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Vibration control and electricity generating device using a number of hula-hoops and generators

Y. Yoshitake[a,*], T. Ishibashi[a], A. Fukushima[b]

[a] *Department of Structural Engineering, Nagasaki University, 1-14 Bunkyoumachi, Nagasaki 852-8521, Japan*
[b] *Namura Ship-building Co. Ltd, 5-1 Shioya Kurokawachou, Imari, Saga 848-0121, Japan*

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Abstract

This paper deals with the vibration control device which simultaneously generates electricity by using absorbed vibration energy. This device consists of a hula-hoop and a generator. In this paper, we researched the case of using a number of devices. As the result of the experiment and the numerical calculation, the following points were made clear: (1) When two devices are used, the component of the centrifugal forces which are perpendicular to the direction of the external force are cancelled by rotating two hula-hoops in opposite directions. Moreover, the optimum quenching according to the amplitude of the external force is possible by varying the number of rotating hula-hoops. (2) The operating curve was shown by composing the resonance curves when the hula-hoops rotate and those when they do not. (3) The amount of generated electric power is proportional to the second power of the external frequency. (4) The experimental and the numerical results were in good qualitative agreement with each other.

1. Introduction

Many dynamic absorbers have been widely used as equipment to control the vibrations of machines and structures [1–3], for example, application of impact dampers to a highway lighting pole [4], vibration control of a ropeway carrier by dynamic absorber [5], vibration control of multi-degree-of-freedom systems by dynamic absorbers [6] and so on. However, all the absorbed energy is thrown away as heat. Moreover, many active dampers [7,8] have been used in machines and structures, for example, tendon control in tall structures [9], application of the active mass damper to an elevator [10], vibration control in a tower structure by a two-dimensional active

*Corresponding author. Tel.: +81-95-819-2589; fax: +81-95-819-2602.
E-mail address: yoshitak@st.nagasaki-u.ac.jp (Y. Yoshitake).

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mass damper [11] and so on. Yet, the external energy is needed in the active dampers. A new vibration control device that utilizes this useless vibration energy is expected.

The authors have researched the vibration control and electricity generating device consisting of a hula-hoop and a generator [12,13]. This device uses the principle of a synchronous rotation of a hula-hoop. The hula-hoop rotates the generator and electricity is generated. In the quenching problem of self-excited vibration [12], it is realized that vibration quenching and generation are possible using the chaotic motion of a hula-hoop. In the quenching problem of forced vibration [13], the amplitudes near the resonance are well controlled by the steady rotation of the hula-hoop. It is thought that if this device is put into practical use, the electrical energy will be able to be used for various purposes, and energy conservation will be realized.

However, when a single hula-hoop is used, there is a possibility that the vibration perpendicular to the excitation direction occurs. Concerning the application of this device to a real system, this vibration must be quenched. Therefore, in this paper, to quench this vibration and to realize the great effects of vibration control according to the amplitude of the external force, the problem of vibration quenching using multiple devices is performed.

2. Background

2.1. Experimental apparatus

The experiment was carried out in the case of attaching four vibration control and electricity generation devices on the forced vibration system. The experimental apparatus is shown in Fig. 1. The main system is the single-degree-of-freedom system which consists of a mass and four plate springs. These are made of aluminum. A single device consists of a hula-hoop which is made of aluminum and a generator. In this paper, hula-hoops 1, 2, 3 and 4 are considered the same and all of the generators are also considered the same. Two devices, where the line connecting those devices becomes perpendicular to the excitation direction, are treated as a set and are attached to the main system. In Fig. 1, hula-hoops 1 and 2 are considered one set, and hula-hoops 3 and 4 another set. There is no difference between the two sets. When the two hula-hoops are rotating, i.e., when the set of hula-hoops of 1 and 2, or the set of hula-hoops of 3 and 4 rotates in mutually

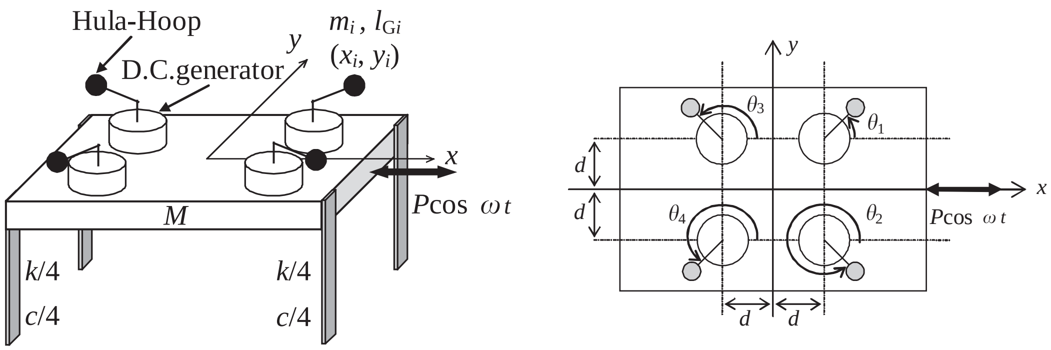


Fig. 1. Experimental apparatus.

