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Using delayed state feedback to stabilize periodic motions of an oscillator

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Abstract

The paper states how to stabilize the periodic motions of a linear, non-positively damped system of single degree of freedom through the use of delayed state feedback. The study indicates that the delayed state feedback works in certain frequency ranges. For a linear undamped system of single degree of freedom, the delayed displacement feedback is able to stabilize almost all periodic motions of the system provided that their fundamental frequencies are higher than the natural frequency of the system, but it only works in a series of narrower and narrower frequency bands lower than the natural frequency. The introduction of delayed velocity feedback can remarkably enlarge the working frequency ranges of delayed displacement feedback. However, even the delayed state feedback cannot stabilize a periodic motion if the corresponding period is an integral multiple of the natural period of system.

The criteria of stability switches prove to be a powerful tool to analyze the stabilization problem of a linear system. However, the stabilization of a linear undamped system of single degree of freedom with delayed velocity feedback only is a degenerate case where the available criteria of stability switches fail to offer any useful information. A detailed study in the paper reveals the complexity of the degenerated case, where the stabilization conditions can be identified according to the second order derivative of the real part of an arbitrary characteristic root.

1. Introduction

When a dynamic system is subject to a periodic excitation, a periodic motion may come into being, asymptotically stable, critically stable or unstable. In engineering, it is usually required to stabilize an unstable periodic motion or a critically stable periodic motion by using proper

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control. The stabilization problem has a fundamental importance in engineering, and hence the stabilization of periodic motions of dynamic systems has drawn much attention over the past decades. Since the seminal idea proposed by Ott et al. [1], many studies have been made on directing the chaotic motion of non-linear dynamic system to an unstable motion embedded in the strange attractor such that the chaotic motion can be utilized. Among the current techniques of stabilization, the delayed feedback proposed by Pyragas looks very simple, but works effectively [2]. This technique has drawn much attention and found various applications [3,4].

When any delayed feedback is introduced, even a linear system of single degree of freedom becomes a dynamic system of infinite dimensions, and the corresponding stability analysis is much more complicated. Hence, it is far from an easy task to stabilize the periodic motion of a linear system through the use of delayed feedback. In a recent letter [5], Le-Ngoc discussed how to determine the stability chart on the plane spanned by time delay and displacement feedback gain for a linear system of single degree of freedom. Krodkiewski and Faragher tried to determine the gains of delayed state feedback numerically and applied their approach to stabilizing the periodic motion of helicopter rotor blades [6,7]. In practice, however, their approach may not always work. In fact, the mechanism of delayed control has not yet been fully understood even for a linear dynamic system of single degree of freedom. Furthermore, the current studies on the stability of delayed dynamic systems do not reveal any concise relations that engineers may be interested in. For example, for a linear dynamic system, the current stabilization conditions do not provide any relations in terms of the physical parameters of the system.

The objective of this study is to gain an insight into the stabilization problem of a linear non-positive damping system of single degree of freedom by using delayed state feedback in order to improve the stability of its periodic motions. The study focuses on the analytic conditions of stabilization on the basis of theory of stability switches of delayed dynamic systems [8,9], and gives some concise conditions for the stabilization. The peculiarity of the study is that all the stabilization conditions are given in terms of the physical parameters of the system.

2. A stabilization problem of delayed state feedback

The study begins with a linear non-positive damping system of single degree of freedom subject to the excitation of period T as follows:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t), \qquad f(t) = f(t+T), \quad T > 0, \tag{1}$$

where $m > 0$, $k > 0$ and $c \leqslant 0$. As well known from the theory of linear vibration, the motion $x(t)$ of the system includes two parts. One is the periodic motion $\bar{x}(t)$ of period T owing to the periodic excitation $f(t)$. The other is a kind of non-decaying motion since $c \leqslant 0$ holds. Thus, the periodic motion $\bar{x}(t)$ is not asymptotically stable and the motion $x(t)$ of system does not settle down to the periodic motion $\bar{x}(t)$. To improve the asymptotical stability of $\bar{x}(t)$, a state feedback with time delay T is introduced so that Eq. (1) reads

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) + u[x(t) - x(t-T)] + v[\dot{x}(t) - \dot{x}(t-T)], \tag{2}$$

where $u, v \in R^1$ are feedback gains of displacement and velocity, respectively. It is obvious that the delayed state feedback does not affect $\bar{x}(t)$, but the disturbance around $\bar{x}(t)$ only. That is, the control force appears only when the motion of system deviates from the periodic motion $\bar{x}(t)$.

The disturbance $\Delta x(t)$ near the periodic motion $\bar{x}(t)$ yields a linear delay differential equation as follows:

$$m\Delta\ddot{x}(t) + c\Delta\dot{x}(t) + k\Delta x(t) = u[\Delta x(t) - \Delta x(t-T)] + v[\Delta\dot{x}(t) - \Delta\dot{x}(t-T)]. \tag{3}$$

The stability of Eq. (3) is governed by the eigenvalue problem of following quasi-polynomial

$$D(\lambda, T) \equiv m\lambda^2 + (c-v)\lambda + (k-u) + (u + v\lambda)\,\mathrm{e}^{-\lambda T} \equiv P(\lambda) + Q(\lambda)\,\mathrm{e}^{-\lambda T}. \tag{4}$$

To deal with the above eigenvalue problem, let

$$\begin{aligned} &P_R(\omega) \equiv \mathrm{Re}\,P(\mathrm{i}\omega) = k - u - m\omega^2, \quad P_I(\omega) \equiv \mathrm{Im}\,P(\mathrm{i}\omega) = (c-v)\omega, \\ &Q_R(\omega) \equiv \mathrm{Re}\,Q(\mathrm{i}\omega) = u, \quad Q_I(\omega) \equiv \mathrm{Im}\,Q(\mathrm{i}\omega) = v\omega \end{aligned} \tag{5}$$

and define the following polynomial as in Refs. [8,9]:

$$\begin{aligned} F(\omega) &\equiv P_R^2(\omega) + P_I^2(\omega) - Q_R^2(\omega) - Q_I^2(\omega) \\ &= (k - u - m\omega^2)^2 + (c-v)^2\omega^2 - u^2 - v^2\omega^2 \\ &= m^2\omega^4 + [2m(u-k) + (c^2 - 2cv)]\omega^2 + k^2 - 2ku. \end{aligned} \tag{6}$$

As analyzed in Refs. [8,9], the linear delay differential equation (3) does not switch its stability if $F(\omega)$ has no real roots. That is, the periodic motion $\bar{x}(t)$ is not asymptotically stable for any period T. The key to stabilize the periodic motion $\bar{x}(t)$ is to properly design the feedback gains $u, v \in R^1$ such that $F(\omega)$ has at least one simple positive root and the linear delay differential equation (3) gains stability through stability switches [8,9] with variation of the time delay.

For simplicity in mathematics, this study focuses on the case when $c = 0$. In general, the study in the case of $c < 0$ involves no surmounting difficulty by nature. In the case of $c = 0$, the periodic motion $\bar{x}(t)$ of uncontrolled system is critically stable, and the polynomial $F(\omega)$ becomes

$$F(\omega) = m^2\omega^4 + 2m(u-k)\omega^2 + k^2 - 2ku. \tag{7}$$

It has two positive real roots $\omega_{1,2}$ as follows:

$$\omega_{1,2}^2 = \frac{1}{2m^2}\left[2m(k-u) \pm \sqrt{4m^2(u-k)^2 - 4m^2(k^2 - 2ku)}\right] = \frac{1}{m}(k - u \pm |u|), \tag{8}$$

namely,

$$\begin{aligned} &\omega_1^2 = \frac{k-2u}{m}, \quad \omega_2^2 = \frac{k}{m}, \quad u < 0; \\ &\omega_1^2 = \omega_2^2 = \frac{k}{m}, \quad u = 0; \\ &\omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{k-2u}{m}, \quad 0 < u < \frac{k}{2}. \end{aligned} \tag{9}$$

Because these roots are independent of the velocity feedback gain, the following sections deal with the systems with delayed displacement feedback first, and then with those equipped with delayed state feedback and with delayed velocity feedback only, respectively.

