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Analysis of non-linear mode shapes and natural frequencies of continuous damped systems

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Abstract

In this paper, the aim is to find the non-linear mode shapes and natural frequencies for a class of one-dimensional continuous damped systems with weak cubic inertia, damping and stiffness non-linearities. This paper presents general formulations for natural frequencies and mode shapes with all non-linearity effects. Initially the non-linear system with general boundary conditions is discretized, and using a two-dimensional manifold, the model of cubic non-linearities is constructed and the general equation of motion which governs non-linear system is derived. The method of multiple scales is then used to extend the non-linear mode shapes and natural frequencies. During this analysis, it is realized that when the natural frequencies of the linear system become equal to the natural frequencies of the non-linear system a one-to-one internal resonance will appear. Also, there is a three-to-one internal resonance which is not dependent on the damping of the system. Finally, general formulations of amplitude for vibrations, natural frequencies and mode shapes of the non-linear system are obtained in parametric forms. Thus, a non-linear problem with some simple integration can be solved. The formulations are capable of handling any non-linearities in inertia, damping, stiffness, or any combination of them under any arbitrary boundary conditions.

1. Introduction

Models are presented in this paper in order to construct the non-linear mode shapes and natural frequencies of one-dimensional continuous damped or undamped systems with weak cubic inertia,

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damping and stiffness non-linearities. It is well known that the natural frequencies of a linear system are invariant quantities, but in geometrically non-linear systems they become variant due to the variation of vibration amplitude.

Several papers report studies of non-linear continuous multi-degree-of-freedom systems. Vakakis and Rand [1,2] obtained the non-linear normal modes of a conservative two-degrees-of-freedom non-linear system. Shaw and Pierre [3,4] obtained the governing differential equations of motion for a weakly non-linear discrete and continuous system. Nayfeh et al. [5,6] studied the non-linear normal modes of continuous undamped systems. They used the method of real manifold approach, the method of complex manifold approach, and the method of multiple scales to construct mode shapes and natural frequencies of the continuous systems with general boundary conditions. They constructed the natural frequencies and mode shapes of the system with combined non-linear inertia and stiffness, using three methods as the verification of the results. In the other works, Nayfeh [7,8] obtained the normal modes of the cantilever and simply supported beams. Here, the non-linearities were due to inertia and stiffness terms. Nayfeh and et al. [9] also considered the damping parameters individually to study the non-linear system response and bifurcations. Foda [10] studied the non-linear free vibration of a beam considering shear deformation and rotary inertia.

The non-linear modes of damped systems are mostly used in the design of vibration absorbers. In such problems, non-linearity of damping should be considered but the non-linearity in inertia and stiffness are generally neglected. For a more precise result, a combined non-linearity should be considered. Pai and Schulz [11] designed a non-linear vibration absorber and improved its stability. Also, Nayfeh and colleagues [12–14] studied non-linear vibration absorbers using the method of multiple scales.

Among the different analytical methods to solve the non-linear problems, the method of multiple scales is one of the most popular. This method, unlike the numerical methods, provides a better understanding and physical insight into the characteristics of the non-linear problems. This method is used by Nayfeh in different papers. Also, Hamdan and et al. [15] compared this method with other available methods in studying the non-linear vibrations.

In this paper, general formulations for the non-linear natural frequencies and mode shapes of a non-linear continuous system with non-linearities in inertia, damping and stiffness and for any boundary conditions are presented. The formulations can be used individually for any type of non-linear parameter of inertia, damping, stiffness, or all together.

2. Problem identification

In non-dimensional form, the equation of motion for a continuous damped system with weak cubic inertia, damping and stiffness non-linearities may be expressed as

$$\ddot{y}(x,t) + c\dot{y}(x,t) + \mathrm{L}[y(x,t)] + \mathrm{N}[\ddot{y},\dot{y},y] = 0 \tag{1}$$

in which c is a damping factor, the dot symbol indicates differentiation with respect to time t. The operator L is a self-adjoint linear spatial operator; and N is a non-linear spatial operator on y.

The homogeneous boundary conditions are given by

$$\mathrm{B}_1[y(x,t)] = 0 \text{ at } x = 0 \quad \text{and} \quad \mathrm{B}_2[y,(x,t)] = 0 \text{ at } x = 1 \tag{2}$$

for which B_1 and B_2 are homogeneous spatial operators.

For a linear problem, i.e., the system with $\mathrm{N}\equiv 0$, there are n natural frequencies ω_n corresponding to the linear mode shapes $p_n(x)$. The inner product between two functions $f(x)$ and $g(x)$ is defined by

$$\langle f(x), g(x) \rangle = \int_0^1 f(z)g(z)\,\mathrm{d}z, \tag{3}$$

Since the operator L is a self-adjoint operator, the mode shapes $p_n(x)$ will be orthogonal. Considering the above definition, the mode shapes $p_n(x)$ may be normalized using the following equation:

$$\langle p_n(x),\ p_m(x) \rangle = \delta_{nm}, \tag{4}$$

where δ_{nm} is the Kronecker delta. Moreover,

$$\langle \mathrm{L}[p_n(x)],\ p_m(x) \rangle = \omega_n^2 \delta_{nm}, \tag{5}$$

$$\langle c p_n(x),\ p_m(x) \rangle = 2\hat{\zeta}\omega_n \delta_{nm}, \tag{6}$$

where $\hat{\zeta}$ is the damping factor.