opposite directions, each component perpendicular to the excitation direction of the centrifugal force has the same magnitude but opposite direction. Therefore, these component forces counteract each other and only the component forces in the direction of the external force remain. As a result, the vibration perpendicular to the external force and any rotation do not occur in the main system and the vibration of the main system is controlled effectively.

Mass M, natural frequency f_n, and viscous coefficient c of the main system measured by the experiment were as follows:

$$M = 664.5\ \text{g}, \quad f_n = 16.2\ \text{Hz}, \quad c = 4.0 \times 10^{-3}\ \text{N s/mm}.$$

Here, the mass of the DC generators is included in the mass M of the main system.

The mass of a hula-hoop is m_i, stem length is l_{Gi}, the inertia moment of a hula-hoop about the axis of the center of gravity is I_{Gi}. These parameters are as follows:

$$m_i = 5.0\ \text{g}, \quad l_{Gi} = 13.2\ \text{mm}, \quad I_{Gi} = 6.293 \times 10^2\ \text{g mm}^2.$$

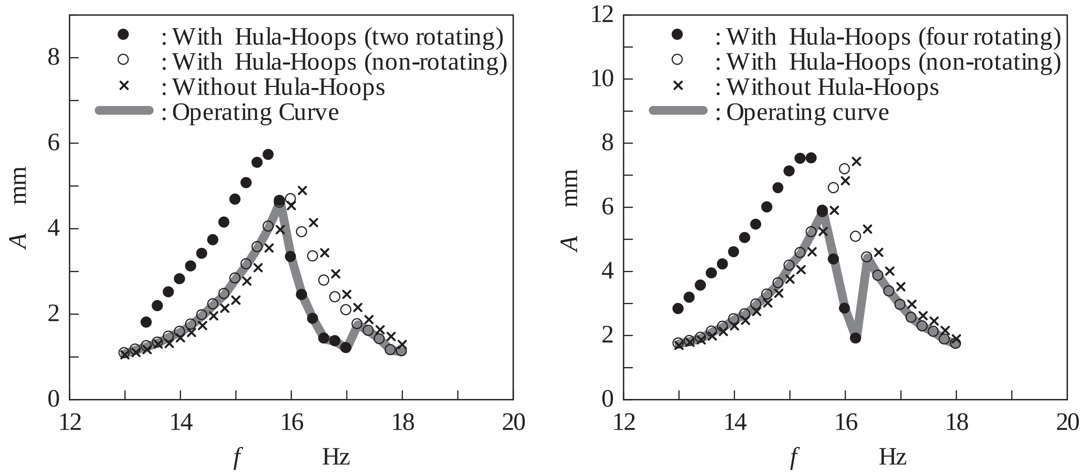
In the experiment, characteristics of the motor used as a power generator are as follows: DC motor RE-140 by Mabuchi Motor Company, rated rotational speed 8000 r.p.m., output 0.84 W.

When the main system is vibrated by the external force, if hula-hoops are given initial velocities, those can rotate synchronizing with the frequency of the external force, and the devices can generate power, while the vibration of the main system is controlled by the absorbed vibration energy.

In this paper, the experiment was carried out in the cases that the amplitude of external force P is 2 and 3 N. The vibration perpendicular to the external force and the rotation of the main system were not observed in the experiment.