3. Delayed displacement feedback

3.1. Negative feedback of delayed displacement ($u<0$, $v=0$)

Given the two positive real roots $\omega_{1,2}$ of $F(\omega)$, as shown in Refs. [8,9], Eq. (3) has following two sets of critical time delays determined from Eq. (4):

$$\tau_{1,2r} = \frac{\theta_{1,2} + 2r\pi}{\omega_{1,2}}, \quad r = 0, 1, 2, \ldots, \tag{10}$$

where

$$\sin\theta_{1,2} = 0, \quad \cos\theta_{1,2} = \frac{m\omega_{1,2}^2 + u - k}{u}. \tag{11}$$

Substituting $\omega_{1,2}^2$ in Eq. (9) when $u<0$ into Eq. (11) gives $\theta_1 = \pi$ and $\theta_2 = 0$ such that

$$\tau_{1r} = (2r+1)\pi\sqrt{\frac{m}{k-2u}}, \qquad \tau_{2r} = 2r\pi\sqrt{\frac{m}{k}}, \; r = 0, 1, 2, \ldots \; . \tag{12}$$

Noting

$$F'(\omega) = 4m\omega(m\omega^2 + u - k), \tag{13}$$

one arrives at

$$F'(\omega_1) = -4m\omega_1 u > 0, \quad F'(\omega_2) = 4m\omega_2 u < 0. \tag{14}$$

These two inequalities imply that a pair of roots of $D(\lambda, T)$ is crossing the imaginary axis from the left complex plane to the right complex plane with an increase of time delay T around τ_{1r}, whereas a pair of roots of $D(\lambda, T)$ comes into the left complex plane from the right complex plane with an increase of time delay T around τ_{2r}. Because the periodic motion of system without the time delay is critically stable since $c = 0$, i.e., $D(\lambda, 0)$ has a pair of pure imaginary roots, it is easy to see that the pure imaginary roots become the roots with negative real parts when $T \in (0, \tau_{10})$ so that the periodic motion of system is asymptotically stable. Eq. (12) indicates that the relation $\tau_{1r+1} - \tau_{1r} < \tau_{2r+1} - \tau_{2r}$ holds. Therefore, $D(\lambda, T)$ must have the roots with positive real parts with an increase of time delay T such that the periodic motion of system becomes unstable at last.

It is obvious that the choice of displacement feedback gain u enables one to reach different rankings of τ_{1r} and τ_{2r}, and to adjust the distribution of roots of $D(\lambda, T)$ as well. To demonstrate this fact, two examples are discussed as following.

The first example is the case when $u = -3k/2$, which leads to the following two sets of critical time delays:

$$\begin{aligned} &\tau_{10} = \frac{\pi}{2}\sqrt{\frac{m}{k}}, \quad \tau_{11} = \frac{3\pi}{2}\sqrt{\frac{m}{k}}, \quad \tau_{12} = \frac{5\pi}{2}\sqrt{\frac{m}{k}}, \quad \tau_{13} = \frac{7\pi}{2}\sqrt{\frac{m}{k}}, \ldots, \\ &\tau_{20} = 0, \quad \tau_{21} = 2\pi\sqrt{\frac{m}{k}}, \quad \tau_{22} = 4\pi\sqrt{\frac{m}{k}}, \quad \tau_{23} = 6\pi\sqrt{\frac{m}{k}}, \ldots \; . \end{aligned} \tag{15}$$

These time delays can be ranked as

$$0 = \tau_{20} < \underline{\tau_{10} < \tau_{11}} < \tau_{21} < \tau_{12} \cdots \; . \tag{16}$$

According to Eqs. (14) and (16), it is easy to find that when $T \in (0, \tau_{10})$, Eq. (4) has a single pair of roots with negative real parts so that the periodic motion of system is asymptotically stable. When T is passing through τ_{10}, a pair of roots with positive real parts emerges because $F'(\omega_1) > 0$ holds. Hence, Eq. (4) has a pair of roots with positive real parts for all $T \in (\tau_{10}, \tau_{11})$. When T is passing through τ_{11}, Eq. (4) gains the second pair of roots with positive real parts when $T \in (\tau_{11}, \tau_{21})$. When T is passing through each value of the critical time delays, the system either reduces or increases a pair of roots with positive real parts, but the number of reduced pairs is less than that of added pairs. As a result, the periodic motion of system is unstable if $T \in (\tau_{10}, +\infty)$. Fig. 1 shows two numerical studies of asymptotically stable motion and unstable motion, respectively.

In the second example, $u = -k/2$ gives rise to two sets of critical time delays

$$\tau_{10} = \frac{\pi}{\sqrt{2}}\sqrt{\frac{m}{k}}, \quad \tau_{11} = \frac{3\pi}{\sqrt{2}}\sqrt{\frac{m}{k}}, \quad \tau_{12} = \frac{5\pi}{\sqrt{2}}\sqrt{\frac{m}{k}}, \quad \tau_{13} = \frac{7\pi}{\sqrt{2}}\sqrt{\frac{m}{k}}, \ldots,$$
$$\tau_{20} = 0, \quad \tau_{21} = 2\pi\sqrt{\frac{m}{k}}, \quad \tau_{22} = 4\pi\sqrt{\frac{m}{k}}, \quad \tau_{23} = 6\pi\sqrt{\frac{m}{k}}, \ldots \,. \tag{17}$$

The following ranking

$$0 = \tau_{20} < \tau_{10} < \tau_{21} < \underline{\tau_{11} < \tau_{12}} < \tau_{22} \cdots \tag{18}$$

indicates that the periodic motion of system is asymptotically stable (see Figs. 2a and c) when $T \in (0, \tau_{10})$ and $T \in (\tau_{21}, \tau_{11})$, and that it is unstable (see Figs. 2b and d) if $T \in (\tau_{10}, \tau_{21})$ and $T \in (\tau_{11}, +\infty)$.

Eq. (12) implies that the larger the displacement feedback gain u, the wider the interval $(0, \tau_{10})$, where the periodic motion is asymptotically stable. The upper limit of the right endpoint of this interval is $\max(\tau_{10}) = \lim_{u \to 0-} \pi\sqrt{m/(k-2u)} = \pi\sqrt{m/k}$. Thus, the real parts of a pair of roots of $D(\lambda, T)$ will approach to zero when $u \to 0-$ provided that $T \in (0, \pi\sqrt{m/k})$ holds.

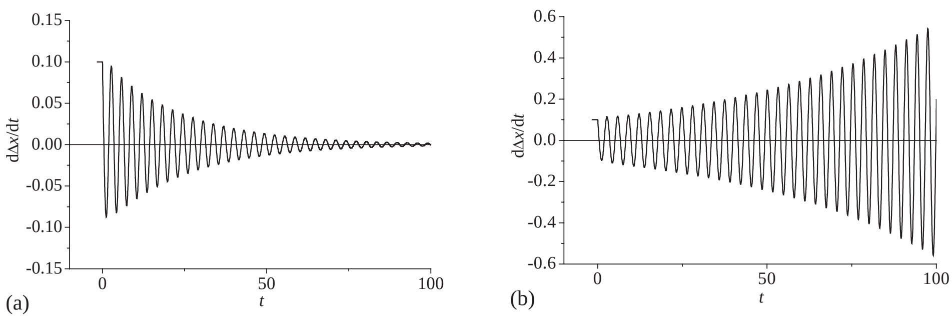


Fig. 1. The velocity variation of a periodic motion under delayed displacement feedback subject to an initial disturbance $\Delta\dot{x}(0) = 0.1$ m/s when $\sqrt{m/k} = 1.0$ s, $u = -1.5$ N/m: (a) $T = 1.5$ s$\in(0, \tau_{10}) \approx (0, 1.571)$ and (b) $T = 1.6$ s$\in(\tau_{10}, \tau_{11}) \approx (1.571, 4.712)$.