To discretize Eqs. (1) and (2) using the expansion theorem, $y(x,t)$ may be expressed as

$$y(x,t) = \sum_{n=1}^{\infty} p_n(x) q_n(t), \tag{7}$$

where $q_n(t)$ are the time-dependent portion of the solution.

Substituting Eq. (7) into Eq. (1) and taking the inner product of the resulting equation with $p_j(x)$, gives

$$\ddot{q}_j + 2\hat{\zeta}\omega_j \dot{q}_j + \omega_j^2 q_j + G_j(q,\dot{q},\ddot{q}) = 0 \ \text{ for } j = 1, 2, \ldots, \tag{8}$$

where

$$G_j(q,\dot{q},\ddot{q}) = \left\langle p_j(x), \mathrm{N}\left[\sum_{m=1}^{\infty} p_m(x) q_m(t),\ \sum_{m=1}^{\infty} p_m(x) \dot{q}_m(t),\ \sum_{m=1}^{\infty} p_m(x) \ddot{q}_m(t)\right]\right\rangle. \tag{9}$$

There are different methods to expand the non-linearity term expressed in Eq. (9). Here, a two-dimensional manifold is used to expand the non-linear terms. To construct the manifold, Eq. (8) is rewritten as a system of two first order equations:

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = r_j, \tag{10}$$

$$\frac{\mathrm{d}r_j}{\mathrm{d}t} = -2\hat{\zeta}\omega_j \dot{q}_j - \omega_j^2 q_j - G_j. \tag{11}$$

The property of this two-dimensional manifold is that when the non-linearity vanishes, there will be k linear modes. Using q_k and p_k parameters, the manifold will be expressed as

$$q_j(t) = Q_{jk}(q_k, r_k) \quad \text{and} \quad r_j(t) = R_{jk}(q_k, r_k). \tag{12}$$

The definition of normal-mode-invariant-manifolds is: a normal mode of motion for a non-linear, autonomous system is a motion which takes place on a two-dimensional invariant manifold in the system's phase space. This manifold has the following properties: it passes through the stable equilibrium point $Q_{jk}(0,0) = 0$ and $R_{jk}(0,0) = 0$ of the system and at that point it is tangent to a plane which is an eigenspace of the system linearized about equilibrium point [4]. Using these properties and Eqs. (10)–(12), the cubic non-linear terms will appear as

$$G_j(q, \dot{q}, \ddot{q}) = g_{1jk}q_k^3 + g_{2jk}q_k\dot{q}_k^2 + g_{3jk}q_k^2\ddot{q}_k + g_{4jk}q_k^2\dot{q}_k + g_{5jk}\dot{q}_k^3 + \cdots, \tag{13}$$

where

$$g_{ijk} = \langle p_j(x), N_i(p_k(x)) \rangle. \tag{14}$$

For more details, the reader should refer to Refs. [3,4,6].

Considering all cubic non-linear terms in Eq. (13), the first term, $g_{1jk}q_k^3$, represents the non-linearity in stiffness or geometry. Existence of this term alone in Eq. (13) makes it the Duffing equation, which has previously been studied. The second and third terms, $g_{2jk}q_k\dot{q}_k^2 + g_{3jk}q_k^2\ddot{q}_k$, are due to non-linearity in inertia. The two last terms, $g_{4jk}q_k^2\dot{q}_k + g_{5jk}\dot{q}_k^3$, represent the damping non-linearities. Nayfeh [6] studied the case of combined cubic inertia and stiffness non-linearities; for which, Eq. (13) is written as

$$G_j(q, \dot{q}, \ddot{q}) = g_{1jk}q_k^3 + g_{2jk}q_k\dot{q}_k^2 + g_{3jk}q_k^2\ddot{q}_k + \cdots. \tag{15}$$

In most cases, the non-linear damping terms are neglected because of complexity in calculations, or due to the fact that their values are negligible. Also, when the effects of non-linear damping terms are considered, other non-linear terms are neglected. However, combinations of all non-linear terms have not yet been studied. In this paper, for the first time, the non-linearity in the form of combined cubic inertia, stiffness and damping as presented in Eq. (13) is studied.

3. Solution using the method of multiple scales

In this part, the method of multiple scales [16–18] is used to determine the periodic solutions of Eq. (8) in order to construct the non-linear normal modes and corresponding natural frequencies. To apply the perturbation method the parameter ε may be used as a small dimensionless parameter to show the weakness of the non-linear terms and consider $\hat{\zeta} = \varepsilon\zeta$ to and rewrite Eq. (8) as

$$\ddot{q}_j + 2\varepsilon\zeta\omega_j\dot{q}_j + \omega_j^2 q_j + \varepsilon G_j(q, \dot{q}, \ddot{q}) = 0 \quad \text{for } j = 1, 2, \ldots, \tag{16}$$

in which q_j may be written as a second order expansion in the form of

$$q_j(t, \varepsilon) = q_{j0}(T_0, T_1) + \varepsilon q_{j1}(T_0, T_1) + \cdots, \tag{17}$$

where T_0 and T_1 are two time scales in a form of $T_0 = t$ as a fast time scale, characterizing motions occurring at one of the natural frequencies, ω_k, and $T_1 = \varepsilon t$ as a slow time scale characterizing a shift in the natural frequencies due to the non-linearity effect.

