} G(t - t_1)s_{ij}(t_1)\mathrm{d}t_1 + 2\eta_s s_{ij}, \quad \Delta p_f = c_f^2\Delta\rho_f, \quad \Delta\rho_f = \rho_f - \rho_{f0},$$

$$s_{ij} = e_{ij} - \frac{1}{3}(\boldsymbol{\nabla}\cdot\mathbf{v})I. \tag{21}$$

The volume viscoelasticity of the liquid is neglected here because it has only a minor effect on the sound propagation in the system under consideration [18]. Eq. (19) can be simplified in view of the following estimates: $\partial^2 v_x/\partial x^2 \sim v_0/L^2$, $(1/r)\,\partial v_x/\partial r \sim v_0/\delta^2 L^2$, $\partial^2 v_x/\partial r^2 \sim v_0/\delta^2 L^2$, where v_0 is the scale of flow velocity in the wave and $\delta = R_1/L \ll 1$. Therefore, it is possible to neglect by $\partial^2 v_x/\partial x^2$ in Eq. (19) with respect to other derivatives. The additional suggestion is concerned with the cross effect of shear and volume viscoelasticity of liquid in the momentum balance equation that is small [10,18] and can also be neglected. As a result, it follows from (19):

$$\rho_{f0}\frac{\partial v_x}{\partial t} = -\frac{\partial p_f}{\partial x} + \int_{-\infty}^{t} G(t-t_1)\left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r}\frac{\partial v_x}{\partial r}\right)\mathrm{d}t_1 + \eta_s\left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r}\frac{\partial v_x}{\partial r}\right). \tag{22}$$

The relative velocity V_x of the liquid with respect to the pipe wall, $V_x = v_x - \dot{u}_x$, the mean flow rate V, pressure P and density ρ of the fluid are introduced as

$$V = \frac{2}{R_1^2 - R_2^2}\int_{R_2}^{R_1} V_x r\,\mathrm{d}r;\quad P = \frac{2}{R_1^2 - R_2^2}\int_{R_2}^{R_1} p_f r\,\mathrm{d}r;\quad \rho = \frac{2}{R_1^2 - R_2^2}\int_{R_2}^{R_1} \rho_f r\,\mathrm{d}r. \tag{23}$$

Then after averaging Eqs. (20)–(22) along the cross-section of the gap:

$$\rho_{f0}\frac{\partial V}{\partial t} = -\frac{\partial P}{\partial x} + \frac{2}{R_1(1-\varepsilon_2^2)}[\tau_{xr|_{R_1}} - \varepsilon_2 \tau_{xr|_{R_2}}],\quad \varepsilon_2 = \frac{R_2}{R_1}, \tag{24}$$

$$\frac{\partial \rho}{\partial t} + \frac{2\rho_{f0}}{R_1(1-\varepsilon_2^2)}[v_{r|_{R_1}} - \varepsilon_2 v_{r|_{R_2}}] + \rho_{f0}\frac{\partial V}{\partial x} = 0, \tag{25}$$

$$P = p_0 + c_f^2(\rho - \rho_{f0}). \tag{26}$$

Eqs. (24) and (25) took account of the fact that $\dot{u}_x \ll V$ and $\partial v_r/\partial x \ll \partial v_x/\partial r$. The last non-equality permits the shear component of the rate deformation tensor in the liquid to be defined as $s_{xr} \approx \frac{1}{2}\partial V_x/\partial r$.

Eqs. (24)–(26) are rewritten in terms of dimensionless variables:

$$\bar{\rho} = \frac{\Delta\rho}{\rho_{f0}} = \rho/\rho_{f0} - 1;\quad \bar{P} = P/p_0 - 1;\quad \bar{V} = \frac{V t_0}{R_1};\quad \bar{v}_{r|_{R_1}} \equiv \bar{v}_{R_1} = (t_0/R_1)(v_r)_{r=R_1},$$

$$\{\bar{\tau}_{w_1}, \bar{\tau}_{w_2}\} = \{\tau_{xr|_{R_1}}, \tau_{xr|_{R_2}}\}/p_0.$$