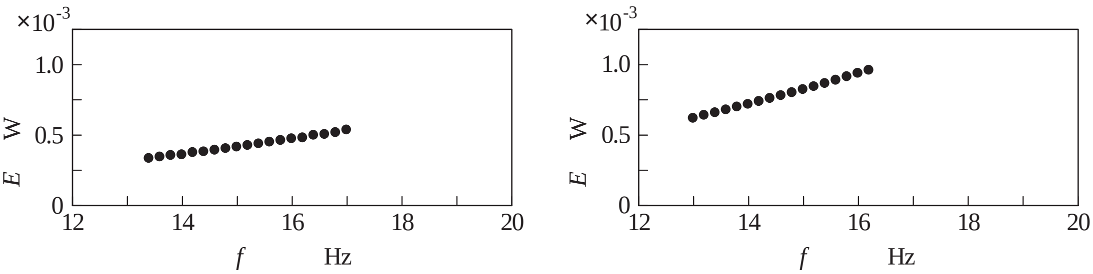
2.2. Resonance curve

The resonance curve obtained from the experiment is shown in Fig. 2. Here, the ordinate A is the displacement amplitude of the main system, and the abscissa f is the frequency of the external force. When the amplitude of the external force P is 2 N, two of four hula-hoops are rotated by giving initial velocities to one set of hula-hoops, and when P is 3 N, all of the hula-hoops are rotated by giving initial velocities. The symbol $\times$ in the figure shows the amplitude in the case of no hula-hoop. On the other hand, the circles mean that hula-hoops are attached to the main system, the symbol $\bullet$ and $\circ$ show the amplitude when the hula-hoops rotate, and those when the hula-hoops do not rotate, respectively. When initial velocities are not given to the hula-hoops and those do not rotate, the resonance frequency becomes slightly lower than that of the main system without a hula-hoop. This is because the mass of the hula-hoops are added to the main system, and the natural frequency of the main system decreases. On the other hand, when the initial velocities are given to the hula-hoops, they rotate and the resonance frequency decreases to some extent. Also, the amplitude decreases sharply near the resonance of the main system without hula-hoops. Thus, these devices can control the amplitude near the resonance point of the main system well. However, the devices are not so useful in the wide frequency region near the resonance of the main system. Moreover, it is necessary to make hula-hoops standstill when the frequency of the external force is lower than the resonance point of the main system. Therefore,



(a) P=2N (Two Hula-Hoops rotating) (b) P=3N (FourHula-Hoops rotating)

Fig. 2. Resonance curve (experiment).



(a) P=2N (Two Hula-Hoops rotating) (b) P=3N (Four Hula-Hoops rotating)

Fig. 3. Generated electric power (experiment).

the gray solid line of the figure is the desirable resonance curve, i.e., the operating curve. As shown in Figs. 2(a) and (b), it turned out that the devices can control the amplitude of the main system optimally by varying the number of rotating hula-hoops according to the external force.

2.3. Power generation

The amount of power generation obtained from the experiment is shown in Fig. 3. The ordinate is the amount of power generation E(W), and the abscissa is the frequency of the external force. In the experiment, the same resistance of $R_f = 1\ \Omega$ is connected between the terminals of each generator and the voltage V between the terminals is measured. Also, the amount of power generation is calculated using the following equation: $E = V^2/R_f$(W). From Fig. 3, it appears that the amount of power generation is proportional to the second power of the frequency of the external force. The relationship between the generated power and the frequency of the external force will be mentioned theoretically in Section 3.3.

3. Theoretical analysis

3.1. Equation of motion

The model shown in Fig. 1 is analyzed in this section. As the positions of the hula-hoops (x_i, y_i) are the functions of $\theta_{i,}$, these are shown as follows $(i = 1, 2, \ldots, 4)$:

$$x_i = x \pm d + l_{Gi} \cos\theta_i \quad (+\colon\ i = 1, 2, \quad -\colon\ i = 3, 4),$$

$$y_i = \pm d + l_{Gi} \sin\theta_i \quad (+\colon\ i = 1, 3, \quad -\colon\ i = 2, 4),$$

$$\dot{x}_i = \dot{x} - l_{Gi}\dot{\theta}_i \sin\theta_i, \quad \dot{y}_i = l_{Gi}\dot{\theta}_i \cos\theta_i. \tag{1}$$

Here, the number of devices is four.

Kinetic energy T, potential energy V and dissipative energy F are shown as follows:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\sum_{i=1}^{4} m_i(\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{2}\sum_{i=1}^{4} I_{Gi}\dot{\theta}_i^2, \tag{2}$$

$$V = \tfrac{1}{2}kx^2, \tag{3}$$

$$F = \frac{1}{2}c\dot{x}^2 + \frac{1}{2}\sum_{i=1}^{4} c_i\dot{\theta}_i^2. \tag{4}$$

The kinetic energy can be expressed by the following equation:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\sum_{i=1}^{4} m_i(\dot{x}^2 - 2l_{Gi}\dot{x}\dot{\theta}_i \sin\theta_i + l_{Gi}^2\dot{\theta}_i^2) + \frac{1}{2}\sum_{i=1}^{4} I_{Gi}\dot{\theta}_i^2. \tag{5}$$

