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The vibration of a non-prismatic beam on an inertial elastic half-plane

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Abstract

The problem of vibration of a non-prismatic beam resting on an inertial elastic half-plane described by classical elastokinetics equations is solved using Chebyshev series approximation. As a result, closed analytical formulae for the sought solution's coefficients—the passive foundation pressure function and the system displacement function—were obtained. The method was applied to solve the problem of harmonically excited vibration.

1. Introduction

The use of variable cross-section beam systems in modern engineering structures has been increasing due to the necessity for the rational shaping and economical designing of structures and for architectonic reasons. Solutions of many static problems, including stability problems, can be found in a monograph by Krynicki and Mazurkiewicz [1]. An analytical solution, consisting in the Fourier series expansion of the displacement function and the application of variational methods, was presented in a paper by Heidebrecht [2]. Fourier series supplemented with power polynomials were applied to solve linear, variable-coefficient differential equations (derived from, e.g., variable cross-section beam vibration problems) in a paper by Ganga Rao and Spyrakos [3]. A rigidity matrix and an inertia matrix for a beam with linearly varying height was determined by Gupta [4]. Non-prismatic beams were also studied by Eisenberger who in Ref. [5] determined rigidity matrix elements for several kinds of non-prismatic beams. Jointly with Reich [6] he applied the finite element method to a static analysis and a dynamic analysis to solve the stability problem, approximating the beam's displacements by polynomials of degree three. In Ref. [7] he presented formulae for rigidity matrix elements for a beam element with variable rigidities described by

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power series. Klasztorny [8] applied the same polynomial approximation to determine matrices for Euler and Timoshenko beam finite elements with variable parameters, resting on the Winkler foundation. The Eisenberger formulae describing rigidity and inertia matrices for bars with varying rigidity and density [7] were extended to bars resting on a two-parameter foundation and subjected to non-potential loads in a paper by Glabisz [9]. They were applied there to solve several stability problems relating to non-prismatic columns, using, similarly as in Refs. [7,8], power series to approximate the displacement function. The author of the present paper applied Chebyshev series to solve the vibration problem for a non-prismatic beam resting on a non-homogenous two-parameter elastic foundation in Refs. [10,11]. The solution in the form of a series with respect to Chebyshev polynomials presented in Ref. [10] was obtained by solving an infinite system of algebraic equations for harmonic vibration. The equations' coefficients were described by closed analytical formulae. The results obtained in Ref. [10] were used in Ref. [11] to determine a dynamic matrix of rigidity for non-prismatic finite elements. The described elements were then employed to solve elementary and more complex problems of stability. The problem of the influence of boundary conditions, polynomial beam density and rigidity expansion coefficients and two-parameter foundation rigidity parameter expansion coefficients on the system's eigenfrequencies was considered by Elishakoff [12]. Wave problems in a sectionally non-prismatic finite beam were investigated by Burr et al. [13]. The vibration of a beam resting on a two-parameter elastic foundation, generated by different kinds of loads, is analyzed in a paper by Ghani Razaqpur and Shah [14], but the analysis is limited to prismatic beams.

In the present paper, the problem of the linear harmonic vibration of a beam with variable strength and geometric parameters, resting on an inertial elastic half-plane described by classical elastokinetics equations, is considered. It is assumed that the beam's variable parameters, such as flexural rigidity, density and load, can be expanded into series with respect to Chebyshev polynomials of the first kind. Using the theorems found in monograph [15] and the results reported in Refs. [10,11], formal relationships between the passive foundation pressure function expansion and the displacement function expansion are determined. The solution of the problem of interactions between the beam and the foundation (the determination of the unknown coefficients of the passive foundation pressure function and the displacement function) is based on the method used by Sejmov [16] to solve the vibration problem for a stiff block foundation resting on an inertial elastic half-plane. After calculating the unknown coefficients of the passive foundation pressure function, the sought displacement function coefficients are determined. The method is illustrated by applying it to a numerical example in which the vibration of a non-prismatic beam, generated by a uniformly distributed harmonic load and a linearly variable asymmetrical load, is considered.

2. Problem formulation

A rectilinear non-prismatic Euler beam of length $2a$, resting on an inertial elastic half-plane and subjected to dynamic normal loads $P(X, t)$ and static axial forces $N(X)$ ($N > 0$ in tension), is considered. The beam and the half-plane are joined with bilateral constraints in the normal direction (Fig. 1).