The solution is sought in the form

$$\{\bar{P}, \bar{V}, \bar{\rho}, \bar{\tau}_{w_1}, \bar{\tau}_{w_2}\} = \{\hat{P}, \hat{V}, \hat{\rho}, \hat{\tau}_1, \hat{\tau}_2\}\exp[\mathrm{i}(\omega\tau - k\zeta)],$$

$$\bar{v}_{R_1} = \mathrm{i}\omega u_1,\quad \bar{v}_{R_2} = \mathrm{i}\omega w_1, \tag{27}$$

which leads to the following equations for complex amplitudes:

$$\mathrm{i}\omega\hat{V} = \mathrm{i}\kappa k\hat{P} + \frac{2\kappa}{1-\varepsilon_2^2}(\hat{\tau}_1 - \varepsilon_2\hat{\tau}_2),\quad \kappa = \frac{\rho_s}{\rho_{f0}}, \tag{28}$$

$$\mathrm{i}\omega\hat{\rho} + \frac{2\mathrm{i}\omega}{1-\varepsilon_2^2}(\hat{u}_1 - \varepsilon_2\hat{w}_1) - \mathrm{i}k\hat{V} = 0, \tag{29}$$

$$\hat{P} = \kappa^{-1}\bar{c}_f^2\hat{\rho}. \tag{30}$$

To close the hydrodynamic problem for the liquid, it is necessary to find transient frictions at the pipe wall and the rod surface in terms of the average flow velocity.

2.4. Transient friction at liquid–solid interfaces

Sound attenuation in the liquid-filled tube is governed mainly by shear stress dynamics [19]. This allows the compressibility of the liquid in evaluations of transient frictions at liquid–solid interfaces to be disregarded. A similar approach was used also in Ref. [20] for the modelling of sound attenuation in a capillary porous media. Note that several approximate relations for $\bar{\tau}_w$ were obtained earlier in Refs. [5,6,21,22]. For example, in Ref. [6] the transient friction at the tube wall was evaluated by suggesting that the wall is flat, which corresponds to a limiting case of a thin viscous boundary layer. The solution, obtained below, is free of this limitation, which is especially important for polymeric liquids that possess frequency-dependent viscosity.

Eq. (22), written in terms of non-dimensional relative velocity, has the form

$$\frac{\partial \bar{V}_x}{\partial \tau} = \bar{K} + \kappa \int_{-\infty}^{\tau} \bar{G}(\tau - \bar{t}_1)\left(\frac{\partial^2 \bar{V}_x}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \bar{V}_x}{\partial \xi}\right)\mathrm{d}\bar{t}_1 + \kappa\bar{\eta}_s\left(\frac{\partial^2 \bar{V}_x}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \bar{V}_x}{\partial \xi}\right),$$

$$\bar{V}_x = V_x t_0/R_1, \quad \bar{K} = -\kappa\frac{\partial(\Delta\bar{p}_f)}{\partial\varsigma} - \ddot{u}_2, \quad \bar{\eta}_s = \eta_s/(p_0 t_0), \quad \bar{G} = G/p_0. \tag{31}$$

Here all unknowns are proportional to $\mathrm{e}^{\mathrm{i}(\omega\tau - k\zeta)}$. It follows from (31) for the complex amplitudes $\hat{V}_x,\ \hat{K}$ of $\bar{V}_x,\ \bar{K}$ that

$$\frac{\mathrm{d}^2\hat{V}_x}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}\hat{V}_x}{\mathrm{d}\xi} + \mu^2\hat{V}_x = -\frac{\hat{K}}{\kappa\bar{\eta}},$$

$$\bar{\eta} = \bar{\eta}_s + (\mathrm{i}\omega)^{-1}G^*, \quad G^* = \int_0^\infty \frac{(\omega\bar{\theta})\bar{F}(\bar{\theta})(\mathrm{i} + \omega\bar{\theta})}{1 + (\omega\bar{\theta})^2}\mathrm{d}\bar{\theta}, \quad \bar{\theta} = \theta/t_0, \quad \mu = \mathrm{i}(\mathrm{i}\omega/\kappa\bar{\eta})^{1/2}. \tag{32}$$

Here G^* is the non-dimensional complex dynamic module of the liquid, and $\bar{F}(\bar{\theta})$ is the non-dimensional spectrum of relaxation times $\bar{\theta}$, which is introduced according to the relation:

$$\bar{G}(\tau - \bar{t}_1) = \int_0^\infty \bar{F}(\bar{\theta})\mathrm{e}^{-(\tau - \bar{t}_1)/\bar{\theta}}\,\mathrm{d}\bar{\theta}. \tag{33}$$

The solution of Eq. (32) has the form

$$\hat{V}_x = \frac{\hat{K}}{\mathrm{i}\omega} + A\mathrm{J}_0(\mu\xi) + B\mathrm{Y}_0(\mu\xi), \tag{34}$$

where J_0 and Y_0 are Bessel functions of first and second kinds of the zero order. The arbitrary constants A and B are to be found from boundary conditions following Eqs. (7) and (18). Within the long-wave approach adopted the boundary conditions can be written in the approximate form

$$\hat{V}_x|_{\xi=1} \approx 0, \quad \hat{V}_x|_{\xi=\varepsilon_2} \approx 0. \tag{35}$$

Note that numerical simulations [12], which took into account the influence of translational displacements of the shell in the wave on the value of the friction force at the interface, have shown that this influence is minor and can be considered negligible.