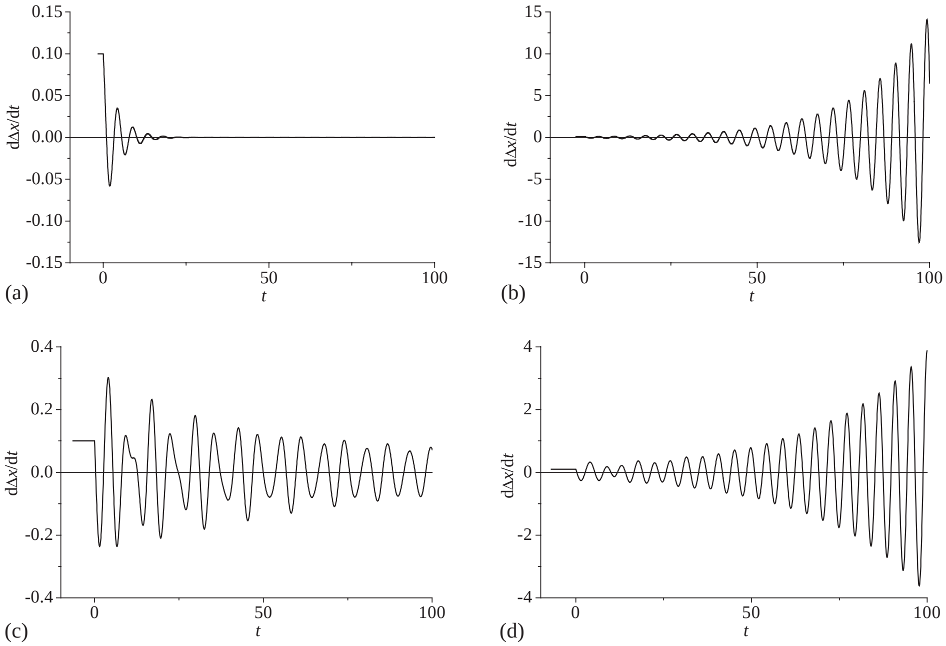


Fig. 2. The velocity variation of a periodic motion under delayed displacement feedback subject to an initial disturbance $\Delta\dot{x}(0) = 0.1$ m/s when $\sqrt{m/k} = 1.0$ s, $u = -0.5$ N/m: (a) $T = 1.5$ s$\in(0, \tau_{10}) \approx (0, 2.221)$, (b) $T = 2.5$ s$\in(\tau_{10}, \tau_{21}) \approx (2.221, 6.283)$, (c) $T = 6.4$ s$\in(\tau_{21}, \tau_{11}) \approx (6.283, 6.664)$ and (d) $T = 7.0$ s$\in(\tau_{11}, \tau_{12}) \approx (6.664, 11.11)$.

3.2. Positive feedback of delayed displacement ($0 < u < k/2$, $v = 0$)

In this case, the roots $\omega_{1,2}$ in Eq. (9) lead to the following two critical time delays:

$$\tau_{1r} = 2r\pi\sqrt{\frac{m}{k}}, \quad \tau_{2r} = (2r+1)\pi\sqrt{\frac{m}{k-2u}}, \quad r = 0, 1, 2, \ldots \; . \tag{19}$$

Eq. (13) gives

$$F'(\omega_1) = 4m\omega_1 u > 0, \quad F'(\omega_2) = -4m\omega_2 u < 0. \tag{20}$$

A pair of roots of $D(\lambda, T)$ comes into the right complex plane from the left complex plane with an increase of time delay T around τ_{1r}, while a pair of roots of $D(\lambda, T)$ comes into the left complex plane from the right complex plane with an increase of time delay T around τ_{2r}. Especially for $T \in (0, \tau_{20})$, $D(\lambda, T)$ gains a pair of roots with positive real parts such that the periodic motion of system is unstable. When $T = \tau_{20}$, this pair of roots comes back to the left complex plane so that the periodic motion becomes asymptotically stable again. Eq. (19) indicates the relation $\tau_{1r+1} - \tau_{1r} < \tau_{2r+1} - \tau_{2r}$. Hence, $D(\lambda, T)$ must have the roots with positive real parts with an increase of time delay T. That is, the periodic motion of system looses the stability at last.

An example of positive feedback of delayed displacement is $u = k/4$, which leads to the two sets of critical time delays

$$\tau_{10} = 0, \quad \tau_{11} = 2\pi\sqrt{\frac{m}{k}}, \quad \tau_{12} = 4\pi\sqrt{\frac{m}{k}}, \quad \tau_{13} = 6\pi\sqrt{\frac{m}{k}}, \ldots,$$
$$\tau_{20} = \sqrt{2}\pi\sqrt{\frac{m}{k}}, \quad \tau_{21} = 3\sqrt{2}\pi\sqrt{\frac{m}{k}}, \quad \tau_{22} = 5\sqrt{2}\pi\sqrt{\frac{m}{k}}, \quad \tau_{23} = 7\sqrt{2}\pi\sqrt{\frac{m}{k}}, \ldots \tag{21}$$

and the following ranking

$$0 = \tau_{10} < \tau_{20} < \underline{\tau_{11} < \tau_{12}} < \tau_{21} < \cdots \ . \tag{22}$$

As a result, the periodic motion of system is asymptotically stable when $T \in (\tau_{20}, \tau_{11})$, and it is unstable when $T \in (0, \tau_{20})$ or $T \in (\tau_{11}, +\infty)$.

Eq. (19) shows that the smaller the displacement feedback gain u, the wider the interval (τ_{20}, τ_{11}), where the periodic motion is asymptotically stable. The lower limit of the left endpoint of this interval yields $\min(\tau_{20}) = \lim_{u\to 0+} \pi\sqrt{m/(k-2u)} = \pi\sqrt{m/k}$. Hence, the negative real parts of a pair of roots of $D(\lambda, T)$ will approach to zero when $u \to 0+$ if $T \in (\pi\sqrt{m/k}, 2\pi\sqrt{m/k})$. Similarly, the positive real parts of a pair of roots of $D(\lambda, T)$ will approach to zero when $u \to 0+$ if $T \in (0, \pi\sqrt{m/k})$.

3.3. Discussions

As analyzed above, the periodic motion of system with $u<0$ is asymptotically stable when $T \in (0, \pi\sqrt{m/(k-2u)}$. The upper limit of the right endpoint of this interval reads $\pi\sqrt{m/k}$. For $0<u<k/2$, the periodic motion of system is asymptotically stable if $T \in (\pi\sqrt{m/(k-2u)}, 2\pi\sqrt{m/k})$. The lower limit of the left endpoint of this interval is $\pi\sqrt{m/k}$. These two facts imply that it is possible to choose proper feedback gain u of delayed displacement to improve the stability of the periodic motion of system when $T \in (0, 2\pi\sqrt{m/k})$ except for the case when $T = \pi\sqrt{m/k}$. In terms of frequency, the natural frequency of system (1) is $f_n = (2\pi)^{-1}\sqrt{k/m}$ and the fundamental frequency f_b of a periodic motion is the reciprocal of the period T, namely, $f_b = 1/T$. It is quite straightforward to conclude that the delayed displacement feedback is able to stabilize almost all periodic motions, except for the periodic motion with the fundamental frequency $f_b = 2f_n$, provided that their fundamental frequencies are higher than the natural frequency of system.

The analysis in Section 3.1 also enables one to stabilize the periodic motion in some frequency bands lower than the natural frequency through proper choice of negative feedback of delayed displacement if $T > 2\pi\sqrt{m/k}$. To demonstrate this fact, one can substitute $u = -\varepsilon k,\ 0<\varepsilon \ll 1$ into Eq. (12) and obtains

$$\tau_{1r} = (2r+1)\pi\sqrt{\frac{m}{k+2\varepsilon k}} = \frac{(2r+1)\pi}{\sqrt{1+2\varepsilon}}\sqrt{\frac{m}{k}}, \qquad \tau_{2r} = 2r\pi\sqrt{\frac{m}{k}}, \quad r = 0, 1, 2, \ldots \ . \tag{23}$$

The ranking of τ_{1r} and τ_{2r} enables one to see that the periodic motion is asymptotically stable when $T \in (\tau_{2r}, \tau_{1r})$ as long as $r < 1/(2\sqrt{1+2\varepsilon}-2)$ holds. For example, $\varepsilon = 0.1$ gives a bound $r < 5.239$. That is, the periodic motion of system is asymptotically stable when $T \in (\tau_{2r}, \tau_{1r})$, $r = 0, 1, 2, 3, 4, 5$. Fig. 3 illustrates an asymptotically stable case and an unstable case, respectively. In those case studies, the absolute value of the displacement feedback gain is very small so that the