Substituting Eq. (17) into Eq. (16) and equating coefficients of the like powers of ε, one may obtain

$$(\varepsilon^0): \frac{\partial^2 q_{j0}}{\partial T_0^2} + \omega_j^2 q_{j0} = 0, \tag{18}$$

$$(\varepsilon^1): \frac{\partial^2 q_{j1}}{\partial T_0^2} + \omega_j^2 q_{j1} = -2\frac{\partial^2 q_{j0}}{\partial T_0\,\partial T_1} - 2\zeta\omega_j \frac{\partial q_{j0}}{\partial T_0} - G_j. \tag{19}$$

Eq. (18) shows the linear form of Eq. (16) as $\varepsilon \to 0$. In this situation the non-linear modes are reduced to kth linear mode. The solution of Eq. (18), considering the boundary conditions, may be written as

$$q_{k0} = A_k(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0} + \bar{A}_k(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0} \tag{20}$$

and

$$q_{j0} = 0 \quad \text{for } j \neq k, \tag{21}$$

where the function A_k is an unknown at this level of approximation and will be determined by eliminating the secular terms from the q_{j1} solution. Substituting Eqs. (20) and (21) into Eq. (19), gives

$$\begin{aligned} \frac{\partial^2 q_{k1}}{\partial T_0^2} + \omega_k^2 q_{k1} = &- 2\mathrm{i}\omega_k\left[A_k'(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0} - \bar{A}_k'(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0}\right] \\ &- 2\mathrm{i}\zeta\omega_k^2\left[A_k(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0} + \bar{A}_k(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0}\right] - G\big[\left[A_k(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0} + \bar{A}_k(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0}\right], \\ &\mathrm{i}\omega_k\left[A_k(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0} - \bar{A}_k(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0}\right], -\omega_k^2\left[A_k(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0} + \bar{A}_k(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0}\right]\big], \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial^2 q_{j1}}{\partial T_0^2} + \omega_j^2 q_{j1} = &- G\big[\left[A_k(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0} + \bar{A}_k(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0}\right], \mathrm{i}\omega_k\left[A_k(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0},\right. \\ &\left. -\bar{A}_k(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0} - \omega_k^2\left[A_k(T_1)\,\mathrm{e}^{\mathrm{i}\omega_k T_0} + \bar{A}_k(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0}\right]\right] \quad \text{for } j \neq k. \end{aligned} \tag{23}$$

Using Eq. (13), Eqs. (22) and (23) may be rewritten as

$$\begin{aligned} \frac{\partial^2 q_{k1}}{\partial T_0^2} + \omega_k^2 q_{k1} = &- \big\{2\mathrm{i}\omega_k A_k' + 2\mathrm{i}\zeta\omega_k^2 A_k + \big[3g_{1kk} + \omega_k^2(g_{2kk} - g_{3kk}) \\ &+\mathrm{i}\omega_k g_{4kk} + 3\mathrm{i}\omega_k^3 g_{5kk}\big]A_k^2\bar{A}_k\big\}\mathrm{e}^{\mathrm{i}\omega_k T_0} \\ &- \big[g_{1kk} - \omega_k^2(g_{2kk} + g_{3kk}) + \mathrm{i}\omega_k g_{4kk} \\ &-\mathrm{i}\omega_k^3 g_{5kk}\big]A_k^3\mathrm{e}^{3\mathrm{i}\omega_k T_0}\big\} + \mathrm{cc}, \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial^2 q_{j1}}{\partial T_0^2} + \omega_j^2 q_{j1} = &- \big[3g_{1kk} + \omega_k^2(g_{2kk} - g_{3kk}) + \mathrm{i}\omega_k g_{4kk} + 3\mathrm{i}\omega_k^3 g_{5kk}\big]A_k^2\bar{A}_k\mathrm{e}^{\mathrm{i}\omega_k T_0} \\ &- \big[g_{1kk} - \omega_k^2(g_{2kk} + g_{3kk}) + \mathrm{i}\omega_k g_{4kk} - \mathrm{i}\omega_k^3 g_{5kk}\big]A_k^3\mathrm{e}^{3\mathrm{i}\omega_k T_0} + \mathrm{cc}, \end{aligned} \tag{25}$$

where cc stands for the complex conjugate of the preceding terms and the prime shows the derivative with respect to T_1. The terms which produce secular terms should be eliminated from Eq. (24) as

$$2\mathrm{i}\omega_k A_k' + 2\mathrm{i}\zeta\omega_k^2 A_k + \left[3g_{1kk} + \omega_k^2(g_{2kk} - g_{3kk}) + \mathrm{i}\omega_k g_{4kk} + 3\mathrm{i}\omega_k^3 g_{5kk}\right] A_k^2 \bar{A}_k = 0. \tag{26}$$

Then, the solution of Eq. (24) may be expressed as

$$q_{k1} = \frac{g_{1kk} - \omega_k^2(g_{2kk} + g_{3kk}) + \mathrm{i}\omega_k g_{4kk} - \mathrm{i}\omega_k^3 g_{5kk}}{8\omega_k^2} A_k^3\, \mathrm{e}^{3\mathrm{i}\omega_k T_0} + \mathrm{cc}. \tag{27}$$

Substituting Eqs. (20) and (27) into Eq. (17), gives

$$q_k = A_k \mathrm{e}^{\mathrm{i}\omega_k T_0} + \varepsilon \frac{g_{1kk} - \omega_k^2(g_{2kk} + g_{3kk}) + \mathrm{i}\omega_k g_{4kk} - \mathrm{i}\omega_k^3 g_{5kk}}{8\omega_k^2} \times A_k^3 \mathrm{e}^{3\mathrm{i}\omega_k T_0} + \mathrm{cc}. \tag{28}$$