It follows from (34) and (35) that:

$$A = -\frac{\hat{K}}{\mathrm{i}\omega}\left\{1 - \frac{\mathrm{Y}_0(\mu)}{\mathrm{Y}_0(\mu\varepsilon_2)}\right\}\left\{\mathrm{J}_0(\mu) - \frac{\mathrm{Y}_0(\mu)\mathrm{J}_0(\mu\varepsilon_2)}{\mathrm{Y}_0(\mu\varepsilon_2)}\right\}^{-1},\ B = -\frac{\hat{K}}{\mathrm{i}\omega \mathrm{Y}_0(\mu\varepsilon_2)}\left\{1 - \frac{\mathrm{J}_0(\mu\varepsilon_2)\left[1 - \frac{\mathrm{Y}_0(\mu)}{\mathrm{Y}_0(\mu\varepsilon_2)}\right]}{\mathrm{J}_0(\mu) - \frac{\mathrm{Y}_0(\mu)\mathrm{J}_0(\mu\varepsilon_2)}{\mathrm{Y}_0(\mu\varepsilon_2)}}\right\}. \tag{36}$$

Relations (34) and (36) at $\varepsilon_2 \to 0$ give

$$\hat{V}_x = \frac{\hat{K}}{\mathrm{i}\omega}\left[1 - \frac{\mathrm{J}_0(\mu\xi)}{\mathrm{J}_0(\mu)}\right], \tag{37}$$

that coincides with a similar solution [10] for a liquid-filled tube without the internal rod.

The aim here is to express the shear stresses at the interfaces in terms of the averaged liquid velocity in the wave V. In order to do this, a velocity gradient is sought at the tube wall and on the rod surface. It follows from (34) that:

$$\left(\frac{\mathrm{d}\hat{V}_x}{\mathrm{d}\xi}\right)_{\xi=1} = -\mu[A\mathrm{J}_1(\mu) + B\mathrm{Y}_1(\mu)],$$
$$\left(\frac{\mathrm{d}\hat{V}_x}{\mathrm{d}\xi}\right)_{\xi=\varepsilon_2} = -\mu[A\mathrm{J}_1(\mu\varepsilon_2) + B\mathrm{Y}_1(\mu\varepsilon_2)]. \tag{38}$$

The amplitude of the averaged velocity $\hat{V}$ can now be evaluated according to (23). It results in

$$\hat{V} = \frac{\hat{K}}{\mathrm{i}\omega} + \frac{2}{\mu(1-\varepsilon_2^2)}\{A[\mathrm{J}_1(\mu) - \varepsilon_2\mathrm{J}_1(\mu\varepsilon_2)] + B[\mathrm{Y}_1(\mu) - \varepsilon_2\mathrm{Y}_1(\mu\varepsilon_2)]\}. \tag{39}$$

Comparing (39) and (38), it is easy to find that:

$$\hat{V} = \frac{\hat{K}}{\mathrm{i}\omega} - \frac{2}{\mu^2(1-\varepsilon_2^2)}\left\{\left(\frac{\mathrm{d}\hat{V}_x}{\mathrm{d}\xi}\right)_{\xi=1} - \varepsilon_2\left(\frac{\mathrm{d}\hat{V}_x}{\mathrm{d}\xi}\right)_{\xi=\varepsilon_2}\right\}. \tag{40}$$

On the other hand, with the use of the approximate relation $s_{xr} \approx \frac{1}{2}\partial v_x/\partial r$, the rheological equation for shear stresses can be written in the form

$$\tau_{xr} = \int_{-\infty}^{t} G(t-t_1)\frac{\partial V_x}{\partial r}\mathrm{d}t_1 + \eta_s\frac{\partial V_x}{\partial r}. \tag{41}$$

The expression for the dimensionless complex amplitude $\hat{\tau}_{xr}$, following from Eq. (41), has the form

$$\hat{\tau}_{xr} = \frac{1}{\mathrm{i}\omega}[G^* + \mathrm{i}\omega\bar{\eta}_s]\frac{\mathrm{d}\hat{V}_x}{\mathrm{d}\xi}. \tag{42}$$