Using the equation of Lagrange, the equations of motion are shown as follows:

$$\left(M + \sum_{i=1}^{4} m_i\right)\ddot{x} - \sum_{i=1}^{4} m_i(l_{Gi}\dot{\theta}_i^2 \cos\theta_i + l_{Gi}\ddot{\theta}_i \sin\theta_i) + c\dot{x} + kx = P\cos\omega t, \tag{6}$$

$$(I_{Gi} + m_i l_{Gi}^2)\ddot{\theta}_i - m_i l_{Gi}\ddot{x}\sin\theta_i + c_i\dot{\theta}_i = 0 \quad (i = 1, 2, \ldots, 4). \tag{7}$$

3.2. Numerical computation method

In numerical computation, the shooting method is used. An outline to calculate a periodic solution of Eqs. (6) and (7) is shown below. This method is the same as the usual shooting method [14–16] except for dealing with the rotational angles of the hula-hoops.

First, Eqs. (6) and (7) are rewritten to the non-linear simultaneous ordinary differential equations of the first degree, relating with parameters $\mathbf{y} = {}^{\mathrm{t}}(x, \theta_1, \ldots, \theta_4, \dot{x}, \dot{\theta}_1, \ldots, \dot{\theta}_4)$. Here, superscript t expresses a transposition sign. Initial value $\mathbf{y}^0 = {}^{\mathrm{t}}(x(0), \theta_1(0), \ldots, \theta_4(0), \dot{x}(0), \dot{\theta}_1(0), \ldots, \dot{\theta}_4(0))$ at $t = 0$ are assumed, the numerical integration for the equations related to $\mathbf{y}$ is carried out for one period $t = T = 2\pi/\omega$, and the solution is set to $\mathbf{y}^1 = {}^{\mathrm{t}}(x(T), \theta_1(T), \ldots, \theta_4(T), \dot{x}(T), \dot{\theta}_1(T), \ldots,$

$\dot{\theta}_4(T)$). Then, the conditions of a periodic solution will serve as the following equation:

$$\mathbf{y}^1 - \mathbf{y}^0 = 0. \tag{8}$$

Considering that the hula-hoop rotates synchronizing with the vibration of the main system, the parameter $\theta_i(T)$ corresponding to the rotational angle θ_i is replaced by $\theta_i(T) - 2\pi j$ ($i = 1, 2, \ldots, 4$). Here, when the hula-hoop rotates in the positive direction, j is 1. When it rotates in the negative direction, j is -1 and when it does not rotate and vibrate, j is 0. Moreover, considering that $\mathbf{y}^1$ is obtained by carrying out numerical integration from the assumed $\mathbf{y}^0$, $\mathbf{y}^1$ is a function of $\mathbf{y}^0$. Therefore, obtaining a periodic solution is equivalent to finding $\mathbf{y}^0$ which satisfies Eq. (8). If the Newton–Raphson method is applied to Eq. (8), the equation for repetition of calculation becomes

$$(\boldsymbol{B} - \boldsymbol{I}_{10})\tilde{\mathbf{y}} = \mathbf{y}_0 - \mathbf{y}_1, \tag{9}$$

where $\boldsymbol{I}_{10}$ is the unit matrix of 10×10, $\tilde{\mathbf{y}}$ is the amount of correction for the next repetition, and $\mathbf{y}^0 + \tilde{\mathbf{y}}$ is the corrected value. The matrix $\boldsymbol{B}$ is constructed by the solutions of variational equations which are calculated from the initial values, i.e., unit vectors which form a unit matrix. When the periodic solution is obtained, the stability of a periodic solution is judged as follows: If all the absolute values of the eigenvalues of matrix $\boldsymbol{B}$ are smaller than a unit when $\mathbf{y}^0$ converges, the solution is judged stable. If at least one of the absolute values is larger than a unit, the solution is judged unstable.

3.3. Characteristic of the generator

The characteristic of a generator was obtained from the experiment. The circuit diagram of a general DC motor is shown in Fig. 4. R_a is the winding resistance of the generator (Ω), V_B is the brush contact potential drop (V), R_f is the resistance connected between the two terminals of the generator, and K_E is the counterelectromotive force constant of the generator (V s/rad).