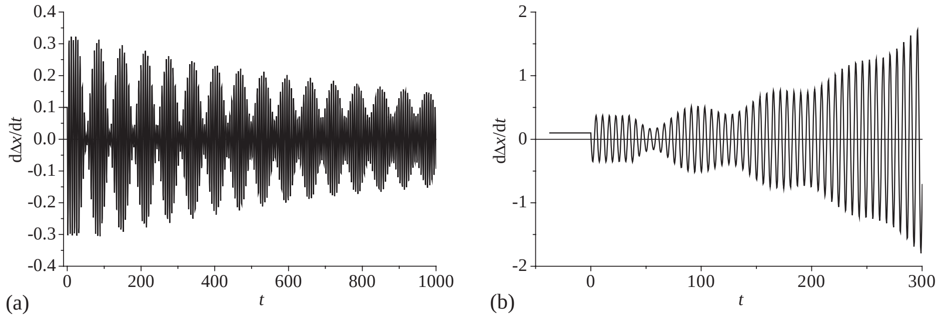


Fig. 3. The velocity variation of a periodic motion under delayed displacement feedback subject to an initial disturbance $\Delta\dot{x}(0) = 0.1$ m/s when $\sqrt{m/k} = 1.0$ s, $u = -0.1$ N/m: (a) $T = 31.5$ s$\in(\tau_{25}, \tau_{15}) \approx (31.42, 31.55)$ and (b) $T = 37.0$ s$\in(\tau_{15}, \tau_{16}) \approx (31.55, 37.28)$.

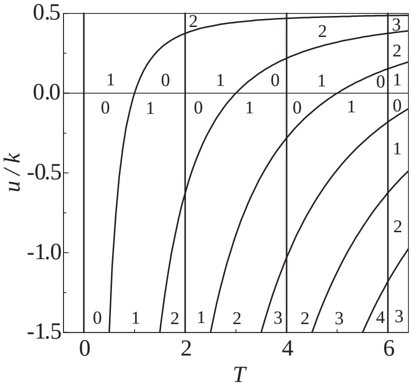


Fig. 4. Stability chart of periodic motions under delayed displacement feedback when $\pi\sqrt{m/k} = 1.0$ s.

disturbance decays or diverges quite slow. That is, the frequency range can only be enlarged at the cost of decreasing the effect of stabilization.

To intuitively show the relation between the period T of a periodic motion and the displacement feedback gain u, it is possible to plot the stability chart as shown in Fig. 4 according to Eqs. (12), (14), (19) and (20). The number in each small region in Fig. 4 represents the pair number of roots of $D(\lambda, T)$, which have positive real parts. In the regions marked by 0, the periodic motion of system can be stabilized. Fig. 4 indicates again that the delayed displacement feedback is able to stabilize almost all periodic motions if their periods satisfy $T \neq r\pi\sqrt{m/k}$, $r = 0, 1, 2, \ldots$.

Furthermore, Fig. 4 shows that a periodic motion undergoes a number of stability switches with an increase of its period T if the absolute value of displacement feedback gain is relatively small. For $u<0$, Eq. (12) gives $\tau_{20} < \tau_{10}$. The periodic motion of system undergoes $2r+1$ stability switches when $\tau_{2r} < \underline{\tau_{1r} < \tau_{1r+1}} < \tau_{2r+1}$ holds with an increase of r. Substituting Eq. (12) into the conditions $\tau_{2r} < \tau_{1r}$ and $\tau_{1r+1} < \tau_{2r+1}$ yields

$$-\frac{(4r+1)k}{8r^2} < u < -\frac{(4r+5)k}{8(r+1)^2}, \quad r = 0, 1, 2, \ldots, \tag{24}$$

which enables the periodic motion of system to undergo $2r+1$ stability switches. Similarly, Eq. (19) gives $\tau_{10}<\tau_{20}$ when $u>0$. With an increase of r, the periodic motion of system undergoes $2(r+1)$ stability switches when $\tau_{1r}<\tau_{2r}<\tau_{1r+1}<\tau_{1r+2}<\tau_{2r+1}$ holds true. The substitution of Eq. (19) into the conditions $\tau_{2r}<\tau_{1r+1}$ and $\tau_{1r+2}<\tau_{2r+1}$ leads to

$$\frac{4r+7}{8(r+2)^2}k<u<\frac{4r+3}{8(r+1)^2}k, \quad r=0,1,2,3,\dots . \tag{25}$$

Under this condition, the periodic motion of system undergoes $2(r+1)$ stability switches. Obviously, no stability switch occurs if $u>3k/8$ holds. That is, the delayed displacement feedback in this case is unable to stabilize any periodic motions.

Given the period T of a periodic motion, the above analysis, with help of those regions marked by 0 in Fig. 4, enables one to choose a proper feedback gain u of delayed displacement to stabilize the periodic motion.

4. Delayed state feedback ($uv\neq 0$)

This section deals with the case when combined feedbacks of delayed displacement and delayed velocity are introduced. If $v\neq 0$, $\theta_{1,2}$ in Eq. (10) yields [8,9]

$$\sin\theta_{1,2}=\frac{v\omega_{1,2}(m\omega_{1,2}^2-k)}{u^2+v^2\omega_{1,2}^2}, \quad \cos\theta_{1,2}=\frac{u(m\omega_{1,2}^2-k+u)+v^2\omega_{1,2}^2}{u^2+v^2\omega_{1,2}^2}. \tag{26}$$

Substituting $\omega_{1,2}^2$ in Eq. (9) into Eq. (26) gives

$$\theta_1=\theta, \quad \theta_2=0, \quad u<0; \quad \theta_1=\theta_2=0, \quad u=0; \quad \theta_1=0, \quad \theta_2=\theta, \quad 0<u<\frac{k}{2}; \tag{27}$$

where θ yields

$$\sin\theta=-\frac{2uv\sqrt{m(k-2u)}}{mu^2+v^2(k-2u)}, \quad \cos\theta=\frac{v^2(k-2u)-mu^2}{mu^2+v^2(k-2u)}. \tag{28}$$

Let θ be a function of state feedback gains (u,v) and denote it by $\theta(u,v)$. It is easy to derive the partial derivatives of $\theta(u,v)$ from Eq. (28)

$$\frac{\partial\theta}{\partial u}=-\frac{4uv^2m(k-u)}{uv\sqrt{m(k-2u)}[mu^2+v^2(k-2u)]}, \quad \frac{\partial\theta}{\partial v}=\frac{4u^2v\sqrt{m(k-2u)}}{uv[mu^2+v^2(k-2u)]}. \tag{29}$$

For $u<0$, Eq. (29) indicates that $\theta(u,v)$ is monotonically decreasing with respect to v. Hence, $\theta(u,v)$ yields $\lim_{v\to-\infty}\theta(u,v)=2\pi$, $\theta(u,-\sqrt{mu^2/(k-2u)})=3\pi/2$, $\theta(u,0)=\pi$, $\theta(u,\sqrt{mu^2/(k-2u)})=\pi/2$ and $\lim_{v\to+\infty}\theta(u,v)=0$. Obviously, the range of $\theta(u,v)$ with a variation of v enables one to enlarge the working range of delayed displacement feedback. For instance, in the case of $T=\pi\sqrt{m/k}$ when the delayed displacement feedback does not work, Eq. (10) gives two sets of critical time delays

$$\tau_{1r}=(\theta+2r\pi)\sqrt{\frac{m}{k-2u}}, \qquad \tau_{2r}=2r\pi\sqrt{\frac{m}{k}}, \quad r=0,1,2,\dots . \tag{30}$$

From Eqs. (30) and (14), it is possible to choose $-k\ll u<0$ and $v<0$ such that $\pi\sqrt{m/k}\in(\tau_{20},\tau_{10})=(0,\theta\sqrt{m/(k-2u)})$ holds and the periodic motion of system is asymptotically

