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Letter to the Editor

A simple formula for estimating fundamental frequencies when two structures are coupled

J.G.M. Kerstens*

Structural Design Group, Technical University Eindhoven, P.O. Box 513, 5600 AB Eindhoven, Netherlands

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1. Introduction

During a design process the structure may need to be modified and therefore a quick and fairly accurate estimate of the fundamental frequency is helpful before embarking on complex and large-scale dynamic analyses. Another field of application is the modification due to placing exciters on structures which will change the fundamental frequencies.

It is shown how the established modal constraint method [1,2] can be applied to estimate the fundamental frequency of a structure consisting of two sub-structures whose fundamental frequencies and modes are known.

The paper is outlined as follows. By way of introduction in Section 2 a brief description of the application of the modal constraint method is given when constraints are added to one base problem. After this excursion in Section 3 the theory of the modal constraint method is extended to coupling an arbitrary number of base structures. Here the differential displacements of the coupled structures at the coupling locations are required to be zero.

In Section 4 the new simple formula is derived and applied to a number of typical examples. The results are compared with "exact" results or with those obtained by alternative formulas, e.g., Dunkerley's formula.

2. The modal constraint method: adding constraints

The energy functional E_{bs} for a vibrating base structure can be written in the following form:

$$E_{bs} = \frac{1}{2}\int_A (Lu)u \, \mathrm{d}A + \frac{1}{2}\rho \int_A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}A, \tag{1}$$

*Tel.: +31-40-2475098; fax: +31-40-2450328.
E-mail address: j.g.m.kerstens@bwk.tue.nl (J.G.M. Kerstens).

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where A is the region of operator L which for a Cartesian co-ordinate system is $L = \nabla^2$. The parameter ρ is the specific mass.

The displacement function u expressed in eigenfunctions u_i of the base problem is given by

$$u = \sum_i q_i(t) u_i, \tag{2}$$

where $q_i(t)$ are generalized co-ordinates

Suppose that at a number of locations the displacements of the plate are required to be zero (e.g., adding point supports) then the constraint energy W_k associated with adding this constraint condition k can be expressed as follows:

$$W_k = \alpha_k \sum_i c_{ki} q_i, \tag{3}$$

where $c_{ki} = u_i(x_k, y_k)$, i.e., the ith modal displacement at location k.

The coefficient α_k in Eq. (3) can be identified as a Lagrangian multiplier representing the generalized force of constraint acting on the base structure

The energy functional for the desired structure E_{ds} can be written as

$$E_{ds} = \frac{1}{2}\int_A (Lu)u \,\mathrm{d}A + \frac{1}{2}\rho \int_A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}A - \sum_k W_k. \tag{4}$$

To obtain an expression for the Lagrangian multiplier α_k it is convenient to define first a constraint function p_k by the following Fourier expansion (see Refs. [1,2]):

$$p_k = \sum_j c_{kj} u_j, \quad \text{where} \quad c_{kj} = \int_A p_k u_j \,\mathrm{d}A. \tag{5}$$

Note that here use has been made of the orthonormality of the eigenfunctions u_i.

An expression for the Lagrangian multiplier α_k can be obtained by applying the constraint function p_k to the energy functional E_{bs} as follows (see Refs. [1,2]):

$$\alpha_k = \int_A (Lu) p_k \,\mathrm{d}A = \int_A L\left(\sum_i u_i q_i\right) \sum_j c_{kj} u_j \,\mathrm{d}A = \sum_j \lambda_j^* c_{kj} q_j, \tag{6}$$

where λ_j^* is the jth eigenvalue of the base structure. Note that $\int_A (Lu_i) u_i \,\mathrm{d}A = \lambda_i^* \int_A (u_i) u_i \,\mathrm{d}A = \lambda_i^*$.

Substituting Eqs. (3) and (6) into Eq. (4) and taking the first variation results in the following expression for the desired structure:

$$\rho \ddot{q}_i + \lambda_i q_i = \sum_k \sum_j \lambda_j^* c_{kj} c_{ki} q_j \tag{7}$$

or in matrix form

$$[[\mathbf{I} - \mathbf{C}^{\mathrm{T}}\mathbf{C}][\boldsymbol{\lambda}^*_{diag}] - \lambda \mathbf{I}]\{\mathbf{q}\} = \mathbf{0}; \quad \lambda = \rho\omega^2. \tag{8}$$

It should be realized that the above matrix is non-symmetric.