When a hula-hoop rotates regularly, the following relation between the terminal voltage V and the rotational angular velocity ω (rad/s) holds [17]:

$$V = R_f(K_E\omega - V_B)/(R_a + R_f). \tag{10}$$

The power generated between terminals E(W) is given by the following equation:

$$E = V^2/R_f. \tag{11}$$

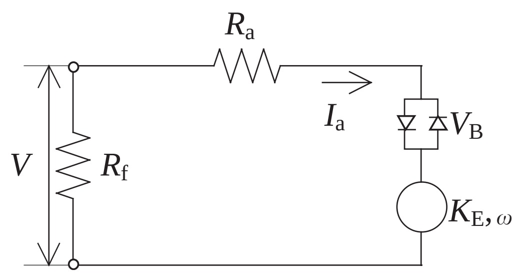


Fig. 4. Circuit of DC generator.

If the brunsh contact potential drop V_B is nearly equal to zero, the terminal voltage is proportional to the rotational speed ω and the generated power is proportional to the second power of it.

Moreover, considering the torque loss T_f by friction, windage loss, and so on, the energy required to rotate a generator is given by the following equation [17]:

$$W_R = T_f\omega + K_E\omega(K_E\omega - V_B)/(R_a + R_f). \tag{12}$$

Therefore, the efficiency of the generator is shown by the following equation from Eqs. (11) and (12):

$$\eta = E/W_R. \tag{13}$$

On the other hand, if all the energies required for power generation are considered as viscous damping energy, the energy defined as E_{ci} for each generator is expressed as follows ($i = 1, 2, \ldots, 4$):

$$E_{ci} = \frac{1}{T}\int_0^T c_i\dot{\theta}_i \, \mathrm{d}\theta_i = \frac{1}{T}\int_0^T c_i\dot{\theta}_i^2 \, \mathrm{d}t. \tag{14}$$

Since it is facile, we express the actually required energy W_R to rotate the generator as viscous damping energy E_{ci}. Then, the following equation is obtained:

$$W_R = E_{ci}. \tag{15}$$

A generator does not necessarily rotate at a constant rotational speed because of the poles of the core of the generator and the number of permanent magnets. Moreover, when the generator is rotated by the hula-hoop, the rotational angular velocity of a generator deviates also during one period by the deviation of the rotational angular velocity of the hula-hoop. In this paper, such a deviation in the rotation of the generator is omitted, and the angular velocity $\dot{\theta}$ is assumed to be equal to ω, the following equation is obtained from Eq. (14):

$$E_c = c_i\omega^2. \tag{16}$$

From Eqs. (15) and (16), the coefficient of viscous damping is given by the following equation:

$$c_i = \frac{W_R}{\omega^2} = \frac{W_R}{(2\pi f)^2}, \tag{17}$$

where f (Hz) is the number of rotations of the generator which is equal to the frequency of the external force. As a result, the sum of the generated power is expressed as follows:

$$E = \sum_{i=1}^{4} \eta_i \frac{1}{T}\int_0^T c_i\dot{\theta}_i^2 \, \mathrm{d}t. \tag{18}$$

Since this study aims to investigate whether the device consisting of a hula-hoop and a generator can control the vibration and generate simultaneously in the regions of the resonance point of a main system or not, the damping caused by generation is treated approximately as mentioned above. The relationship between the rotational frequency f of the generator and the coefficient of viscous damping c_i obtained from Eq. (17) is shown in Fig. 5. Here, the values of

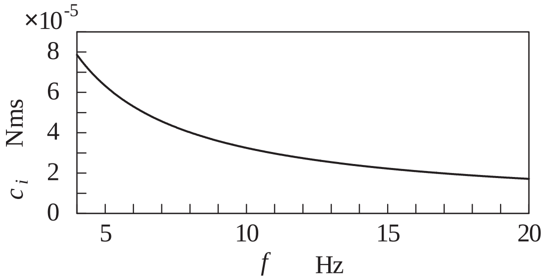


Fig. 5. Damping coefficients of generators.

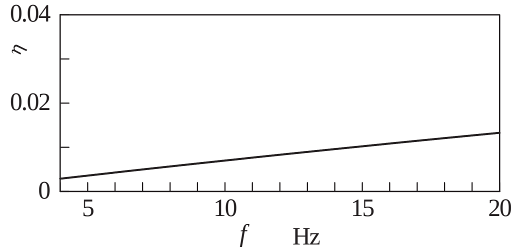


Fig. 6. Efficiencies of generators.

K_E, V_B, R_a, and T_f obtained from the experiment of the generator are as follows [8]:

$$K_E = 1.5 \times 10^{-3}\ \text{V s/rad}, \quad V_B \fallingdotseq 0\ \text{V}, \quad R_a = 1\ \Omega, \quad T_f = 2.9 \times 10^{-4}\ \text{N m},$$

Here, as the torque loss T_f is a function of frequency f, the value at $f = 16.2$ Hz is shown as an example. The efficiency of the generator obtained from Eq. (13) is shown in Fig. 6. Because a high rotation type DC motor is used as the generator, the efficiency is low.