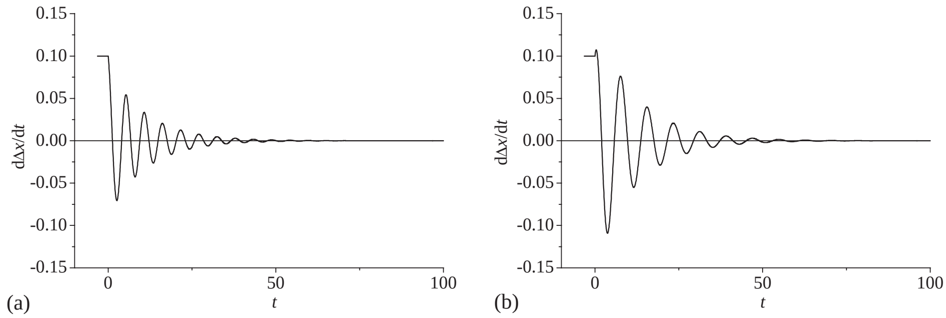


Fig. 5. The velocity variation of a periodic motion under delayed state feedback subject to an initial disturbance $\Delta\dot{x}(0) = 0.1$ m/s when $\sqrt{m/k} = 1.0$ s, $T = 3.142$ s: (a) $u = -0.125$ N/m, $v = -0.112$ N s/m and (b) $u = 0.125$ N, $v = -0.144$ N s/m.

stable. Fig. 5a demonstrates the case when $u = -k/8$ and $v = -\sqrt{mu^2/(k-2u)}$, which result in $\tau_{10} = (3\pi/\sqrt{5})\sqrt{m/k} > \pi\sqrt{m/k}$.

For $0<u<k/2$, Eq. (29) shows that $\theta(u,v)$ is monotonically increasing with respect to v. Thus, $\theta(u,v)$ yields $\lim_{v\to-\infty}\theta(u,v) = 0$, $\theta(u, -\sqrt{mu^2/(k-2u)}) = \pi/2$, $\theta(u,0) = \pi$, $\theta(u, \sqrt{mu^2/(k-2u)}) = 3\pi/2$ and $\lim_{v\to+\infty}\theta(u,v) = 2\pi$. According to the following two sets of critical time delays given by Eq. (10):

$$\tau_{1r} = 2r\pi\sqrt{\frac{m}{k}}, \qquad \tau_{2r} = (\theta + 2r\pi)\sqrt{\frac{m}{k-2u}}, \quad r = 0,1,2,\ldots \tag{31}$$

and Eq. (20), one can choose $0<u\ll k/2$ and $v<0$ such that $\pi\sqrt{m/k}\in(\tau_{20},\tau_{11}) = (\theta\sqrt{m/(k-2u)}, 2\pi\sqrt{m/k})$ holds true and the periodic motion of system is asymptotically stable. Fig. 5b shows the case when $u = k/8$ and $v = -\sqrt{mu^2/(k-2u)}$, which give $\tau_{20} = (\pi/\sqrt{3})\sqrt{m/k}<\pi\sqrt{m/k}$.

Given the velocity feedback gain v, one can first obtain the relation between $\theta(u,v)$ and u from Eqs. (28) and(29), and then substitute it into Eqs. (30) and(31) to derive the relation between τ_{1r} and u, and the relation between τ_{2r} and u. Plotting the curves of τ_{1r} and τ_{2r} with respect to u in Fig. 6, one determines the number of roots of $D(\lambda, T)$ with positive real parts in each small region, and finally obtains the stability charts for different combinations of state feedback gains. Like Fig. 4, the number in each small region in Fig. 6 represents the pair number of roots with positive real parts. In the regions marked by 0, the periodic motion of system can be stabilized. The comparison between Figs. 4 and 6 indicates that the addition of delayed velocity feedback remarkably enlarges the stability regions. For a strong delayed velocity feedback, it is possible to stabilize any periodic motion if its period yields $T\neq 2r\pi\sqrt{m/k}$, $r = 0,1,2,\ldots$ no matter whether the delayed displacement feedback is positive or negative.

5. Delayed velocity feedback ($u = 0$, $v\neq 0$)

Eq. (9) shows that $F(\omega)$ has two repeated roots $\omega_1 = \omega_2 = \sqrt{k/m}$ when $u = 0$. Meanwhile, Eq. (13) gives $F'(\omega_{1,2}) = 0$ in this case. Hence, the system with delayed velocity feedback only is a

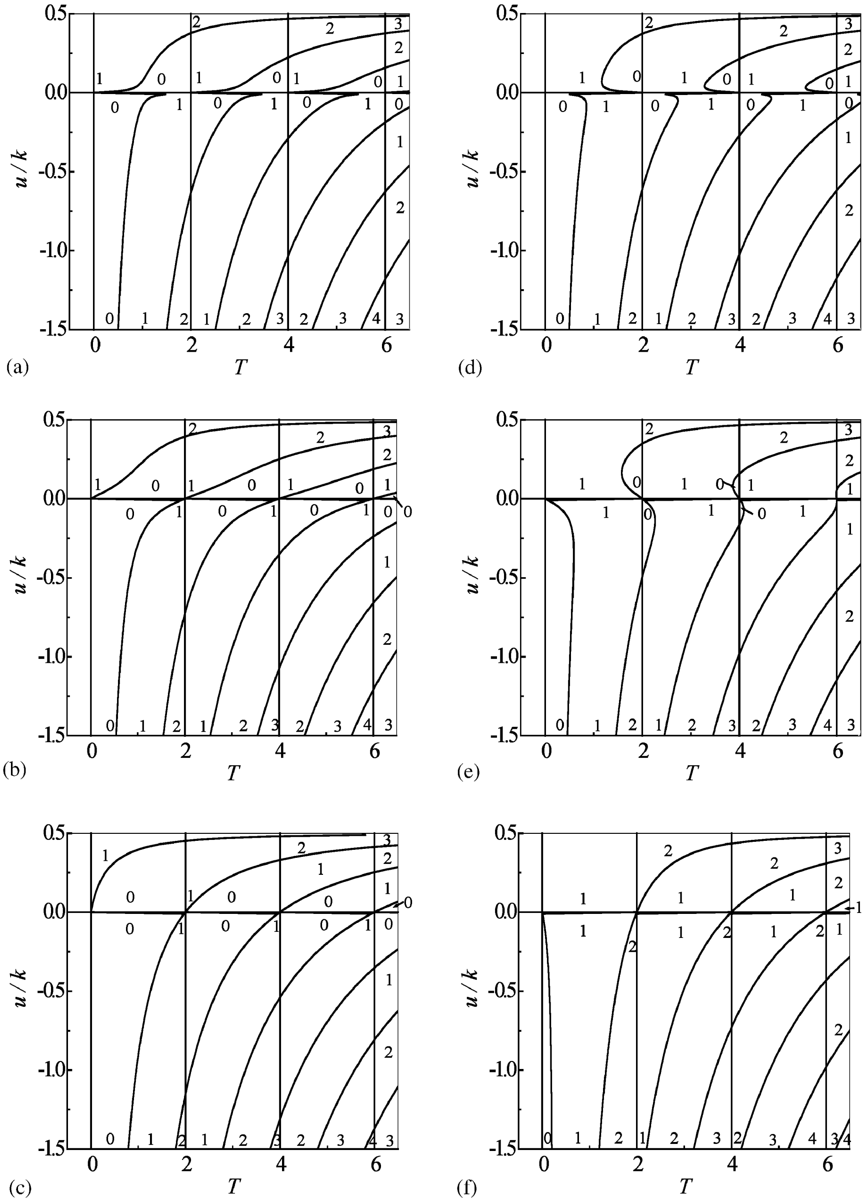


Fig. 6. Stability charts of periodic motions under delayed state feedback when $\pi\sqrt{m/k} = 1.0$ s: (a) $v = -0.01$, (b) $v = -0.1$, (c) $v = -1.0$, (d) $v = 0.01$, (e) $v = 0.1$ and (f) $v = 1.0$.

degenerated case where the method in Refs. [8,9] fails to analyze the stability switches of Eq. (3). For this reason, a direct analysis will be made in this section for the relation between the feedback gain v of delayed velocity and the stabilization condition $\alpha < 0$, with $\lambda = \alpha + i\omega$ being an arbitrary characteristic root of Eq. (4).