However, Eq. (8) can be rewritten in a symmetric matrix form as follows:

$$[[\mathbf{I} - \mathbf{C}^{\mathrm{T}}\mathbf{C}] - \lambda[\mathbf{1}/\boldsymbol{\lambda}^*_{diag}]]\{\bar{\mathbf{q}}\} = \mathbf{0}; \quad \{\mathbf{q}\} = [\mathbf{1}/\boldsymbol{\lambda}^*_{diag}]\{\bar{\mathbf{q}}\}; \quad \lambda = \rho\omega^2, \tag{9}$$

where $\mathbf{I}$ is the unity matrix

Note that only the constraint matrix **C** needs to be orthonormalized columnwise as in the above derivation of the equations the eigenfunctions u_i are already assumed to be orthonormal. Then the following equation should hold:

$$[\mathbf{C}][\mathbf{C}]^{\mathrm{T}} = \mathbf{I}. \tag{10}$$

3. Coupling of base structures

In the case where N base structures are coupled at M locations the energy functional of the desired structure E_{ds} is (see Eq. (1))

$$E_{ds} = \sum_{n=1}^{N} \left[\frac{1}{2} \int_{A_n} (L_n u_n) u_n \, \mathrm{d}A_n + \frac{1}{2} \rho_n \int_{A_n} \left(\frac{\partial u_n}{\partial t} \right)^2 \mathrm{d}A_n \right] - \sum_{k=1}^{M} W_k, \tag{11}$$

where the eigenfunction u_n of the nth base structure has the following expansion:

$$u_n = \sum_i u_{ni} q_{ni}(t). \tag{12}$$

Similar to the previous section the constraint energy associated with coupling condition k where base structures m and n are involved is as follows:

$$W_k = \alpha_k \left(\sum_i c_{kmi} q_{mi} - \sum_j c_{knj} q_{nj} \right). \tag{13}$$

Note that here coupling is defined as requiring the differential displacements at coupling condition k to be zero.

According to the previous section the Lagrangian multiplier α_k can be expanded as follows (see Ref. [2]):

$$\alpha_k = \sum_i \lambda^*_{mi} c_{kmi} q_{mi} - \sum \lambda^*_{nj} c_{knj} q_{nj}, \tag{14}$$

where c_{kmi} is the ith modal displacement of structure m at location k and λ^*_{mi} is the ith eigenvalue of base structure m.

Substituting Eqs. (12)–(14) in Eq. (11) yields into expressions for a general case.

Suppose there are two base structures 1 and 2 coupled at only one location then the expression for constraint energy W_k is

$$W_1 = \alpha_1 \left(\sum_i c_{11i} q_{1i} - \sum_j c_{12j} q_{2j} \right) \tag{15}$$

or in matrix form

$$W_1 = (\{\mathbf{q}_1\}^{\mathrm{T}} \{\mathbf{c}_{11}\} - \{\mathbf{q}_2\}^{\mathrm{T}} \{\mathbf{c}_{12}\}) \alpha_1 = \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \mathbf{c}_{11} \\ -\mathbf{c}_{12} \end{Bmatrix} \alpha_1. \tag{16}$$

The Lagrangian multiplier α_k reduces in this case to

$$\alpha_1 = \sum_i c_{11i} q_{1i} - \sum_j c_{12j} q_{2j} \tag{17}$$

or in matrix form

$$\boldsymbol{\alpha}_1 = \begin{Bmatrix} \mathbf{c}_{11} \\ -\mathbf{c}_{12} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix}. \tag{18}$$

Substituting Eqs. (16)–(18) in Eq. (11) and taking the first variation gives

$$\begin{bmatrix} \boldsymbol{\lambda}^*_{1;diag} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}^*_{2;diag} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} - \omega^2 \begin{bmatrix} \boldsymbol{\rho}_{1;diag} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\rho}_{2;diag} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} - \begin{Bmatrix} \mathbf{c}_{11} \\ -\mathbf{c}_{12} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \mathbf{c}_{11} \\ -\mathbf{c}_{12} \end{Bmatrix} \begin{bmatrix} \boldsymbol{\lambda}^*_{1;diag} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}^*_{2;diag} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{19}$$

In a coupled situation the above equation can be written in a more compact matrix notation

$$[\boldsymbol{\lambda}^*_{diag}]\{\mathbf{q}\} - \omega^2[\boldsymbol{\rho}_{diag}]\{\mathbf{q}\} - \{\mathbf{C}\}^{\mathrm{T}}\{\mathbf{C}\}[\boldsymbol{\lambda}^*_{diag}]\{\mathbf{q}\} = \{\mathbf{0}\}; \qquad [\boldsymbol{\lambda}^*_{diag}] = [\boldsymbol{\omega}^{*2}_{diag}\boldsymbol{\rho}_{diag}] \tag{20}$$

or rewritten

$$\{([\mathbf{I}] - \{\mathbf{C}\}^{\mathrm{T}}\{\mathbf{C}\}) - \omega^2[1/\boldsymbol{\omega}^{*2}_{diag}]\}\{\mathbf{q}\} = \{\mathbf{0}\}. \tag{21}$$

Now it is time to apply the theory of the modal constraint method to a number of examples and compare the results with those obtained by other contending formula.