The solution of Eq. (25) becomes

$$q_{j1} = \frac{3g_{1kk} + \omega_k^2(g_{2kk} - g_{3kk}) + \mathrm{i}\omega_k g_{4kk} + 3\mathrm{i}\omega_k^3 g_{5kk}}{\omega_k^2 - \omega_j^2} \times A_k^2 \bar{A}_k \mathrm{e}^{\mathrm{i}\omega_k T_0} + \frac{g_{1kk} - \omega_k^2(g_{2kk} + g_{3kk}) + \mathrm{i}\omega_k g_{4kk} - \mathrm{i}\omega_k^3 g_{5kk}}{9\omega_k^2 - \omega_j^2} \times A_k^3 \mathrm{e}^{3\mathrm{i}\omega_k T_0} + \mathrm{cc}. \tag{29}$$

Using Eqs. (21) and (29), the non-linear mode shapes are obtained as

$$q_j = \varepsilon q_{j1}. \tag{30}$$

Eq. (29) demonstrates the internal resonances with no dependency on damping. There are two internal resonances; the first occurs when $\omega_j = \omega_k$ which is called a one-to-one internal resonance, and the second is a three-to-one internal resonance, i.e., $\omega_j = 3\omega_k$.

In order to obtain the non-linear natural frequencies, A_k may be expressed in polar form as

$$A_k = \tfrac{1}{2} a_k \mathrm{e}^{\mathrm{i}\beta_k}. \tag{31}$$

Then by separating Eq. (26) into the real and imaginary parts,

$$a_k' + \zeta\omega_k a_k + \frac{(g_{4kk} + 3g_{5kk}\omega_k^2)a_k^3}{8} = 0, \tag{32}$$

$$\omega_k \beta_k' a_k = \frac{(3g_{1kk} + \omega_k^2(g_{2kk} - 3g_{3kk}))a_k^3}{8}. \tag{33}$$

In case of the presence of non-linear stiffness or inertia or a combination of these two non-linearities, but in absence of damping, Eq. (32) will appear as

$$a_k' = 0, \tag{34}$$

which means the amplitude of the vibration is constant and using fact in Eq. (33), gives

$$\beta_k = \frac{3g_{1kk} + \omega_k^2(g_{2kk} - 3g_{3kk})}{8\omega_k} \varepsilon t a_k^2 + \beta_{k0}, \tag{35}$$

where β_{k0} is a constant. Rewriting β_k in the form

$$\beta_k = \omega_{Nk} t + \beta_{k0}, \tag{36}$$

where ω_{Nk} is the non-linear natural frequency and using Eq. (35), provides the non-linear natural frequency as

$$\omega_{Nk} = \omega_k + \frac{3g_{1kk} + \omega_k^2(g_{2kk} - 3g_{3kk})}{8\omega_k}\varepsilon a_k^2, \tag{37}$$

which is equivalent to the equation obtained by Nayfeh and et al. for an undamped system [5].

However, here, where there is a combination of non-linear inertia, stiffness and damping, a_k will be a function of time. Using Eq. (32), the dependency of the amplitude of vibration on time may be shown as

$$a_k^2 = \frac{8a_{k0}^2\zeta\omega_k}{8\zeta\omega_k e^{2\zeta\omega_k\varepsilon t} + a_{k0}^2(g_{4kk} + 3\omega_k^2 g_{5kk})(e^{2\zeta\omega_k\varepsilon t} - 1)}, \tag{38}$$

where and a_{k0} is a constant which shows the maximum amplitude of the vibration at the beginning of the oscillations. Substituting Eq. (38) into (33), β may be obtained with respect to time as

$$\beta_k = \frac{3g_{1kk} + \omega_k^2(g_{2kk} - 3g_{3kk})}{8\omega_k} \times \left[\frac{4\ln\left[8\zeta\omega_k e^{2\zeta\omega_k\varepsilon t} + a_{k0}^2(g_{4kk} + 3\omega_k^2 g_{5kk})(e^{2\zeta\omega_k\varepsilon t} - 1)\right]}{\varepsilon(g_{4kk} + 3\omega_k^2 g_{5kk})} + \frac{8\zeta\omega_k t}{(g_{4kk} + 3\omega_k^2 g_{5kk})}\right] + \beta_{k0}, \tag{39}$$

which leads to

$$\omega_{Nk} = \frac{3g_{1kk} + \omega_k^2(g_{2kk} - 3g_{3kk})}{8\omega_k} \times \left[\frac{4\ln\left[8\zeta\omega_k e^{2\zeta\omega_k\varepsilon t} + a_{k0}^2(g_{4kk} + 3\omega_k^2 g_{5kk})(e^{2\zeta\omega_k\varepsilon t} - 1)\right]}{\varepsilon t(g_{4kk} + 3\omega_k^2 g_{5kk})} + \frac{8\zeta\omega_k}{(g_{4kk} + 3\omega_k^2 g_{5kk})}\right] + \omega_k. \tag{40}$$

Eqs. (39) and (40) show that β_k and ω_{Nk} are logarithmically dependent on the time. However, when the amplitude of vibration is a constant and using Eq. (33), ω_{Nk} is directly dependent on time, which is to be expected.