The combination of (40) and (42) leads to the relation:

$$\frac{2}{1-\varepsilon_2^2}(\hat{\tau}_1 - \varepsilon_2\hat{\tau}_2) = \mu\bar{\eta}\left(\frac{\hat{K}}{\mathrm{i}\omega} - \hat{V}\right). \tag{43}$$

Eq. (43) has clear physical meaning: it represents the integral form of a momentum balance equation for the liquid in the gap. The "driving force" $\hat{K}$ in Eq. (43) can be expressed in terms of $\hat{V}$ from Eq. (39):

$$\frac{\hat{K}}{\mathrm{i}\omega} = \left\{1 - \frac{2}{\mu(1-\varepsilon_2^2)}[\bar{A}\mathrm{J}_1(\mu) + \bar{B}\mathrm{Y}_1(\mu) - \varepsilon_2(\bar{A}\mathrm{J}_1(\mu\varepsilon_2) + \bar{B}\mathrm{Y}_1(\mu\varepsilon_2))]\right\}^{-1}\hat{V},$$

$$\{\bar{A}, \bar{B}\} = [-(\mathrm{i}\omega)^{-1}\hat{K}]\,\{A, B\}. \tag{44}$$

With this formula relation (43) takes the final form

$$\hat{\tau}_1 - \varepsilon_2\hat{\tau}_2 = -4\bar{\eta}D\hat{V},$$

$$D = -\frac{1}{4}\frac{\mu T(\mu,\varepsilon_2)}{1 - 2\mu^{-1}(1-\varepsilon_2^2)^{-1}T(\mu\varepsilon_2)},$$

$$T(\mu,\varepsilon_2) = \bar{A}[\mathrm{J}_1(\mu) - \varepsilon_2\mathrm{J}_1(\mu\varepsilon_2)] + \bar{B}[\mathrm{Y}_1(\mu) - \varepsilon_2\mathrm{Y}_1(\mu\varepsilon_2)]. \tag{45}$$

3. Dispersion equation

The boundary conditions for normal stresses at the interfaces contain the deviatoric stress component τ_{rr}. Similar to [5,6], it can be defined approximately as follows:

$$\tau_{rr|R_1} \approx \tau_{rr|R_2} = \tau_R \equiv \frac{1}{\pi R_1^2(1-\varepsilon_2^2)}\int_{R_2}^{R_1}\tau_{rr}\cdot 2\pi r\,\mathrm{d}r. \tag{46}$$

In the same manner, the averaged pressure disturbance in the gap is used for Δp_f (note that the characteristic time of pressure levelling across the gap is much less than the characteristic time of the wave propagation). It allows the boundary conditions for normal stresses to be written in the form

$$\hat{\sigma}_{rr} = -\hat{p}_c = \hat{\tau}_R - \hat{P}, \tag{47}$$

where $\hat{\sigma}_{rr}, \hat{\tau}_R$ are non-dimensional complex amplitudes of σ_{rr}, τ_R. It follows from (46) together with rheological Eq. (21), mass balance Eq. (20) and kinematic conditions (27) for radial velocities $\bar{v}_{R_1}, \bar{v}_{R_2}$ in the liquid at the interfaces that:

$$\hat{\tau}_R = \frac{2\kappa\bar{\eta}}{3\bar{c}_f^2}\mathrm{i}\omega\hat{P} + \frac{2\mathrm{i}\omega\bar{\eta}}{(1-\varepsilon_2)}(\hat{u}_1 - \hat{w}_{1R}), \tag{48}$$

where $\hat{w}_{1R} = \hat{w}_{1|\xi=\varepsilon_2}$. As a result the boundary condition (47) takes the form

$$\hat{p}_c = \left(1 - \frac{2\mathrm{i}\omega\kappa\bar{\eta}}{3\bar{c}_f^2}\right)\hat{P} - \frac{2\mathrm{i}\omega\bar{\eta}}{1-\varepsilon_2}(\hat{u}_1 - \hat{w}_{1R}) = -\hat{\sigma}_{rrR}, \quad \hat{\sigma}_{rrR} = \hat{\sigma}_{rr|\xi=\varepsilon_2}. \tag{49}$$