3.4. Resonance curve and bifurcation of solution

The resonance curve obtained from the numerical computation by the shooting method is shown in Fig. 7. The ordinate A is the displacement amplitude of the main system, and the abscissa f is the frequency of the external force. The dashed line in the figure is the solution in the case without a hula-hoop, and the solid thin line is the stable solution in the case of non-rotating hula-hoops. In the case of rotating hula-hoops the solid thick line and the dotted line are, respectively, the stable and unstable solutions: A gray solid line is an operation curve.

Although there are conditions for the initial velocities of hula-hoops to rotate generators, they are not severe. When the displacement amplitude of the main system is larger than some value, the hula-hoops rotate without initial velocities. Therefore, they are moved quite easily by giving small initial velocities. Specifically, in the result of a numerical computation, the initial velocities of the hula-hoop at $f = 16$ Hz in Fig. 7 (a) is $\dot{\theta} = 0.09981$ rad/s.

The general conditions that ensure the solution of the hula-hoops rotating is stable are as follows; the amplitude of the main system must be large, the stem length of a hula-hoop has to be

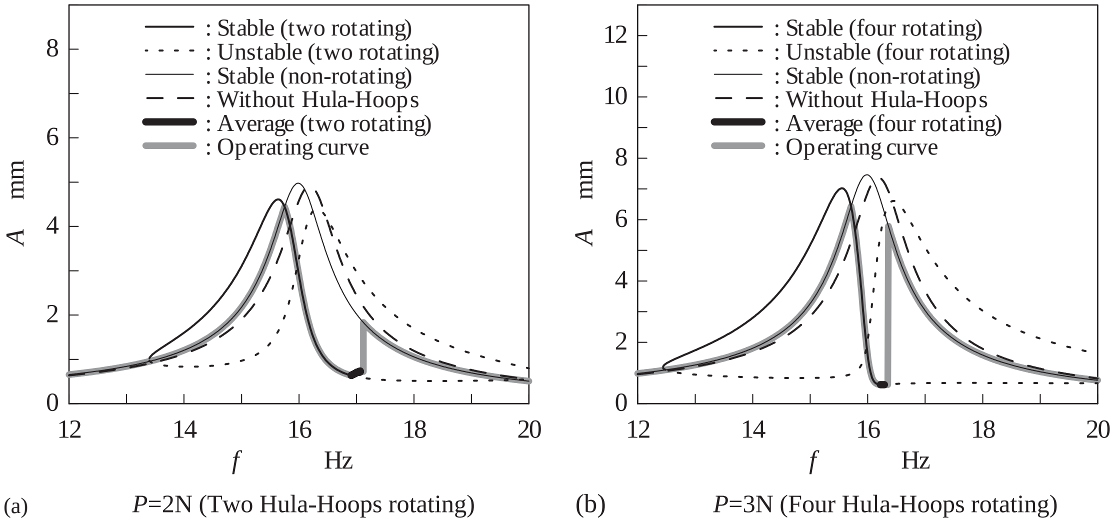


Fig. 7. Resonance curve (calculation).

short, the mass of a hula-hoop is light, and the viscous damping coefficient c_2 of a generator is small. However, a small viscous damping coefficient means less power generation.

Figs. 7(a) and (b) show that these devices control the amplitude near the resonance point of the main system. This result is well in agreement with the experimental result of Fig. 2 qualitatively. It is the strong point of this device that although the vibration control region is narrow, this device can generate power as described in the following section.

When the external force P is 2 N and two hula-hoops rotate, the hula-hoops can rotate to the higher frequency than in the case in which P is 3 N and the four hula-hoops rotate. This numerical computational result is in good agreement with the experimental result. On the other hand, the amplitude of the main system at the resonance is controlled only to about one half, when P is 2 N.