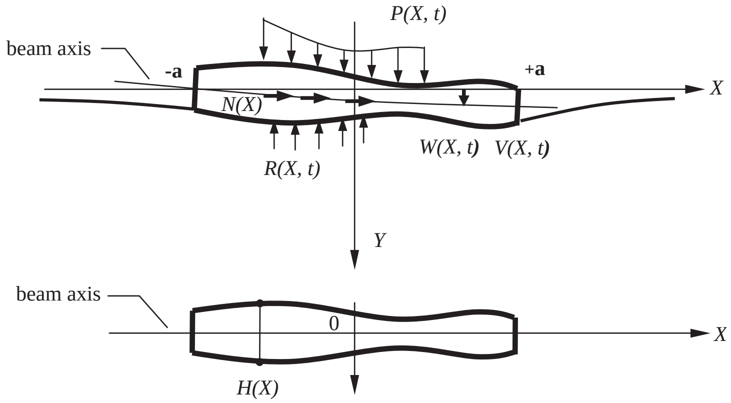


Fig. 1. Diagram of system.

The beam linear transverse vibration is described by the following differential equation:

$$\frac{\partial^2}{\partial X^2}\left(EJ(X)\frac{\partial^2 W}{\partial X^2}\right) - \frac{\partial}{\partial X}\left(N(X)\frac{\partial W}{\partial X}\right) + \rho_B(x)\frac{\partial^2 W}{\partial t^2} = P(X,t) - R(X,t), \tag{1}$$

where W is the displacement perpendicular to the bar's axis, E is Young's modulus, J is a moment of inertia of the bar's cross-section, ρ_B is the mass per unit length and $R(X,t)$ an elastic foundation reaction function.

The cross-sectional forces—bending moments and shearing forces—are given by the formulae

$$M(X,t) = -EJ\frac{\partial^2 W}{\partial X^2}, \quad T(X,t) = -\frac{\partial}{\partial X}\left(EJ\frac{\partial^2 W}{\partial X^2}\right) + N\frac{\partial W}{\partial X}. \tag{2}$$

The inertial elastic half-plane displacement functions are described by the two conjugate partial differential equations

$$\begin{aligned}
&\mu\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) + (\lambda+\mu)\left(\frac{\partial^2 U}{\partial X\,\partial Y} + \frac{\partial^2 V}{\partial Y^2}\right) = -\rho\frac{\partial^2 V}{\partial t^2},\\
&\mu\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) + (\lambda+\mu)\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 V}{\partial X\,\partial Y}\right) = -\rho\frac{\partial^2 U}{\partial t^2}
\end{aligned} \tag{3}$$

and the boundary conditions

$$\begin{aligned}
&\sigma_{YY}(X,0,t) = -R(X)\mathrm{e}^{\mathrm{i}\omega t} \quad \text{for} - a \leqslant X \leqslant + a,\\
&\sigma_{XY}(X,0,t) = 0 \quad \text{for} - \infty \leqslant X \leqslant + \infty,
\end{aligned} \tag{4}$$

where $U(X)$, $V(X)$ are, respectively, horizontal and vertical displacements of the half-plane, λ, μ are Lamé constants and ρ is the elastic foundation's density.

Because of the constraints introduced between the beam and the half-plane, the following relation holds for the displacements:

$$V(X,0,t) = W(X,t) \quad \text{for} - a \leqslant X \leqslant + a. \tag{5}$$

Further considerations are limited to harmonic vibration. If the relations

$$x = X/a, \quad y = Y/a,$$

$$
\begin{aligned}
&W(X,t) = W(X)e^{i\omega t} = aw(x)e^{i\omega t},\\
&V(X,0,t) = V(X,0)e^{i\omega t} = av(x,0)e^{i\omega t},\\
&U(X,0,t) = U(X,0)e^{i\omega t} = au(x,0)e^{i\omega t},\\
&P(X,t) = P(X)e^{i\omega t} = (P_0/a)p(x)e^{i\omega t},\\
&R(X,t) = R(X)e^{i\omega t} = (P_0/a)r(x)e^{i\omega t}
\end{aligned} \tag{6}
$$

are put into Eqs. (1)–(4), this equation of the beam's vibration is obtained:

$$
\begin{aligned}
&\overline{EJ}(x)\frac{\partial^4 w}{\partial x^4} + \left(2\frac{\partial \overline{EJ}(x)}{\partial x}\right)\frac{\partial^3 w}{\partial x^3} + \left(\frac{\partial^2 \overline{EJ}(x)}{\partial x^2} - n\bar{N}(x)\right)\frac{\partial^2 w}{\partial x^2}\\
&\quad - n\frac{\partial \bar{N}(x)}{\partial x}\frac{\partial w}{\partial x} - \omega^2 g\overline{\rho_B}(x)w = n(p(x) - r(x))
\end{aligned} \tag{7}
$$