Finally, using Eqs. (28) and (30), the non-linear mode shapes of the systems with non-linearities in inertia, damping and stiffness, may be expressed as

$$y_k = p_k(x)\left\{a_k\cos(\beta_k) + \varepsilon a_k^3\left[\frac{g_{1kk} - \omega_k^2(g_{2kk} + g_{3kk})}{32\omega_k^2}\cos(3\beta_k) - \frac{g_{4kk} - \omega_k^2 g_{5kk}}{32\omega_k}\sin(3\beta_k)\right]\right\} \tag{41}$$

and

$$y_j = p_k(x)\frac{\varepsilon a_k^3}{4}\left\{\frac{3g_{1kk} + \omega_k^2(g_{2kk} - g_{3kk})}{\omega_k^2 - \omega_j^2}\cos(\beta_k) - \frac{\omega_k g_{4kk} + 3\omega_k^3 g_{5kk}}{\omega_k^2 - \omega_j^2}\sin(\beta_k)\right.$$
$$\left. + \frac{g_{1kk} - \omega_k^2(g_{2kk} + g_{3kk})}{9\omega_k^2 - \omega_j^2}\cos(3\beta_k) - \frac{\omega_k g_{4kk} - \omega_k^3 g_{5kk}}{9\omega_k^2 - \omega_j^2}\sin(3\beta_k)\right\}. \quad (42)$$

Eqs. (37)–(42) are general formulations which can be used in order to construct the non-linear mode shapes and natural frequencies of the systems with any combination of non-linearities in inertia, damping and stiffness, with different boundary conditions. Of course, in order to obtain non-linear natural frequencies, there are two equations, (37) and (40). Eq. (37) should be used only in the absence of damping and in other situations Eq. (40) is suggested. Using these formulations, the analytical solution of the non-linear problems can easily be obtained by simple integration, For a better understanding of the results of this paper, some applications will be presented.

4. Applications

As an application, the non-linear natural frequencies and mode shapes for a hinged–hinged beam resting on a non-linear viscous foundation is obtained. Now, this application will be solved for two different cases of existence of non-linear damping alone and a combination of non-linear stiffness and damping.

Case a: In this case, the partial differential equation and boundary conditions governing the vibration motion in a non-dimensional form will be

$$\ddot{y} + y^{\text{iv}} + \varepsilon\alpha(1 - y^2)\dot{y} = 0, \quad (43)$$

$$y = y'' = 0 \quad \text{at } x = 0 \text{ and } 1. \quad (44)$$

It is seen that the equation of motion has the non-linearity in form of Van Der Pol equation. Comparing Eqs. (44) and (1), then

$$\mathrm{N}[\ddot{y}, \dot{y}, y] = -\alpha y^2\dot{y}. \quad (45)$$

The linear mode shapes and corresponding natural frequencies are given by

$$p_k(x) = \sqrt{2}\sin(k\pi x) \quad \text{and} \quad \omega_k^2 = k^4\pi^4. \quad (46)$$

Considering Eq. (44), it is seen that $\mathrm{N}_1 = \mathrm{N}_2 = \mathrm{N}_3 = \mathrm{N}_5 = 0$ and

$$\mathrm{N}_4 = [p(x)] = -\alpha p(x)^3 = -2\sqrt{2}\alpha\sin^3 k\pi x. \quad (47)$$

Therefore, using Eq. (14); $g_{1kk} = g_{2kk} = g_{3kk} = g_{5kk} = 0$ and

$$g_{4kk} = \langle p_k(x), \mathrm{N}_4(p_k(x))\rangle = -\tfrac{3}{2}\alpha. \quad (48)$$

Using Eq. (40), the non-linear natural frequencies are

$$\omega_{Nk} = \omega_k = k^2\pi^2. \quad (49)$$

Using Eqs. (38), the amplitude will be

$$a_k^2 = \frac{8a_{k0}^2\omega_k}{8\omega_k e^{\alpha\omega_k\varepsilon t} - 3a_{k0}^2(e^{\alpha\omega_k\varepsilon t} - 1)}. \tag{50}$$

Fig. 1 illustrates changes of the amplitude of the first mode versus time variations. The vibration amplitude vanishes after about 14 s.

Using Eq. (40), the mode shapes will be obtained as

$$y_k = \sqrt{2}\sin(k\pi x)\left[a_k\cos(\omega_k t + \beta_{k0}) + \varepsilon a_k^3 \frac{3\alpha}{64\omega_k}\sin(3\omega_k t + 3\beta_{k0})\right], \tag{51}$$

where β_{k0} is a constant. Fig. 2 illustrates vibrations of the mid-point of the beam in its first mode versus time variations. Like amplitude, the first mode vibration in Fig. 2 shows that the vibrations

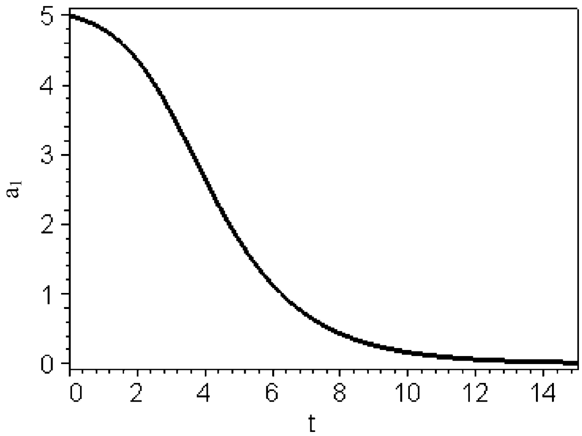

Fig. 1. Changes of amplitude versus time variation. Considering $a_{k0} = 5$, $\alpha = 1$, $\varepsilon = 0.1$.