The dispersion equation follows from the full system of equations for complex amplitudes and boundary conditions, formulated above. After some routine algebra it is possible to bring it to the following form:

$$az^2 - bz + d = 0, \quad z = \bar{c}^{-2}, \quad \bar{c} = \omega/k, \tag{50}$$

$$a = \mathrm{i}\omega \bar{E}_s \varepsilon_1 Q\left(\frac{1 - v_s^2 Q^{-1}}{1 - v_s^2}\right)\cdot\left[1 - \frac{2\mathrm{J}_1(\beta\varepsilon_2)\mathrm{i}\omega\bar{\eta}}{L(1-\varepsilon_2)}\right] - \frac{\omega^2\bar{\eta}}{1-\varepsilon_2},$$

$$b = \mathrm{i}\omega\varepsilon_1 Q + \frac{2\varepsilon_1 Q\omega^2\bar{\eta}\mathrm{J}_1(\beta\varepsilon_2)}{L(1-\varepsilon_2)} - \frac{(1-v_s^2)\omega^2\bar{\eta}}{(1-\varepsilon_2)\bar{E}_s} + \varepsilon_1\bar{E}_s QN$$
$$\times\left[\frac{1}{\bar{c}_f^2} - \frac{\mathrm{J}_1(\beta\varepsilon_2)}{L}\left(\frac{2\varepsilon_2}{(1-\varepsilon_2^2)\kappa} + \frac{2\mathrm{i}\omega\bar{\eta}}{\bar{c}_f^2(1-\varepsilon_2)}\right)\right]\left(\frac{1 - v_s^2 Q^{-1}}{1 - v_s^2}\right) + \frac{\mathrm{i}\omega\bar{\eta}N}{\bar{c}_f^2(1-\varepsilon_2)} + \frac{N}{(1-\varepsilon_2^2)\kappa},$$

$$d = \frac{\varepsilon_1 QN}{\bar{c}_f^2} - \frac{\varepsilon_1 QN\mathrm{J}_1(\beta\varepsilon_2)}{L}\left(\frac{2\varepsilon_2}{(1-\varepsilon_2^2)\kappa} + \frac{2\mathrm{i}\omega\bar{\eta}}{\bar{c}_f^2(1-\varepsilon_2)}\right)$$
$$+ \frac{(1-v_s^2)N}{\bar{E}_s}\left(\frac{\mathrm{i}\omega\bar{\eta}}{\bar{c}_f^2(1-\varepsilon_2)} + \frac{1}{(1-\varepsilon_2^2)\kappa}\right), \quad N = \mathrm{i}\omega + \frac{8\kappa\bar{\eta}D}{1-\varepsilon_2^2}, \quad Q = 1 - \bar{E}_s^{-1}\omega^2(1-v_s^2),$$

$$L = -(\bar{\lambda}_r + 2\bar{\mu}_r)\lfloor \beta\mathrm{J}_0(\beta\varepsilon_2) - \varepsilon_2^{-1}\mathrm{J}_1(\beta\varepsilon_2)\rfloor - \bar{\lambda}_r\varepsilon_2^{-1}\mathrm{J}_1(\beta\varepsilon_2).$$

Eq. (50) took into account that in the long-wave region the following relations hold:

$$1 - \tfrac{2}{3}\mathrm{i}\omega\kappa\bar{\eta}\bar{c}_f^{-2} \approx 1, \quad 1 + \tfrac{1}{3}\varepsilon_1^2 k^4 - \bar{E}_s^{-1}\omega^2(1-v_s^2) \approx 1 - \bar{E}_s^{-1}\omega^2(1-v_s^2).$$

Eq. (50) takes on the form of dispersion equation [12], which was derived for the case of the rigid central rod, when $E_r \to \infty$.

Eq. (50) leads to known results in the limiting case $\omega \to 0$ for ideal liquid in the gap ($\bar{\eta} = 0$). In this limit Eq. (16) takes the quasi-static form

$$\frac{\partial^2\hat{w}_1}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\hat{w}_1}{\partial\xi} - \frac{1}{\xi^2}\hat{w}_1 = 0 \tag{51}$$

and its solution, bounded at $\xi = 0$, is

$$\bar{w}_1 = C_1\xi. \tag{52}$$

As a result, the dispersion Eq. (50) can be written as follows:

$$\frac{1}{\bar{c}^2} = \frac{1}{\bar{c}_f^2} + \frac{1}{\bar{c}_r^2}, \quad \bar{c}_r^2 = \frac{\kappa(1-\varepsilon_2^2)\varepsilon_1\bar{E}_s}{1 + \varepsilon_2^2\varepsilon_1\bar{E}_s(\bar{\lambda}_r + \bar{\mu}_r)^{-1}}. \tag{53}$$