Moreover, in the numerical calculation, the solution that hula-hoops rotate without synchronizing with the external force was found. The bifurcation diagram that a stable periodic solution shifts to unstable in the case of $P = 2$ N is shown in Fig. 8. The Poincare sections of the hula-hoop at $f = 17.0$ and 17.05 Hz are shown in Figs. 9(a) and (b), respectively. In Fig. 8, the ordinate is the non-dimensional angular velocity of a hula-hoop at the moment that the phase of the external force is zero, and the abscissa f is the frequency of the external force. The ordinate and the abscissa in Fig. 9 are the non-dimensional angular velocity and angle of the hula-hoop, respectively, at the moment that the phase of external force is zero. The absolute values of the characteristic multiplier at $f = 16.91$ Hz are 1 in complex. The maximum Lyapunov exponent keeps zero until $f = 17.04$ Hz, and that at $f = 17.05$ Hz is 0.007. Thus, in the numerical calculation, if the frequency of the external force is increased, the periodic solution becomes aperiodic solution at $f = 17.04$ Hz by Hopf bifurcation, and it changes to chaos and finally the hula-hoops stop. Yet, such bifurcation was not able to be made definite in the experiment. When the solutions are the almost periodic motion or chaos, the averaged amplitudes for 100 periods of external force are shown as the solid thickest line in Fig. 7.

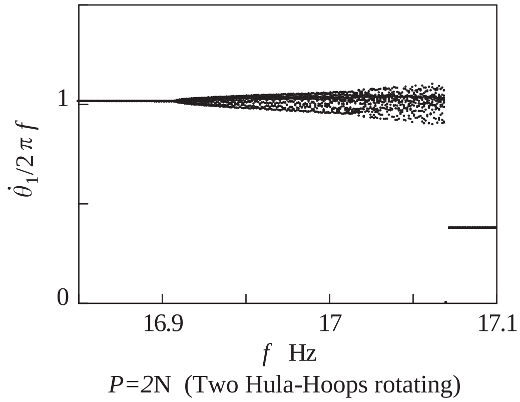


Fig. 8. Bifurcation diagram (calculation).

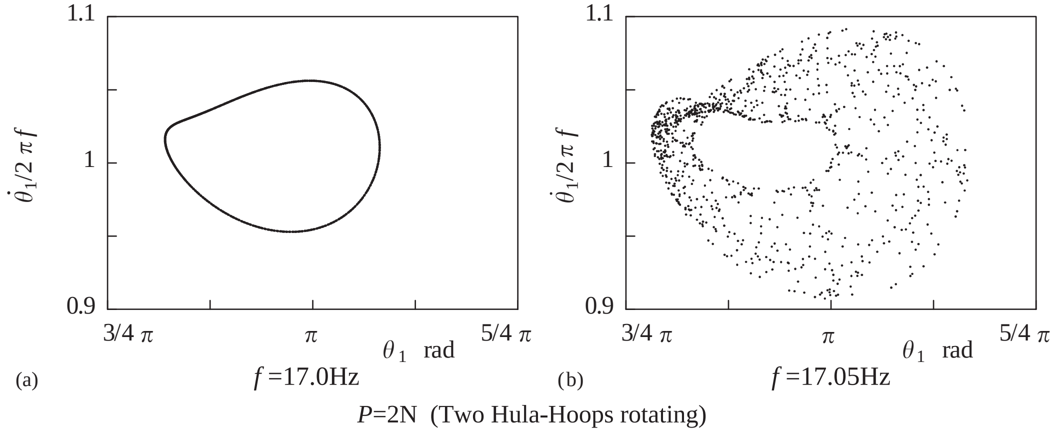


Fig. 9. Poincare section (calculation).

3.5. Generation of electricity

The amount of power generation calculated numerically is shown in Fig. 10. This amount of power generation is calculated using Eq. (18). Namely, the numerically integrated value is multiplied by the efficiency of the generator. Moreover, in the situation of chaos, the averaged amounts of power generation for 100 periods of external force are adopted. In the figure, it is made clear that the generated power is proportional to the second power of the external frequency shown as Eq. (16). These results for $P = 2$ and 3 N are in good qualitative agreement with the experimental results shown in Fig. 3.

Regarding the resonant point, as the amplitude of the main system at the point is large, the hula-hoops rotate easily, and the large capacity generator can be rotated at the point.

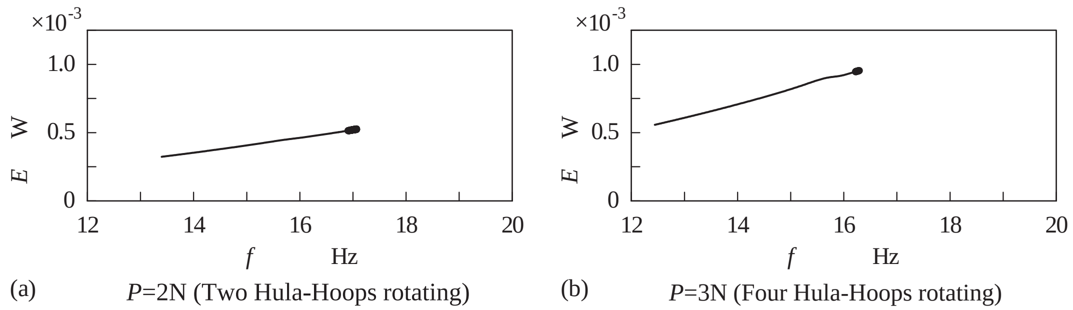


Fig. 10. Generated electric power (calculation).