Substituting $c = 0$, $u = 0$ and $\lambda = \alpha + i\omega$ into Eq. (4) yields

$$D(\alpha + i\omega, T) = m(\alpha + i\omega)^2 + k + v(\alpha + i\omega)[e^{-(\alpha+i\omega)T} - 1] = 0, \quad (32)$$

where one obtains corresponding real and imaginary parts:

$$m(\alpha^2 - \omega^2) + k - v\alpha + v\alpha e^{-\alpha T}\cos\omega T + v\omega e^{-\alpha T}\sin\omega T = 0,$$
$$2m\alpha\omega - v\omega + v\omega e^{-\alpha T}\cos\omega T - v\alpha e^{-\alpha T}\sin\omega T = 0. \quad (33)$$

Solving Eq. (33) for $e^{-\alpha T}\cos\omega T$ and $e^{-\alpha T}\sin\omega T$, one has

$$e^{-\alpha T}\cos\omega T = -\frac{(m\alpha - v)(\alpha^2 + \omega^2) + k\alpha}{v(\alpha^2 + \omega^2)},$$
$$e^{-\alpha T}\sin\omega T = \frac{\omega[m(\alpha^2 + \omega^2) - k]}{v(\alpha^2 + \omega^2)}. \quad (34)$$

For the critical time delays $T = \tau_r \equiv 2r\pi\sqrt{m/k}$, $r = 0, 1, 2, \ldots$, which are determined from Eqs. (27) and (30), and an arbitrary feedback gain v of delayed velocity, it is easy to verify that $(\alpha, \omega) = (0, \pm\sqrt{k/m})$ are two roots of Eq. (34). That is, Eq. (32) always has a pair of pure imaginary roots in this case. This fact, which coincides with Eq. (9), indicates again that the delayed velocity feedback is unable to stabilize any periodic motion if the period satisfies $T = \tau_r = 2r\pi\sqrt{m/k}$, $r = 0, 1, 2\ldots$.

For $T \neq \tau_r = 2r\pi\sqrt{m/k}$, $r = 0, 1, 2, \ldots$, the possibility of stabilization is analyzed as follows. Eq. (34) can be recast, with help of triangle relations, into

$$G_1(\alpha, \omega) \equiv v^2(\alpha^2 + \omega^2)^2 e^{-2\alpha T} - \{[(m\alpha - v)(\alpha^2 + \omega^2) + k\alpha]^2 + \omega^2[m(\alpha^2 + \omega^2) - k]^2\} = 0,$$
$$G_2(\alpha, \omega) \equiv \tan\omega T - \frac{\omega[k - m(\alpha^2 + \omega^2)]}{(m\alpha - v)(\alpha^2 + \omega^2) + k\alpha} = 0. \quad (35)$$

Furthermore, $G_1(\alpha, \omega) = 0$ can be rewritten as

$$e^{-2\alpha T} = 1 + \frac{a_2 - 2a_1\alpha v}{(\alpha^2 + \omega^2)v^2}, \quad (36)$$

where

$$a_1 \equiv m(\alpha^2 + \omega^2) + k > 0,$$
$$a_2 \equiv m\alpha^2(m\alpha^2 + 2m\omega^2 + 2k) + (k - m\omega^2)^2 \geqslant 0. \quad (37)$$

Eqs. (36) and (37) result in the fact that $\alpha < 0$ holds if $v < 0$. Otherwise, if for any $v^* < 0$ there is an $\alpha^* \geqslant 0$ such that Eq. (36) holds, then either $T = \tau_r = 2r\pi\sqrt{m/k}$ holds for some integer r if $\alpha^* = 0$, or $e^{-2\alpha^* T} < 1$ and $1 + (a_2 - 2a_1\alpha^* v^*)/[(\alpha^*)^2 + \omega^2](v^*)^2 > 1$ for $\alpha^* > 0$. These two cases contradict either $T \neq \tau_r = 2r\pi\sqrt{m/k}$, $r = 0, 1, 2, \ldots$ or $v^* < 0$ and Eq. (37). The above analysis indicates that the stabilization condition $\alpha < 0$ holds for an arbitrary root of Eq. (32) when the negative feedback $v < 0$ of delayed velocity is put into use. However, either $\alpha > 0$ or $\alpha < 0$ may be true when $v > 0$. To illustrate these results, the root (α, ω) of $G_1(\alpha, \omega) = 0$ on the complex plane is shown in Fig. 7 for 6 typical cases.

In what follows, a further analysis needs to be made for the case when $v > 0$. The analysis begins from the special case when the period T is extremely short. As proved in Ref. [8], $|\alpha|$ should be a very small quantity in this case since two roots of Eq. (32) are just slightly perturbed from the pure

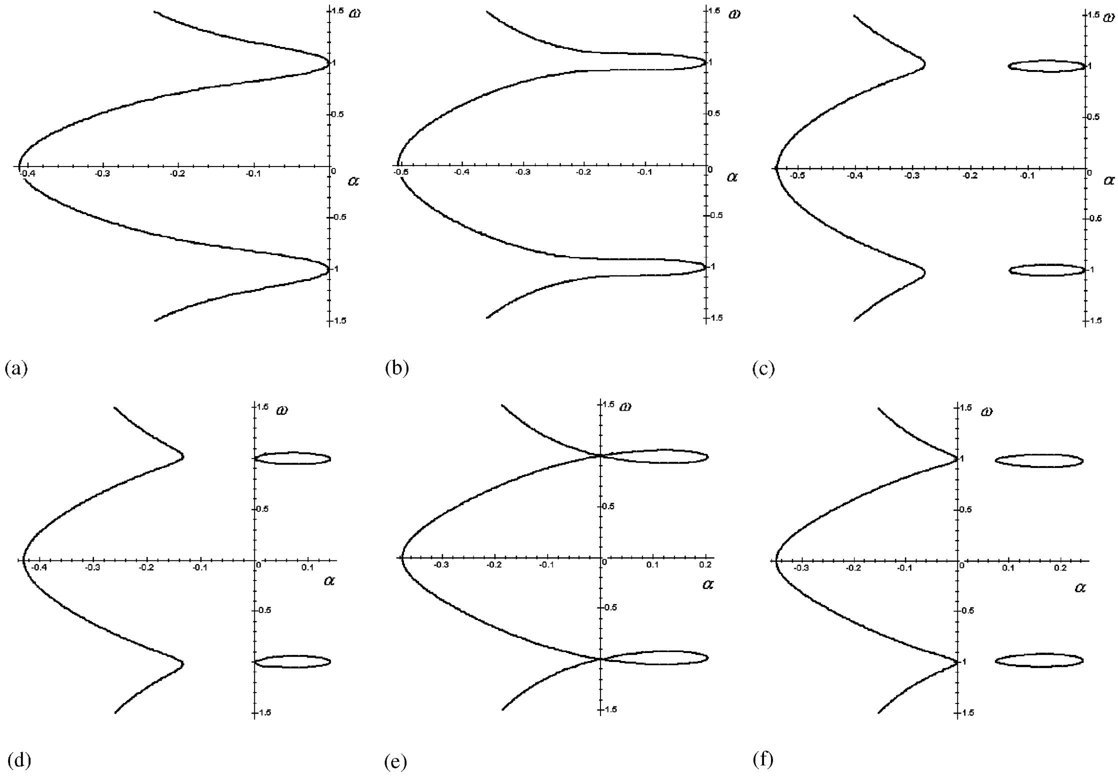


Fig. 7. Root (α, ω) of $G_1(\alpha, \omega) = 0$ on the complex plane for 6 typical values of v when $\sqrt{m/k} = 1.0$ s, $c = 0$, $u = 0$, $T = \tau_1 \approx 6.284$ s: (a) $v = -0.2$, (b) $v = -0.1$, (c) $v = -0.08$, (d) $v = 0.2$, (e) $v = 0.3182 \approx 2m/T$, and (f) $v = 0.4$.

imaginary roots $(\alpha, \omega) = (0, \pm\sqrt{k/m})$. This fact gives rise to $\mathrm{e}^{-2\alpha T} \approx 1$, and $2a_1\alpha v \approx a_0$ follows. Consequently, $\operatorname{sgn} v = \operatorname{sgn} \alpha$ holds. Hence, the stabilization condition $\alpha < 0$ fails to hold for any root of Eq. (32) if $v > 0$ and T is very short. When T increases, the periodic motion keeps unstable provided that T varies in $(0, \tau_1)$, and it may undergo a stability switch only when T is passing through a critical time delay $\tau_r = 2r\pi\sqrt{m/k}$, $r = 1, 2, \ldots$.