This formula can be considered as a generalization of the Korteweg relation [23] for the water hammer speed c_K in a water-filled thin-walled tube without an internal rod ($\varepsilon_2 = 0$):

$$\frac{1}{\bar{c}_K^2} = \frac{1}{\bar{c}_f^2} + \frac{1}{\kappa\varepsilon_1\bar{E}_s}. \tag{54}$$

It follows from (53) that the presence of the rod lowers the sound speed $\bar{c}$, i.e. the greater the value of ε_2, the smaller the sound speed. For the deformable rod $\bar{c}$ is smaller than for a rigid one. Note that deformations of the rod in the wave influence the low frequency velocity only through the ratio of elastic constants, namely, $\bar{E}_s(\bar{\lambda}_r + \bar{\mu}_r)^{-1}$.

4. Analysis of dispersion equation

Dispersion Eq. (50) was studied numerically for a discrete spectrum model with relaxation times, distributed according to the Spriggs law [24] $\theta_k = \theta_1/k^{\alpha_1}$. In this case the complex dynamic module G^* has the form [17]

$$G^* = \mathrm{i}\omega \frac{\bar{\eta}_p - \bar{\eta}_s}{z(\alpha_1)} \sum_{k=1}^{\infty} \frac{k^{\alpha_1} - \mathrm{i}\omega\bar{\theta}_1}{k^{2\alpha_1} + (\omega\bar{\theta}_1)^2}, \tag{55}$$

where $z(\alpha_1)$ is the Riemann zeta function of the spectral distribution parameter α_1 and $\bar{\theta}_1$ is the main non-dimensional relaxation time of the liquid. To dilute polymeric solutions, the value of $\bar{\theta}_1$ can be estimated from the Rouse theory [16] combined with semi-empirical Martin relation [25] $\eta_p/\eta_s = 1 + \tilde{c}_p \exp(k_M\tilde{c}_p)$:

$$\bar{\theta}_1 = 0.608\bar{\eta}_s A \exp(k_M\tilde{c}_p),$$

$$A = [\eta]Mp_0/R_G, \quad \tilde{c}_p = c_p[\eta], \quad [\eta] = \lim_{c_p \to 0} \frac{\eta_p - \eta_s}{c_p\eta_s}. \tag{56}$$

Here $[\eta]$ is characteristic viscosity of the solution which can be found from Mark–Houwink relation [25].

$$[\eta] = KM^a, \tag{57}$$

where K and a are constants for a given polymer-solvent pair in a certain range of molecular mass variations.

In Figs. 1–4, the non-dimensional sound speed $C = \omega/\mathrm{Re}\{k\}$ and attenuation of sound $\chi = -\mathrm{Im}\{k\}$, as computed from Eq. (50) taking Eqs. (55) and (56) into account, are plotted versus dimensionless frequency ω and relative rod radius ε_2. All curves in the Figs. 1 and 2 correspond to polymeric solution in a highly viscous solvent with $\eta_s = 0.1$ Pa s, $\eta_p = 3.79$ Pa s, $\tilde{c}_p = 5$,

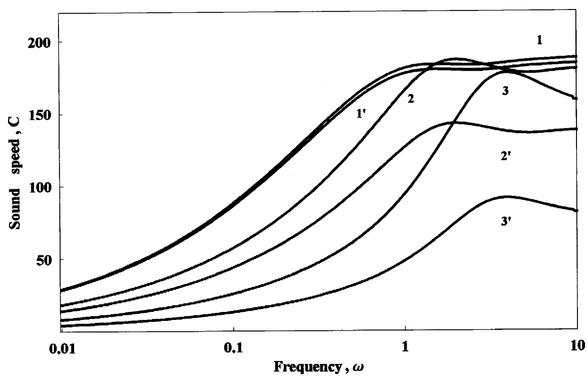

Fig. 1. Sound dispersion in the system for different gap width. Curves 1–3 correspond to $\varepsilon_2 = 0.1$, 0.4 and 0.7, respectively.