4. The application of the modal constraint method: the new formula

In this section simple formulae will be derived for the estimation of the first eigenvalue of two coupled structures.

The following symbols are used in this section.

ω_1^* is the first radian frequency of base structure 1 (eigenvalue: $\lambda_1^* = \omega_1^{*2}$), ϕ_1 is the modal displacement at the coupling location of base structure 1, ω_2^* is the first radian frequency of base structure 2 (eigenvalue: $\lambda_2^* = \omega_2^{*2}$), ϕ_2 is the modal displacement at the coupling location of base structure 2, ω is the estimate of the first radian frequency of the coupled structures

The constraint matrix based on the principles treated in the previous section is as follows:

$$\{\mathbf{C}\}^{\mathrm{T}} = \begin{Bmatrix} \phi_1 \\ -\phi_2 \end{Bmatrix}. \tag{22}$$

After orthonormalization

$$\{\mathbf{C}\}^{\mathrm{T}} = \begin{Bmatrix} \tilde{\phi}_1 \\ -\tilde{\phi}_2 \end{Bmatrix}; \qquad \tilde{\phi}_1 = \phi_1/(\phi_1^2 + \phi_2^2)^{1/2}; \qquad \tilde{\phi}_2 = \phi_2/(\phi_1^2 + \phi_2^2)^{1/2}. \tag{23}$$

Recalling the eigenvalue equation for coupled systems in the previous section and requiring the determinant of the eigenvalue equation to vanish then the following expression is obtained:

$$\det\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{Bmatrix}\tilde{\phi}_1\\ -\tilde{\phi}_2\end{Bmatrix}\{\,\tilde{\phi}_1 \quad -\tilde{\phi}_2\,\} - \omega^2\begin{bmatrix}1/\omega_1^{*2} & 0\\ 0 & 1/\omega_2^{*2}\end{bmatrix}\right\} = 0. \tag{24}$$

Or after some elaboration

$$\begin{vmatrix}(\tilde{\phi}_2^2 - \dfrac{\omega^2}{\omega_1^{*2}}) & -\tilde{\phi}_1\tilde{\phi}_2\\ -\tilde{\phi}_2\tilde{\phi}_1 & (\tilde{\phi}_1^2 - \dfrac{\omega^2}{\omega_2^{*2}})\end{vmatrix} = 0. \tag{25}$$

Applying the determinant operation and solving the above equation gives

$$\omega^2 = \frac{(\phi_2^2\omega_1^{*2} + \phi_1^2\omega_2^{*2})}{(\phi_1^2 + \phi_2^2)}. \tag{26}$$

Note that, as stated before, ϕ_1 and ϕ_2 must originate from orthonormal eigenfunctions or eigenvectors.

Applying both formulas to couple a square simply supported vibrating plate and a lumped mass in the middle (see Fig. 1) with the following properties: mass point mass 1.0 kg; $a = b = 1.0$ m; $D = 1.0\,\mathrm{N\,m^2}$; $\rho h = 1.0\,\mathrm{kg/m^2}$ will give the following results:

$$\omega^2 = \left(1^2 4\pi^4 + 2^2 0\right)/\left(2^2 + 1^2\right) = 77.93\,\mathrm{rad^2/s^2}.$$

For a simply supported vibrating rectangular plate

$$\phi_1 = \frac{2}{\sqrt{ab}}\sin\left(\pi\frac{x}{a}\right)\ \sin\left(\pi\frac{y}{b}\right) \quad \text{and} \quad \omega_1^2 = \frac{D\pi^4}{\rho}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2. \tag{27}$$

Note that $\phi_1(x = 0.5a, y = 0.5b) = 2.0; \phi_2 = \sqrt{m} = 1.0$.

Applying Dunkerley's formula gives

$$\omega^2 = 70.59\,\mathrm{rad^2/s^2}$$

which is a well-known lower bound.

The result obtained with the aid of a finite element model (ANSYS) with 20×20 SHELL 93 mesh is

$$\omega^2 = 71.73\mathrm{rad^2/s^2}.$$

Based on the principles described before a formula for decoupling between the two previous coupled systems can be also derived.