and the cross-section forces (2) are expressed by the formulae

$$
\begin{aligned}
&m(x) = \frac{M(ax)a}{EJ_0} = -\overline{EJ}\frac{\partial^2 w}{\partial x^2},\\
&t(x) = \frac{T(ax)\,a^2}{EJ_0} = -\left(\frac{\partial}{\partial x}\overline{EJ}\right)\frac{\partial^2 w}{\partial x^2} - \overline{EJ}\frac{\partial^3 w}{\partial x^3} + n\bar{N}\frac{\partial w}{\partial x},
\end{aligned} \tag{8}
$$

where

$$
EJ = EJ_0\overline{EJ}, \quad N = P_0\bar{N}, \quad \rho_B = \rho_0\overline{\rho_B}, \quad n = \frac{a^2 P_0}{EJ_0}, \quad g = \frac{a^4\rho_0}{EJ_0} \tag{9}
$$

and EJ_0, EA_0, ρ_0, P_0 are reference quantities. To simplify the notation, EJ, N, ρ_B instead of $\overline{EJ}$, $\bar{N}$, $\bar{\rho}_B$ will be used.

When relation (6) is used, the equations describing the vibration of the half-plane are expressed by the formulae

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \beta^2\frac{\partial^2 u}{\partial y^2} + (1-\beta^2)\frac{\partial^2 v}{\partial x\partial y} = -\beta^2\kappa^2 u,\\
&\frac{\partial^2 v}{\partial x^2} + \beta^2\frac{\partial^2 v}{\partial y^2} + (1-\beta^2)\frac{\partial^2 u}{\partial x\partial y} = -\beta^2\kappa^2 v
\end{aligned} \tag{10}
$$

and the boundary conditions assume the form

$$
\begin{aligned}
&\sigma_{yy} = -P_0 r(x)e^{i\omega t}/a \quad \text{for} -1 \leqslant x \leqslant +1,\\
&\sigma_{yy} = 0 \quad \text{for} -\infty \leqslant x \leqslant +\infty,
\end{aligned} \tag{11}
$$

where $\beta^2 = c_2^2/c_1^2 = \mu/(\lambda+2\mu), \kappa = \omega a/c_2$ and $c_1^2 = (\lambda+2\mu)/\rho$, $c_2^2 = \mu/\rho$ are the velocities of propagation of, respectively, longitudinal and transverse waves in the foundation.

3. Chebyshev series expansion of elastic half-plane displacement function

The method of potentials and the Fourier transformation are applied to solve the system of Eqs. (10). Without going into details of this classical technique of solving elastokinetics problems, the final solution is

$$v(x,0) = -\frac{P_0}{\sqrt{2\pi}a\mu}\int_{-\infty}^{\infty}\frac{\kappa^2\sqrt{\alpha^2-\beta^2\kappa^2}\,\bar{r}(\alpha)}{(2\alpha^2-\kappa^2)^2-4\alpha^2\sqrt{\alpha^2-\beta^2\kappa^2}\sqrt{\alpha^2-\kappa^2}}\,\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \tag{12}$$

where $\bar{r}(\alpha)$ is the Fourier transform of pressure function $r(x)$. Unknown passive foundation pressure function $r(x)$ in the form (see Ref. [16, p. 70])

$$r(x) = (1-x^2)^{-1/2}\sum_{l=0}^{\infty}{}' r_l \mathrm{T}_l(x), \tag{13}$$

where $\mathrm{T}_l(x)$ are Chebyshev polynomials of the first kind and the modified sum symbol Σ' defines the following operation $\sum_{l=0}^{\infty}{}' a_l = \frac{1}{2}a_0 + a_1 + a_2 + a_3 + \cdots$, is sought.

When the Fourier transform is performed on Eq. (12) and the integral formulae

$$\begin{aligned}&\frac{1}{\pi}\int_{-1}^{+1}(1-x^2)^{-1/2}\mathrm{T}_{2l}(x)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x = (-1)^l \mathrm{J}_{2l}(\alpha),\\ &\frac{1}{\pi}\int_{-1}^{+1}(1-x^2)^{-1/2}\mathrm{T}_{2l+1}(x)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x = \mathrm{i}(-1)^l \mathrm{J}_{2l+1}(\alpha),\end{aligned} \tag{14}$$