To examine the case when T is passing through a critical time delay $\tau_r = 2r\pi\sqrt{m/k}$, $r = 1, 2, \ldots$, let the roots of Eq. (32) be the functions in period T and denote them by $\lambda_i(T)$, $i = 1, 2, \ldots$. Then, there exists at least one root $\lambda_j(T)$ of Eq. (32) such that $\operatorname{Re}\lambda_j(\tau_r) = 0$ holds. Moreover, straightforward computation from Eq. (32) gives

$$\left.\frac{\mathrm{d}(\operatorname{Re}\lambda_j)}{\mathrm{d}T}\right|_{(\omega,T)=(\sqrt{k/m},\tau_r)} = 0, \qquad 2m - v\tau_r = 2(m - r\pi v\sqrt{m/k}) \neq 0, \quad r = 1, 2, \ldots \tag{38}$$

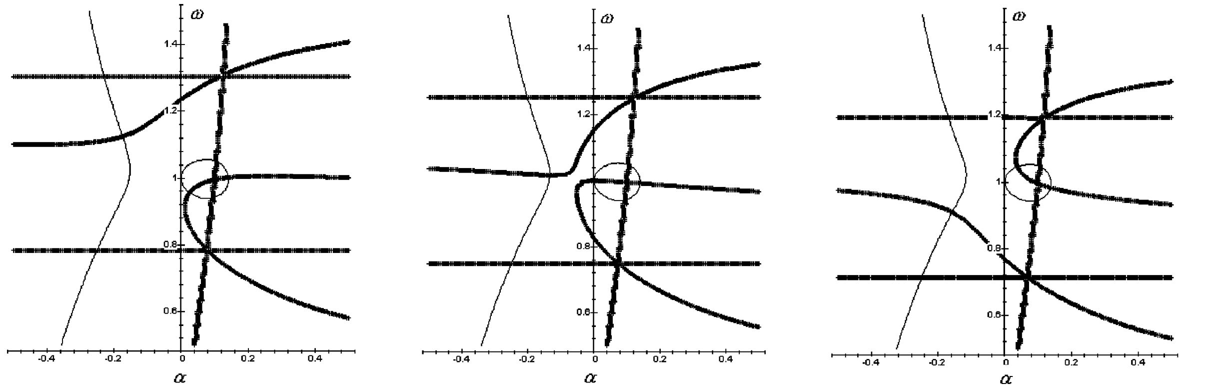
and

$$\left.\frac{\mathrm{d}^2(\operatorname{Re}\lambda_j)}{\mathrm{d}T^2}\right|_{(\omega,T)=(\sqrt{k/m},\tau_r)} = \frac{4mkv}{(2m - v\tau_r)^3} = \frac{mkv}{2(m - r\pi v\sqrt{m/k})^3}, \quad r = 1, 2, \ldots. \tag{39}$$

If $|T - \tau_r|$ is very small, the following Taylor approximation for $\mathrm{Re}\,\lambda_j(T)$ holds

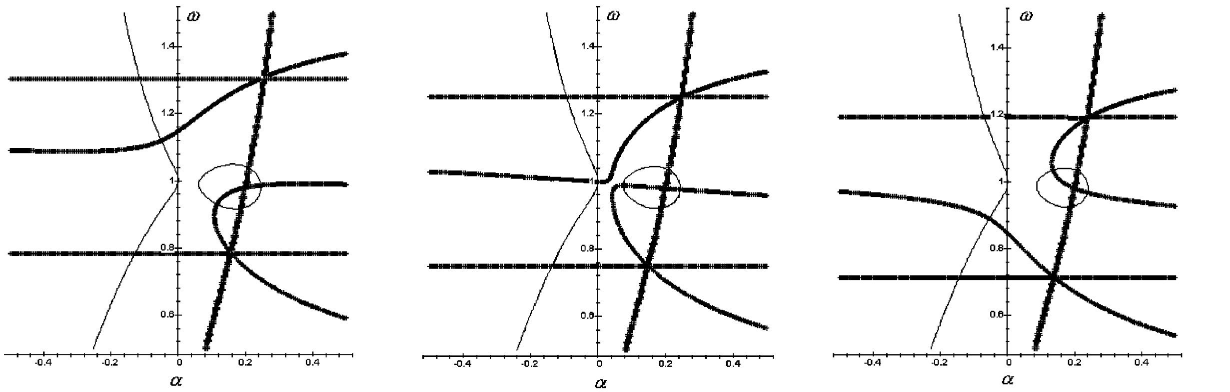
$$\mathrm{Re}\,\lambda_j(T) \approx \frac{4mkv}{(2m - v\tau_r)^3}(T - \tau_r)^2, \quad r = 1, 2, \ldots, \tag{40}$$

provided that $\tau_r \neq 2m/v$. Assume that $\tau_{r^*-1} < 2m/v < \tau_{r^*}$. Then at the neighborhood of the critical time delay τ_r, $r = 0, 1, \ldots, r^* - 1$, the branch of characteristic root $\lambda_j(T)$ given by Eq. (40) satisfies $\mathrm{Re}\,\lambda_j(T) \geqslant 0$. Hence, the branch of $\mathrm{Re}\,\lambda_j(T)$ comes from the right half-complex plane when T approaches any critical time delay τ_r for $r \leqslant r^* - 1$, grazes the imaginary axis when $T = \tau_r$, and returns to the right half-complex plane again when T leaves τ_r. This case is demonstrated in Fig. 8 for three typical time delays near $\tau_1 = \pi$. On the other hand, at the neighborhood of a critical time delay τ_r for $r \geqslant r^*$, the branch of characteristic root $\lambda_j(T)$ always satisfies $\mathrm{Re}\,\lambda_j(T) \leqslant 0$. As shown in



(a) (b) (c)

Fig. 8. Distribution of roots $\lambda = \alpha + \mathrm{i}\omega$, i.e., the intersections of thin curve $G_1(\alpha, \omega) = 0$ and thick curve $G_2(\alpha, \omega) = 0$, near $(0, \sqrt{k/m})$ on the complex plane with a variation of period T around $\tau_1 \approx 6.284$ s, when $\sqrt{m/k} = 1.0$ s, $c = 0$, $u = 0$, $v = 0.2$ N s/m $< 2m/\tau_1$: (a) $T = 6.0$ s, (b) $T = \tau_1 \approx 6.284$ s and (c) $T = 6.6$ s.



(a) (b) (c)

Fig. 9. Distribution of roots $\lambda = \alpha + \mathrm{i}\omega$, i.e., the intersections of thin curve $G_1(\alpha, \omega) = 0$ and thick curve $G_2(\alpha, \omega) = 0$, near $(0, \sqrt{k/m})$ on the complex plane with a variation of period T around $\tau_1 \approx 6.284$ s, when $\sqrt{m/k} = 1.0$ s, $c = 0$, $u = 0$, $v = 0.4$ N s/m $> 2m/\tau_1$: (a) $T = 6.0$ s, (b) $T = \tau_1 \approx 6.284$ s and (c) $T = 6.6$ s.