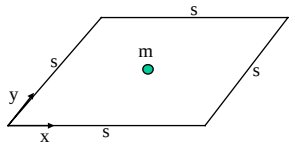

Fig. 1. Simply supported square plate with lumped mass.

For two systems with one decoupling location formula (26) changes into

$$\omega^2 = \frac{(\phi_2^2\omega_1^{*2} - \phi_1^2\omega_2^{*2})}{(\phi_1^2 - \phi_2^2)}. \tag{28}$$

Removing the mass in the previous example using the results from finite element analyses:

$$\phi_1 = 0.91119; \quad \omega^2 = 71.73\,\mathrm{rad}^2/\mathrm{s}^2,$$

$$\omega^2 = \frac{1^2{*}71.73 - 0.91119^2{*}0}{1^2 - 0.91119^2} = 422.6\,\mathrm{rad}^2/\mathrm{s}^2 \tag{29}$$

which is close to $4\pi^4 \approx 389.5\,\mathrm{rad}^2/\mathrm{s}^2$

When applying more than two degrees of freedom, e.g., three degrees of freedom, Eq. (26) has to be extended.

Suppose there are two structures with the following properties:

Structure 1: two degrees of freedom with co-ordinate axes x_1 and x_2.

The first eigenvalue with corresponding modal displacements at the coupling location is

$$\lambda_1^*: \begin{Bmatrix} \phi_{11} \\ \phi_{12} \end{Bmatrix} \quad \text{and for the second eigenvalue } \lambda_2^*: \begin{Bmatrix} \phi_{21} \\ \phi_{22} \end{Bmatrix}.$$

Structure 2: one degree of freedom with co-ordinate axis x_3:

$$\lambda_3^*: \{\phi_3\}.$$

When the origins of co-ordinate axes x_2 and x_3 are coupled then the relevant modal displacements are ϕ_{12}, ϕ_{22} and ϕ_3. Based on this the following constraint matrix can be constructed:

$$\{\mathbf{C}\}^{\mathrm{T}} = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ -\phi_3 \end{Bmatrix}. \tag{30}$$

After orthonormalization

$$\{\mathbf{C}\}^{\mathrm{T}} = \begin{Bmatrix} \tilde{\phi}_{12} \\ \tilde{\phi}_{22} \\ -\tilde{\phi}_3 \end{Bmatrix}. \tag{31}$$

The expression $[\mathbf{I} - \mathbf{C}^{\mathrm{T}}\mathbf{C}][\boldsymbol{\lambda}^*_{diag}] - \lambda[\mathbf{I}]$ (see Eq. (21)) reads then as follows:

$$\begin{bmatrix} (1-\tilde{\phi}_{12}^2)\lambda_1^* - \lambda & \tilde{\phi}_{12}\tilde{\phi}_{22}\lambda_2^* & \tilde{\phi}_{12}\tilde{\phi}_3\lambda_3^* \\ \tilde{\phi}_{22}\tilde{\phi}_{12}\lambda_1^* & (1-\tilde{\phi}_{22}^2)\lambda_2^* - \lambda & \tilde{\phi}_{22}\tilde{\phi}_3\lambda_3^* \\ \tilde{\phi}_3\tilde{\phi}_{12}\lambda_1^* & \tilde{\phi}_3\tilde{\phi}_{22}\lambda_2^* & (1-\tilde{\phi}_3^2)\lambda_3^* - \lambda \end{bmatrix}. \tag{32}$$

The determinant of this matrix expression is required to vanish to obtain the eigenvalues of the coupled structures.

Suppose that $\lambda_1^* = 0$ (there exists one rigid body mode) then the determinant of the above matrix expression reduces to

$$\begin{vmatrix} -\lambda & \tilde{\phi}_{12}\tilde{\phi}_{22}\lambda_2^* & \tilde{\phi}_{12}\tilde{\phi}_3\lambda_3^* \\ 0 & (1-\tilde{\phi}_{22}^2)\lambda_2^* - \lambda & \tilde{\phi}_{22}\tilde{\phi}_3\lambda_3^* \\ 0 & \tilde{\phi}_3\tilde{\phi}_{22}\lambda_2^* & (1-\tilde{\phi}_3^2)\lambda_3^* - \lambda \end{vmatrix} = 0. \tag{33}$$

The above expression results in

$$-\lambda \begin{vmatrix} (1-\tilde{\phi}_{22}^2)\lambda_2^* - \lambda & \tilde{\phi}_{22}\tilde{\phi}_3\lambda_3^* \\ \tilde{\phi}_3\tilde{\phi}_{22}\lambda_2^* & (1-\tilde{\phi}_3^2)\lambda_3^* - \lambda \end{vmatrix} = 0. \tag{34}$$