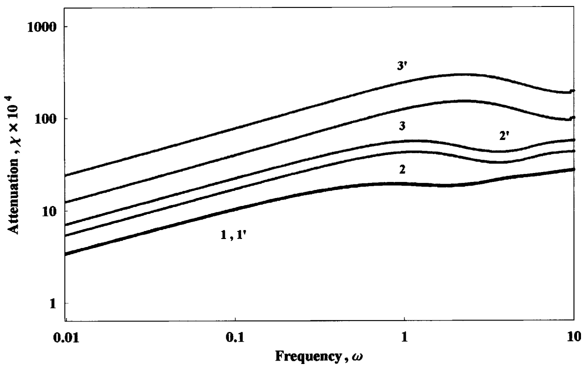

Fig. 2. Attenuation of sound versus frequency for different gap width. Curves 1–3 correspond to the same variants as in Fig. 1.

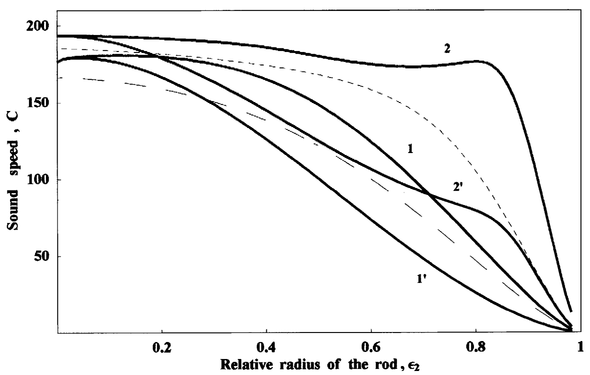

Fig. 3. Sound speed in the system versus relative rod radius, $\omega = 1$.

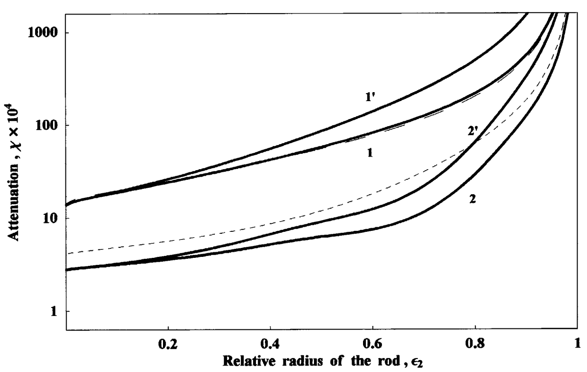

Fig. 4. Attenuation of sound versus relative rod radius, $\omega = 1$.

$A = 200$, $k_M = 0.4$, $\rho_{f0} = 1000\ \mathrm{kg/m^3}$, $\theta_1 = 0.89 \times 10^{-3}$ s (the parameter values were estimated for usually considered range of their variations for high polymer solutions in organic solvents [25,26]). Curves 1 and 1′ in Figs. 3 and 4 correspond to the same liquid, whereas curves 2 and 2′ were calculated for a 2.5% solution of polysterene in toluene (a low-viscous solvent) with $\rho_{f0} = 850\ \mathrm{kg/m^3}$, $\eta_p = 0.5$ Pa s, $\eta_s = 0.5 \times 10^{-3}$ Pa s, $\theta_1 = 10^{-2}$ s [18]. Data represented by curves 1–3 in Figs. 1–4 were obtained for the shell and rod, both made from aluminum; the curves 1′, 2′, 3′ correspond to the aluminum shell and polyurethane rod ($\rho_r = 1.2 \times 10^3\ \mathrm{kg/m^3}$, $E_r = 10^7\ \mathrm{N/m^2}$, $\nu_r = 0.49$ [27]). The dashed and dotted lines in Figs. 3 and 4 correspond to an aluminum made shell and rod; they were calculated for pure viscous liquids with a solution viscosity ($\eta = \eta_p = 3.79$ Pa s for the dashed lines and $\eta = \eta_p = 0.5$ Pa s for the dotted lines, respectively). All simulations were performed for $R_1 = 10^{-2}$ m, $\varepsilon_1 = 0.05$, $c_f = 1500$ m/s, $\alpha_1 = 2$, $E_s = 7 \times 10^{10}\ \mathrm{N/m^2}$, $\nu_s = 0.34$, $\rho_s = 2.7 \times 10^3\ \mathrm{kg/m^3}$, $p_0 = 10^5$ Pa.

The results of simulations show that the system under consideration is characterized by strong sound dispersion. When $\omega \to 0$ also $\bar{\eta} \to \bar{\eta}_p$ and $C \to 0$. This is explained by the transition from an inertial to a creeping flow regime of the liquid. Solution (45) in this frequency region has the same form as the relation for a difference between shear stresses in a steady laminar flow of pure viscous liquid in a cylindrical gap [28]. Viscoelasticity of the liquid manifests itself in the frequency interval, close to $\bar{\omega} \sim \bar{\theta}_1^{-1}$, where it causes the increase in sound speed and attenuation reduction (Figs. 3 and 4). Note that curve 1 in Fig. 4 corresponds to liquid with much less relaxation time than curve 2. It explains why the dashed line and solid line 1 in this Figure are very close (for curve 1, the inverse relaxation time is $\bar{\theta}_1^{-1} \ll 1$).