(where $\mathrm{J}_1(x)$ is the Bessel function of the first kind) are used, then

$$\begin{aligned}\bar{r}(\alpha) &= \frac{1}{\sqrt{2\pi}}\sum_{l=0}^{\infty}{}' r_l\int_{-\infty}^{\infty}(1-x^2)^{-1/2}\mathrm{T}_l(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x\\ &= \left(\frac{2}{\pi}\right)^{1/2}\left(\sum_{l=0}^{\infty}{}'(-1)^l r_{2l}\mathrm{J}_{2l}(\alpha) + \mathrm{i}\sum_{l=0}^{\infty}(-1)^l r_{2l+1}\mathrm{J}_{2l+1}(\alpha)\right).\end{aligned} \tag{15}$$

If transform (15) is substituted into formula (12),

$$\begin{aligned}v(x,0) = &-\frac{P_0}{2a\mu}\sum_{l=0}^{\infty}{}'(-1)^l\int_{-\infty}^{\infty}\frac{\kappa^2\sqrt{\alpha^2-\beta^2\kappa^2}\,r_{2l}\mathrm{J}_{2l}(\alpha)}{(2\alpha^2-\kappa^2)^2-4\alpha^2\sqrt{\alpha^2-\beta^2\kappa^2}\sqrt{\alpha^2-\kappa^2}}\,\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha\\ &-\frac{P_0\mathrm{i}}{2a\mu}\sum_{l=0}^{\infty}(-1)^l\int_{-\infty}^{\infty}\frac{\kappa^2\sqrt{\alpha^2-\beta^2\kappa^2}\,r_{2l+1}\mathrm{J}_{2l+1}(\alpha)}{(2\alpha^2-\kappa^2)^2-4\alpha^2\sqrt{\alpha^2-\beta^2\kappa^2}\sqrt{\alpha^2-\kappa^2}}\,\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha.\end{aligned} \tag{16}$$

Displacement function (16) (see Ref. [16, p. 71]) is expanded into a series with respect to the Chebyshev polynomials of the first kind

$$v(x) = \sum_{l=0}^{\infty}{}' a_l[v]\mathrm{T}_l(x) = \sum_{l=0}^{\infty}{}' v_l\mathrm{T}_l(x), \tag{17}$$

where

$$\sum_{l=0}^{\infty}{}' a_l[f] = \tfrac{1}{2}a_0[f] + a_1[f] + a_2[f] + \cdots. \quad (18)$$

By definition, the expansion's coefficients $a_l[v]$ are given by

$$a_l[v] = \frac{2}{\pi}\int_{-1}^{1}(1-x^2)^{-1/2}v(x)\mathrm{T}_l(x)\,\mathrm{d}x. \quad (19)$$

Relation (14) and the formula

$$\mathrm{J}_v(\mathrm{e}^{m\pi\mathrm{i}}z) = \mathrm{e}^{mv\pi\mathrm{i}}\mathrm{J}_v(z) \quad (20)$$

are used and the substitution $\xi = \alpha/\kappa$ is introduced to obtain

$$a_{2n}[v] = v_{2n} = -\frac{P_0}{a\mu}\sum_{l=0}^{\infty}{}'(-1)^{l+n}r_{2l}\int_0^{\infty}\frac{2\sqrt{\xi^2-\beta^2}\,\mathrm{J}_{2l}(\kappa\xi)\mathrm{J}_{2n}(\kappa\xi)}{(2\xi^2-1)^2-4\xi^2\sqrt{\xi^2-\beta^2}\sqrt{\xi^2-1}}\,\mathrm{d}\xi,$$

$$a_{2n+1}[v] = v_{2n+1} = -\frac{P_0}{a\mu}\sum_{l=0}^{\infty}(-1)^{l+n}r_{2l+1}\int_0^{\infty}\frac{2\sqrt{\xi^2-\beta^2}\,\mathrm{J}_{2l+1}(\kappa\xi)\mathrm{J}_{2n+1}(\kappa\xi)}{(2\xi^2-1)^2-4\xi^2\sqrt{\xi^2-\beta^2}\sqrt{\xi^2-1}}\,\mathrm{d}\xi. \quad (21)$$

Infinite integrals (21) are subjected to further transformation. To simplify the notation, the symbols

$$\varphi_{n,l} = 2\int_0^{\infty}F(x)\mathrm{J}_l(\kappa\xi)\mathrm{J}_n(\kappa\xi)\mathrm{d}\xi, \quad l-n=2m \quad (22)$$

are introduced where

$$F(\xi) = \frac{\left(\sqrt{\xi^2-\beta^2}\right)}{\left((2\xi^2-1)^2-4\xi^2\sqrt{\xi^2-\beta^2}\sqrt{\xi^2-1}\right)}. \quad (23)$$

The additional condition in formula (22) means that numbers l, n specifying the orders of the Bessel functions, are simultaneously even or simultaneously uneven. It is assumed that $l \geqslant n$.