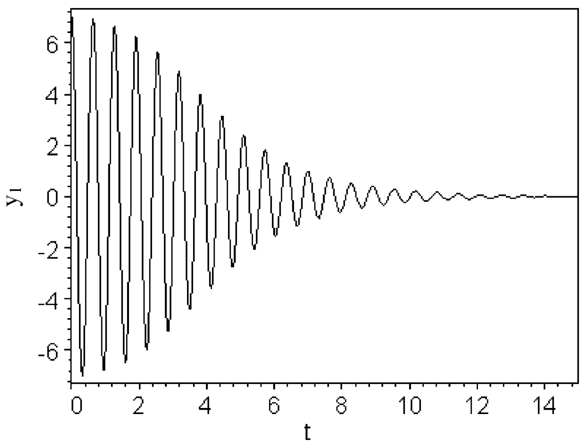

Fig. 2. The first mode vibrations of the mid-point of the beam versus time. Considering $a_{k0} = 5$, $\alpha = 1$, $\varepsilon = 0.1$, $\beta_{k0} = 0$.

vanish after about 14 s. It is seen that the frequency of vibration stay constant (the time period of oscillations are constant), but this situation will be different in the next case.

This example shows how easily the modal properties of a non-linear system can be obtained by the formulations offered in this paper.

Case b: This case obtains non-linear natural frequencies and mode shapes of a hinged–hinged geometrically non-linear beam resting on a non-linear viscous foundation for which the partial differential equation and boundary conditions governing the vibration motion in a non-dimensional form are

$$\ddot{y} + y^{\text{iv}} + \lambda y + \varepsilon\left[\alpha(1 - y^2)\dot{y} + \gamma y^3\right] = 0, \tag{52}$$

$$y = y'' = 0 \quad \text{at} \;\; x = 0 \;\; \text{and} \;\; 1. \tag{53}$$

It is seen that the equation of motion is a combination of non-linearity in stiffness and damping. Comparing Eqs. (52) and (8), then

$$\text{N}[\ddot{y}, \dot{y}, y] = \gamma y^3 - \alpha y^2 \dot{y}. \tag{54}$$

The linear mode shapes and corresponding natural frequencies are given by

$$p_k(x) = \sqrt{2}\sin(k\pi x) \;\; \text{and} \;\; \omega_k^2 = k^4\pi^4 + \lambda. \tag{55}$$

Considering Eq. (54), $\text{N}_2 = \text{N}_3 = \text{N}_5 = 0$ and

$$\text{N}_1[p(x)] = \gamma p(x)^3 = 2\sqrt{2}\gamma \sin^3 k\pi x, \tag{56}$$

$$\text{N}_4[p(x)] = -\alpha p(x)^3 = -2\sqrt{2}\alpha \sin^3 k\pi x. \tag{57}$$

Therefore, $g_{2kk} = g_{3kk} = g_{5kk} = 0$ and

$$g_{1kk} = \langle p_k(x), \text{N}_1(p_k(x)) \rangle = -\tfrac{3}{2}\gamma, \tag{58}$$

$$g_{4kk} = \langle p_k(x), \text{N}_4(p_k(x)) \rangle = -\tfrac{3}{2}\alpha. \tag{59}$$

Using Eq. (40), the non-linear natural frequencies are

$$\omega_{Nk} = \frac{3\gamma}{2}\left[\frac{\ln\left[4\alpha\omega_k e^{\alpha\omega_k\varepsilon t} - 1.5a_{k0}^2\alpha(e^{\alpha\omega_k\varepsilon t} - 1)\right]}{\varepsilon t\alpha\omega_k} + 1\right] + \omega_k. \tag{60}$$

Fig. 3 illustrates changes of the first non-linear natural frequency versus time variations.

Using Eq. (41), the mode shapes will be obtained as

$$y_k = \sqrt{2}\sin(k\pi x)\left\{a_k\cos(\beta_k) + \frac{3}{64}\varepsilon a_k^3\left[\frac{\gamma}{\omega_k^2}\cos(3\beta_k) + \frac{\alpha}{\omega_k}\sin(3\beta_k)\right]\right\}, \tag{61}$$

where a_k is the same as Eq. (50) and using Eq. (39) β_k is

$$\beta_k = \frac{3\gamma}{2}\left[\frac{\ln\left[4\alpha\omega_k e^{\alpha\omega_k\varepsilon t} - 1.5a_{k0}^2\alpha(e^{\alpha\omega_k\varepsilon t} - 1)\right]}{\varepsilon\alpha\omega_k} + t\right] + \beta_{k0}. \tag{62}$$

Fig. 4 illustrates vibrations of the mid-point of the beam in its first mode versus time variations. It is seen that the frequency of vibration varies with time.