4. Conclusion

From the experiment and the numerical calculation, the following were made clear:

(1) When a number of devices are used, the optimum quenching according to the amplitude of the external force is possible by varying the number of rotating hula-hoops. Moreover, the component of the centrifugal forces which are perpendicular to the direction of the external force are cancelled by a set of devices.
(2) The operating curve was shown by composing the resonance curve of the rotating hula-hoops and that of non-rotating hula-hoops.
(3) Generated electric power is proportional to the second power of the external frequency.
(4) The experimental and the numerical results were in good qualitative agreement with each other.

References

[1] J.P. Den Hartog, *Mechanical Vibrations*, McGraw-Hill, New York, 1956.
[2] H.B. Hunt, *Dynamic Vibrations*, Mechanical Engineering Publications Ltd, London, 1979.
[3] B.K. Korenev, L.M. Reznikov, *Dynamic Vibration Absorbers*, Wiley, New York, 1993.
[4] I. Jo, T. Kaneko, S. Nagatsu, C. Takahashi, M. Kimura, Development of highway lighting pole with resistance to wind vortex-induced oscillations, Technical Report of Kawasaki Steel 20-4, 1988, pp. 308–314 (in Japanese).
[5] H. Matsuhisa, R. Gu, Y. Wang, O. Nishihara, S. Sato, Vibration control of a ropeway carrier by passive dynamic vibration absorber, *JSME International Journal, Series C* 38 (4) (1995) 657–662.
[6] K. Seto, K. Iwanami, Y. Takita, Vibration control of multi-degree-of-freedom systems by dynamic absorbers: 1st report, on the design method for dynamic absorbers, *Transactions of JSME* 50 (458) (1984) 1962–1969 (in Japanese).
[7] D. Karnopp, M.J. Crosby, R.A. Harwood, Vibration control using semi-active force generators, *Journal of Engineering for Industry, Transactions of American Society of Mechanical Engineers Series B* 96 (4) (1974) 619–626.
[8] H.H.E. Leipholz, M. Abdel-Rohman, *Control of Structures*, Martinus Nijhoff, Dordrecht, 1986.
[9] J. Roorda, Tendon control in tall structures, *American Society of Civil Engineers, Journal of Structural Division* 101 (ST3) (1975) 505–521.
[10] Y. Okada, R. Okashita, Adaptive control of an active mass damper to reduce structural vibration, *JSME International Journal* 33 (3) (1990) 435–440.

[11] K. Seto, Y. Takita, A. Nagamatu, Vibration control in a tower structure by a two-dimensional active mass damper with an LQ control system, *Transactions of JSME* 54 (501) (1988) 1084–1089 (in Japanese).
[12] Y. Yoshitake, A. Sueoka, T. Moriyama, M. Yamasaki, Quenching of self-excited vibrations and generation of electricity by using a hula-hoop, *Transactions of JSME* 66 (646) (2000) 1785–1792 (in Japanese).
[13] Y. Yoshitake, A. Sueoka, A. Fukushima, K. Yamawaki, H. Akamine, Research on vibration control and electricity generating device: control of a forced single-degree-of-freedom vibrating system, *Transactions of JSME* 66 (650) (2000) 3233–3241 (in Japanese).
[14] T.J.J.R. Aprille, T.N. Trick, A computer algorithm to determine the steady-state response of nonlinear oscillators, on circuit theory, *IEEE Transactions of Circuit Theory* 19 (4) (1972) 354–360.
[15] M. Oestreich, N. Hinrichs, K. Popp, Bifurcation and stability analysis for a non-smooth friction oscillator, *Archive of Applied Mechanics* 66 (1996) 301–314.
[16] Y. Yoshitake, A. Sueoka, H. Tamura, T. Hai, Direct numerical integral method to determine periodic solution of nonlinear systems: equations of motion with discontinuous functions, *Transactions of JSME* 59 (561) (1993) 1428–1435 (in Japanese).
[17] A. Kusko, *Solid-State DC Motor Drives*, MIT Press, Cambridge, MA, 1969.