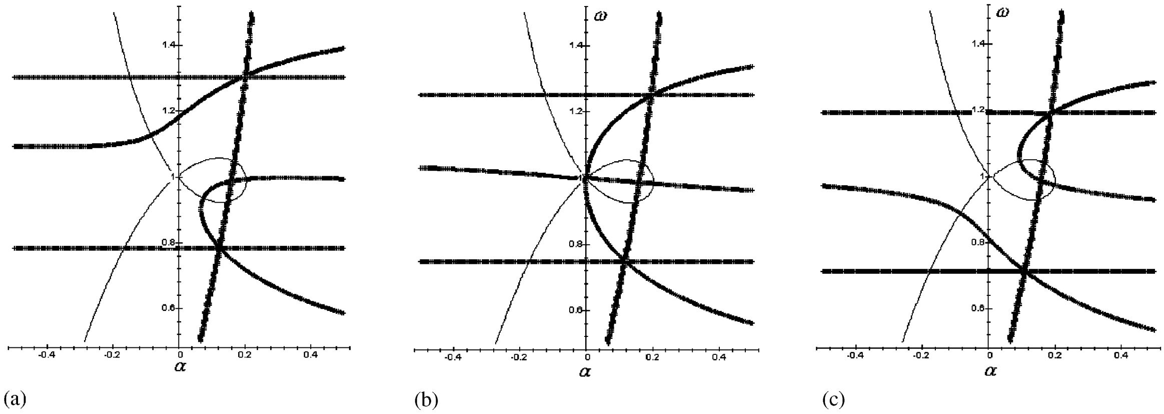


Fig. 10. Distribution of roots $\lambda = \alpha + i\omega$, i.e., the intersections of thin curve $G_1(\alpha,\omega) = 0$ and thick curve $G_2(\alpha,\omega) = 0$, near $(0, \sqrt{k/m})$ on the complex plane with a variation of period T around $\tau_c = 6.284$ s, when $\sqrt{m/k} = 1.0$ s, $c = 0$, $u = 0$, $v = 0.3182$ N s/m $\approx 2m/\tau_c$: (a) $T = 6.0$ s, (b) $T = \tau_c = 6.284$ s and (c) $T = 6.6$ s.

Fig. 9, the branch of $\mathrm{Re}\,\lambda_j(T)$ in this case grazes the imaginary axis from the left half-complex plane when T is passing through τ_r. As a result, if $\tau_r \neq 2m/v$ holds for all $r = 1, 2, \ldots$, the number of characteristic roots, of Eq. (32), with positive parts does not change with an increase of period T from zero to the positive infinity. Therefore, if the positive feedback gain v is chosen not to be $2m/\tau_r$ for any $r = 1, 2, \ldots$, then the periodic motion with period $T \neq \tau_r$ is always unstable.

Now, the final attention is paid to the case when v is chosen to be $2m/\tau_r$ for a specific critical time delay τ_r, denoted by τ_c. In this case, $\mathrm{d}(\mathrm{Re}\,\lambda_j)/\mathrm{d}T$ and $\mathrm{d}^2(\mathrm{Re}\,\lambda_j)/\mathrm{d}T^2$ do not exist at the critical time delay. The singularity of $\mathrm{d}(\mathrm{Re}\,\lambda_j)/\mathrm{d}T$ and $\mathrm{d}^2(\mathrm{Re}\,\lambda_j)/\mathrm{d}T^2$ implies the occurrence of a local bifurcation of $\lambda_j(T)$ or $G_1(\alpha_j, \omega_j) = 0$ with respect to T or v as shown in Fig. 7e. The roots of Eq. (32) near $(\alpha, \omega) = (0, \sqrt{k/m})$ are shown on the complex plane in Fig. 10 with the variation of period T near τ_c, where $\tau_c = 2m/v = \pi$. Obviously, Eq. (32) has two repeated roots $\lambda_i = \lambda_j = \mathrm{i}\sqrt{k/m}$ on the imaginary axis when $T = \tau_c = \pi$ in Fig. 10b. When the period T deviates from τ_c, one of the roots of Eq. (32) leaves the axis for the left half-plane, and the other for the right half-plane, as shown in Figs. 10a and c. In other words, two solution branches of Eq. (32) are getting closer and closer from the left half-plane and the right half-plane, respectively, when the period T approaches to τ_c, and collide each other at $(\alpha, \omega) = (0, \sqrt{k/m})$ when $T = \tau_c$.

In summary, the negative feedback of delayed velocity is able to stabilize the periodic motion provided that the period yields $T \neq \tau_r = 2r\pi\sqrt{m/k}$, $r = 0, 1, 2, \ldots$, while the positive feedback of delayed velocity fails to stabilize any periodic motions.

To demonstrate the above analysis, Fig. 11 gives two case studies, one is asymptotically stable and the other is unstable. Fig. 12 demonstrates the control effect when the periods of periodic motions are relatively long. In Fig. 12a, the period is not any integral multiple of natural period of system so that the periodic motion is stabilized. In Fig. 12b, however, the period is 4 times of 2π, the natural period of system. Hence, the delayed velocity feedback can only keep the critical stability of periodic motion, and fails to stabilize the periodic motion.

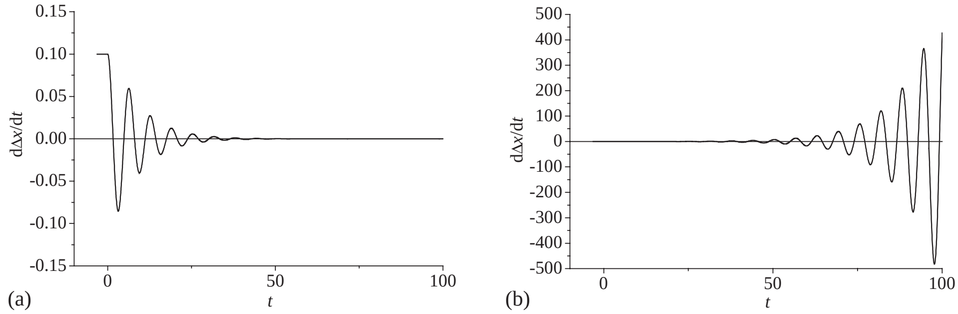


Fig. 11. The velocity variation of a periodic motion under delayed velocity feedback subject to an initial disturbance $\Delta\dot{x}(0) = 0.1$ m/s when $\sqrt{m/k} = 1.0$ s, $T = 3.142$ s: (a) $v = -0.1$ N s/m and (b) $v = 0.1$ N s/m.

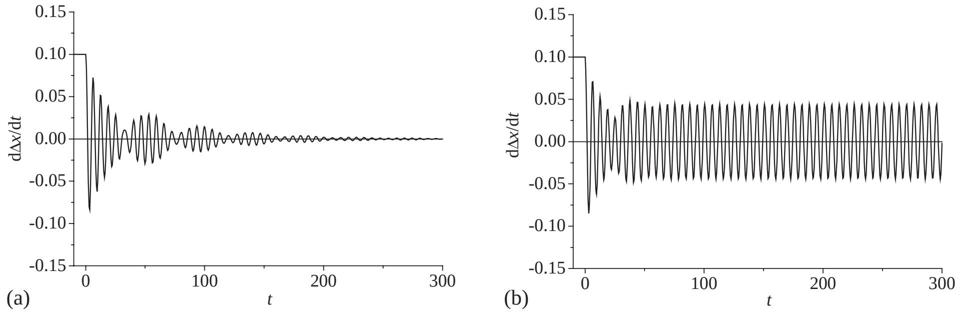


Fig. 12. The velocity variation of a periodic motion under delayed velocity feedback subject to an initial disturbance $\Delta\dot{x}(0) = 0.1$ m/s when $\sqrt{m/k} = 1.0$ s, $v = -0.1$ N s/m: (a) $T = 28.0$ s and (b) $T = 25.13$ s $\approx 8\pi$ s.

6. Conclusions

The delayed state feedback is an effective technique to improve the stability of periodic motions of a linear system with non-positive damping. The present study indicates that the working range of this technique does not cover the whole frequency domain. For a linear undamped system of single degree of freedom, the delayed displacement feedback can stabilize almost all periodic motions provided that their fundamental frequencies are higher than the natural frequency of system, but it only works in a number of narrower and narrower frequency bands lower than the natural frequency. The introduction of delayed velocity feedback can remarkably enlarge the working frequency ranges of delayed displacement feedback. Nevertheless, even the delayed state feedback is not able to stabilize any periodic motion if its period is an integral multiple of the natural period.

The criteria, based on the first order derivative of the real part of a characteristic root, of stability switches prove to be a powerful tool to analyze the stabilization problem and to plot the stability charts. However, the stabilization of a linear undamped system of single degree of freedom with delayed velocity feedback only is a degenerate case where the available criteria of stability switches fail to offer any useful information. In this case, the second order derivative of the real part of an arbitrary characteristic root plays an important role in identifying the trend of the real part of the characteristic root with respect to the system parameters, and enables one to sort out the stabilization conditions.

Finally, Eq. (3) also describes the regenerative chatter in some processes of metal cutting and the above conclusions are useful for the suppression of regenerative chatter.

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