Comparing these two cases (Figs. 2 and 4), it is seen that non-linearities in damping alone can not make natural frequencies non-linear. However, in the presence of non-linear inertia or

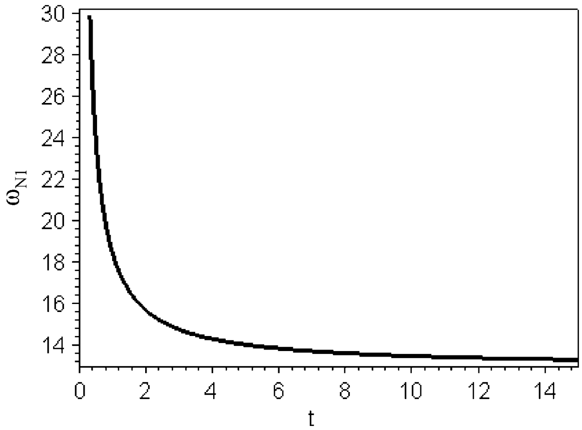

Fig. 3. Changes of first natural frequency versus time variation. Considering $a_{k0} = \alpha = \gamma = 1$, $\varepsilon = 0.1$.

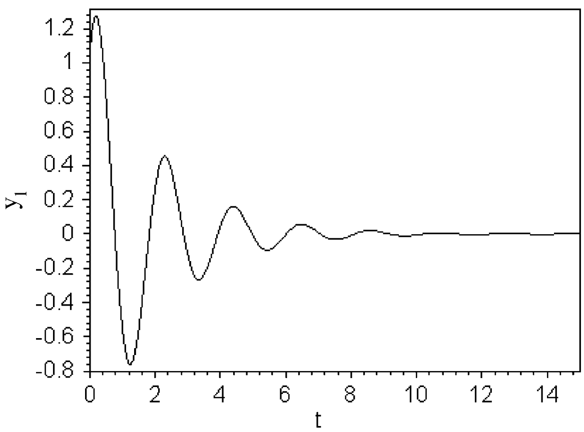

Fig. 4. Vibrations of the middle of the beam in its first mode due to time variations. Considering $a_{k0} = \alpha = \gamma = 1$.

stiffness terms, the non-linear damping terms will also affect the natural frequencies. This has been also expressed in Eq. (40).

These two applications show the efficiency of the formulation obtained in order to calculate the non-linear natural frequencies and mode shapes of a non-linear system. The linear mode shapes and natural frequencies of the system can be found and then by using Eq. (14), the g_{ijk} coefficients will be obtained. Substituting these coefficients in Eqs. (37)–(42), the non-linear natural frequencies and mode shapes of the system are constructed.

In this application a system with combined non-linearities in inertia and stiffness will be studied. For verification of the model, the non-linear modes of a metallic cantilever beam which has been studied by Nayfeh et al. [7] are considered. In the non-dimensional form, the equation of motion and boundary conditions governing vibration of a metallic cantilever beam are given by [19]

$$\ddot{y} + y^{\mathrm{iv}} + \left[y'\left(y'y''\right)'\right]' + \left[y' \int_1^x \int_0^x (\dot{y}'^2 + y'\ddot{y}')\,\mathrm{d}x\,\mathrm{d}x\right]' = 0, \tag{63}$$

$$y = y' = 0 \quad \text{at } x = 0, \tag{64}$$

$$y'' = y''' = 0 \quad \text{at } x = 1, \tag{65}$$

where the prime indicates the differentiation with respect to displacement, x. Non-linear terms are

$$\mathrm{N}[\ddot{y}, \dot{y}, y] = \left[y'\left(y'y''\right)'\right]' + \left[y' \int_1^x \int_0^x (\dot{y}'^2 + y'\ddot{y}')\,\mathrm{d}x\,\mathrm{d}x\right]'. \tag{66}$$

The linear mode shapes and corresponding natural frequencies are given by

$$p_k(x) = \cosh(z_k x) - \cos(z_k x) + [\sin(z_k x) - \sinh(z_k x)] \times \frac{\cosh(z_k x) + \cos(z_k x)}{\sin(z_k x) + \sinh(z_k x)}, \tag{67}$$

$$\omega_k = z_k^2 \tag{68}$$

and the z_k are the roots of

$$1 + \cos(z_k)\cosh(z_k) = 0. \tag{69}$$

The first and second natural frequencies are

$$\omega_1 = 3.51601 \text{ and } \omega_2 = 22.03449,$$

which are obtained by graphical methods.

Non-linear mode shapes of Eq. (63) were obtained in a paper by Nayfeh et al. and the mode shapes of linear and non-linear systems were illustrated for the second mode shape, as it is shown in Fig. 5 [7]. As a comparison, using the formulations obtained in this paper, the linear and non-linear mode shapes of Eq. (63) are calculated.

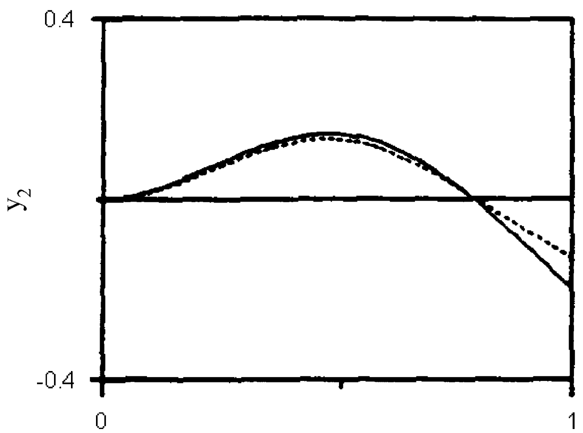

Fig. 5. The linear (——) and non-linear (....) mode shapes for the second mode [7].