Or for $\lambda \neq 0$:

$$a\lambda^2 + b\lambda + c = 0, \tag{35}$$

where

$$\begin{aligned} a &= 1, \\ b &= -\{(1-\tilde{\phi}_{22}^2)\lambda_2^* + (1-\tilde{\phi}_3^2)\lambda_3^*\}, \\ c &= \lambda_2^*\lambda_3^*(1-\tilde{\phi}_{22}^2 - \tilde{\phi}_3^2). \end{aligned} \tag{36}$$

The first eigenvalue for the coupled structures is

$$\lambda = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{37}$$

The above result will be applied to the problem shown in Fig. 2.

It should be noted that adding one mode to the above problem renders a solution of the problem that cannot be written in a closed form.

The eigenvalues and eigenvectors are as follows:

$$\lambda_1^* = 0; \ \begin{Bmatrix} 1/2\sqrt{6} \\ 1/2\sqrt{6} \end{Bmatrix}; \quad \lambda_2^* = 6; \ \begin{Bmatrix} 1/2\sqrt{6} \\ -1/2\sqrt{6} \end{Bmatrix}; \quad \lambda_3^* = 1; \ \{1\}.$$

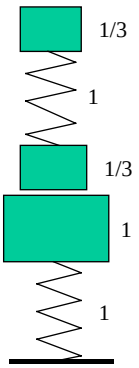

Fig. 2. Two mass–spring systems to be coupled.

The constraint matrix is

$$\mathbf{C}^{\mathrm{T}} = \begin{Bmatrix} 1/2\sqrt{6} \\ -1/2\sqrt{6} \\ 1 \end{Bmatrix}.$$

Orthonormalized

$$\mathbf{C}^{\mathrm{T}} = \begin{Bmatrix} 1/4\sqrt{6} \\ -1/4\sqrt{6} \\ 1/2 \end{Bmatrix}^{\mathrm{T}} = \begin{Bmatrix} \tilde{\phi}_{12} \\ \tilde{\phi}_{22} \\ \tilde{\phi}_{3} \end{Bmatrix}^{\mathrm{T}}.$$

Then finally

$$a = 1; \quad b = -\tfrac{9}{2}; \quad c = \tfrac{9}{4}.$$

The first eigenvalue is

$$\lambda = \omega^2 = \tfrac{9}{4} - \tfrac{3}{4}\sqrt{5} = 0.573$$

which is equal to the exact first eigenvalue.

Applying the formula of Dunkerley for this case gives

$$\lambda = \omega^2 = 0.500$$

which is indeed a lower bound for the exact eigenvalue.

In the previous example where a lumped mass is attached to a vibrating plate only the first eigenvalue and eigenfunction of the simply supported plate has been taken into account.

The result obtained can be improved when also the second eigenvalue and eigenfunction is included.

For this problem Eqs. (32)–(37) will be applied.

The constraint matrix is as follows:

$$\{\mathbf{C}\}^{\mathrm{T}} = \begin{Bmatrix} 2 \\ 2 \\ -1 \end{Bmatrix}^{\mathrm{T}}; \quad \begin{Bmatrix} \lambda_1^* = 4\pi^4 \\ \lambda_2^* = 342\pi^4 \\ \lambda_3^* = 0 \end{Bmatrix}.$$

After orthonormalization

$$\{\mathbf{C}\}^{\mathrm{T}} = \begin{Bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{Bmatrix}^{\mathrm{T}}.$$

Substitution in Eq. (36) gives

$$a = 1; \quad b = 37.77\,\pi^4; \quad c = 28.44\,\pi^4.$$

The first (improved) eigenvalue is then

$$\lambda = \omega^2 = 74.75\,\mathrm{rad}^2/\mathrm{s}^2.$$

Compared with $\lambda = \omega^2 = 77.93\,\mathrm{rad}^2/\mathrm{s}^2$ this is a noticeable improvement.

5. Concluding remarks

The application of the modal constraint method for connecting continuous structures yields sufficiently accurate upper bound estimates for the fundamental frequency.

Exact fundamental eigenfrequencies for lumped mass/spring systems can be obtained when they will be fully described by a total of maximum three modes.

Dunkerley's formula on the other hand does not render exact eigenvalues for such systems and provide lower bounds .

The new formulae in conjunction with the formula of Dunkerley allows inclusion of the exact fundamental eigenfrequency.

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[2] J.G.M. Kerstens, Vibration of complex structures: the modal constraint method, *Journal of Sound and Vibration* 76 (4) (1981) 467–480.