Sound speed decreases with gap narrowing, whereas attenuation grows. The larger the value of ε_2, the greater the difference between results, corresponding to hereditary and pure viscous models for the liquid (solid curve 2 and dotted line in Fig. 3). For very narrow gaps, this effect vanishes as a result of the closeness of the C values to zero.

Elastic properties of the rod highly influence both dispersion and attenuation of sound waves. It follows from Fig. 1 that this influence is enhanced with the gap narrowing. The basic result here has clear physical meaning that the smaller the elastic modulus of the rod material, the smaller the corresponding sound speed. As is seen from Fig. 2, lowering the rod's elasticity modulus is accompanied also by an increase in the wave attenuation.

The theoretical analysis presented in this paper shows, in particular, that the central rod serves to illustrate the viscoelasticity effects on sound propagation in the waveguide. It can find application, for instance, in the rheological characterization of polymeric liquids by acoustic means [29]. To apply this technique, it is necessary to understand the effect of liquid rheology and the waveguide parameters on the dispersion properties of the longitudinal wave propagation in the system, and the model developed above contributes to such understanding.

5. Conclusions

The model of sound wave propagation in deformable tubes with polymeric liquids and elastic central rods has been developed in this study. The derived dispersion equation accounts for elasticity of the pipe wall and internal rod, compressibility of the liquid and its hereditary properties. The problem is solved in a quasi one-dimensional approximation for the low frequency range where the long-wave approach can be used. In the partial case of ideal liquid in the absence of an internal rod, the dispersion equation coincides with the Korteweg relation for water hammer speed.

Numerical analysis has shown that the width of the gap between the pipe wall and the rod highly influences both the sound speed and its attenuation. The narrowing of the gap enhances the effects of the viscoelastic properties of the liquid, leading to sound speed growth and attenuation reduction. This effect vanishes when $\varepsilon_2 \to 1$. Elastic properties of the rod can be disregarded only for small values of ε_2; otherwise they influence essentially the wave propagation. The growth of the rod elastic module leads to an increase of the sound speed and a weakening of the attenuation. The effect is enhanced with the gap narrowing.

Appendix A. Nomenclature

u_x, u_r	displacement of the shell middle surface in longitudinal and transverse directions
v_x, v_r	liquid velocity components in longitudinal and transverse directions
$\bar{V}$	non-dimensional average relative velocity of the liquid in longitudinal direction
V_x	relative velocity of the liquid with respect to the pipe wall
R_1	radius of the shell middle surface
R_2	radius of the rod
L	pipe length
l	wavelength
p	pressure
Δp	contact pressure
e_{ij}	rate deformation tensor in liquid
s_{ij}	deviator of the tensor e_{ij}
$G(t - t_1)$	relaxation function for the liquid
$F(\theta)$	spectrum of relaxation times θ
t	time
t_0	characteristic time, $R_1(\rho_s/p_0)^{1/2}$
h	half of the shell width
E	Young's modulus
C	dimensionless speed of sound waves
R_G	universal gas constant
c_f	sound speed in the liquid
c_K	Korteweg speed of water hammer
k_M	Martin constant
c_p	polymer concentration in solution
M	molecular mass of the polymer
k	non-dimensional wave number

Greek letters

ν	Poisson ratio
ε_1	relative width of the shell, h/R_1

ε_2	relative radius of the rod, R_2/R_1
$\lambda_r,\ \mu_r$	Lame constants of the rod
κ	relative density of the tube material, ρ_s/ρ_{f0}
ρ	density
η_s	solvent viscosity
η_p	Newtonian viscosity of polymeric solution
$[\eta]$	characteristic viscosity of polymeric solution
ω	non-dimensional frequency
ξ	dimensionless radial co-ordinate, r/R_1
ζ	dimensionless axial co-ordinate, x/R_1
τ	non-dimensional time, t/t_0
τ_{ij}	deviatoric stress tensor in liquid
$\mathbf{\nabla}$	divergence

Subscripts

0	equilibrium state
s	shell
f	fluid
r	rod

Superscripts

-	non-dimensional quantity
^	complex amplitude of perturbations in the wave

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