The non-linear terms can be written as

$$\mathrm{N}_1 = [y'(y'y'')']', \tag{70}$$

$$\mathrm{N}_2 = \left[y' \int_1^x \int_0^x \dot{y}'^2 \mathrm{d}x\, \mathrm{d}x\right]', \tag{71}$$

$$\mathrm{N}_3 = \left[y' \int_1^x \int_0^x \left(y'\ddot{y}'\right) \mathrm{d}x\, \mathrm{d}x\right]'. \tag{72}$$

Therefore, the non-linear coefficients can be obtained as

$$g_{1kk} = \int_0^1 p_k[p_k'(p_k'p_k'')']'\, \mathrm{d}x, \tag{73}$$

$$g_{2kk} = g_{3kk} = \int_0^1 p_k \left[p_k' \int_1^x \int_0^x p_k'^2\, \mathrm{d}x\, \mathrm{d}x\right]' \mathrm{d}x, \tag{74}$$

$$g_{4kk} = g_{5kk} = 0. \tag{75}$$

For the second mode, $g_{122} = 1642.14866$ and $g_{222} = g_{322} = 32.07737$.

Considering Eq. (75) and (32) it is found a_k = constant. Using Eq. (37), the second non-linear natural frequency will be

$$\omega_{N2} = 22.03449 - 14.875479 a_k^2. \tag{76}$$

For this system, Nayfeh et al. presented the following formulation for calculating the non-linear natural frequencies [7]

$$\omega_{Nk} = \omega_k + \frac{1}{8\omega_k}(3g_{1kk} - 2\omega_k^2 g_{2kk})\, \varepsilon a_k^2 \tag{77}$$

Using Eq. (77), the non-linear natural frequency will be the same as the results in Eq. (76). To make a better comparison with the results of this paper and Ref. [7], refer to Eq. (37) again. In this application, using Eq. (74), it is seen that $g_{2kk} = g_{3kk}$. Upon substitution that in Eq. (37), Eq. (77) which is presented by Nayfeh et al. [7] will be obtained. So, not only the results will precisely match with results of Nayfeh et al. [7], but also the formulation of this paper is more general.

Also using Eq. (41) and non-linear coefficients for the second mode, the non-linear mode shape for the second mode can be obtained

$$y_2 = \left\{\cosh(z_2 x) - \cos(z_2 x) + [\sin(z_2 x) - \sinh(z_2 x)]\frac{\cosh(z_2 x) + \cos(z_2 x)}{\sin(z_2 x) + \sinh(z_2 x)}\right\} \times \left\{a_2 \cos(\omega_{N2} t) + \varepsilon(-1.89914) a_2^3 \cos(3\omega_{N2} t)\right\}. \tag{78}$$

Now, using Eq. (78) and considering $a_2 = 1$ which is similar to Ref. [7], the second mode shape of linear and non-linear systems are illustrated in Fig. 6. Comparing Figs. 5 and 6, it can be seen how similar they are. Also the linear and non-linear mode shapes can be compared.

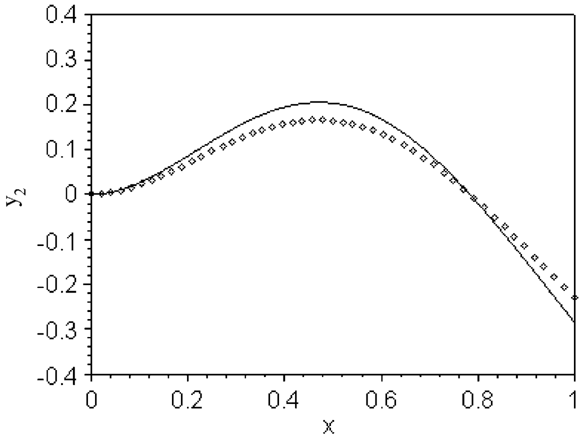

Fig. 6. The linear (——) and non-linear (◇ ◇ ◇) mode shapes for the second mode using Eq. (78).

5. Conclusions

In this paper, using the perturbation method, that is, the method of multiple scales, the non-linear mode shapes and natural frequencies of the one-dimensional continuous damped systems are formulated. One of the advantages of using this technique is the ability to reveal the inherent physics of the problem and parametric formulations for amplitude of vibrations, natural frequencies and mode shapes are found. This allows the time dependency of the vibration amplitudes and the non-linear natural frequencies of damped systems to be demonstrated. A better controllability of the system will thus be possible. There are a one-to-one and a three-to-one internal resonances which are not dependent on damping, i.e., existence of inertia and stiffness non-linearities can cause these internal resonances. Here, when there is a combination of non-linear inertia, stiffness and damping, it is seen that non-linear natural frequencies are logarithmically dependent on time. Finally, the solution is such that it can be applied to any combination of non-linear terms of inertia, damping and stiffness for any boundary conditions of non-linear beams. Two applications in the form of non-linear hinged-hinged beams and non-linear cantilever beams are considered and the non-linear mode shapes and natural frequencies of the beams are obtained. The applications not only compare the solution method offered in this paper with other papers, but they also show, using the formulations of mode shapes and natural frequencies, how easily a non-linear problem can be solved with simple integrations. This method may also be extended to two-dimensional systems.

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Appendix A. Nomenclature

a	amplitude of vibration in the polar system
A	amplitude of vibration
c	damping coefficient
g_{ijk}	coefficient
L	linear spatial operator
α, γ, λ	constant coefficients
ω_i	natural frequencies
N	non-linear spatial operator
p, q	discretized mode shapes dependent on position and time.
t	time
x	position
ε	perturbation coefficient
ζ	damping factor

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