In formula (22) the substitution

$$2\mathrm{J}_n(x) = \mathrm{H}_n^{(1)}(x) + \mathrm{H}_n^{(2)}(x) \quad (24)$$

is performed where

$$\mathrm{H}_n^{(1)}(x) = \mathrm{J}_n(x) + \mathrm{i}\mathrm{N}_n(x), \quad \mathrm{H}_n^{(2)}(x) = \mathrm{J}_n(x) - \mathrm{i}\mathrm{N}_n(x) \quad (25)$$

are Hankel functions of the first and second kind, and this relation is obtained

$$\varphi_{n,l} = \int_0^{\infty}F(\xi)\mathrm{J}_l(\kappa\xi)\mathrm{H}_n^{(1)}(\kappa\xi)\,\mathrm{d}\xi + \int_0^{\infty}F(\xi)\mathrm{J}_l(\kappa\xi)\mathrm{H}_n^{(2)}(\kappa\xi)\,\mathrm{d}\xi. \quad (26)$$

Integrals (26) are calculated by moving the integration onto a complex plane. The appropriate integration contours are shown in Figs. 2 and 3. The signs and the characteristic values of

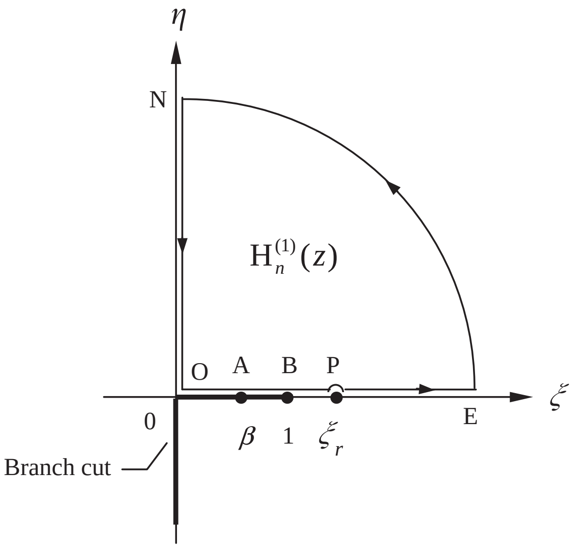


Fig. 2. Integration contour in case when integrand includes function $H_n^{(1)}(z)$. A, B are branch points and P is pole.

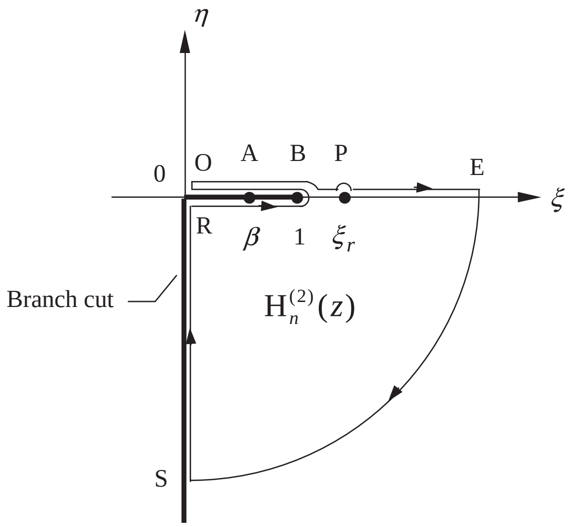


Fig. 3. Integration contour in case when integrand includes function $H_n^{(2)}(z)$. A, B are branch points and P is pole.

multi-valued functions $v_1 = \sqrt{z^2 - \beta^2}$, $v_2 = \sqrt{z^2 - 1}$, where $z = \xi + i\eta$, in formula (23) are presented in Fig. 4. Using known theorems relating to integration on a complex plane, one gets

$$\begin{aligned}\varphi_{n,l} &= \int_0^\infty F(\xi) J_l(\kappa\xi) H_n^{(1)}(\kappa\xi)\,d\xi + \int_0^\infty F(\xi) J_l(\kappa\xi) H_n^{(2)}(\kappa\xi)\,d\xi \\ &= \int_0^\infty I_1\,d\xi + \int_0^\infty I_2\,d\xi = \int_{OBR} I_2\,dz + \int_O^N I_1\,dz + \int_R^S I_2\,dz - 2\pi i \sum \text{res}\, I_2. \qquad (27)\end{aligned}$$

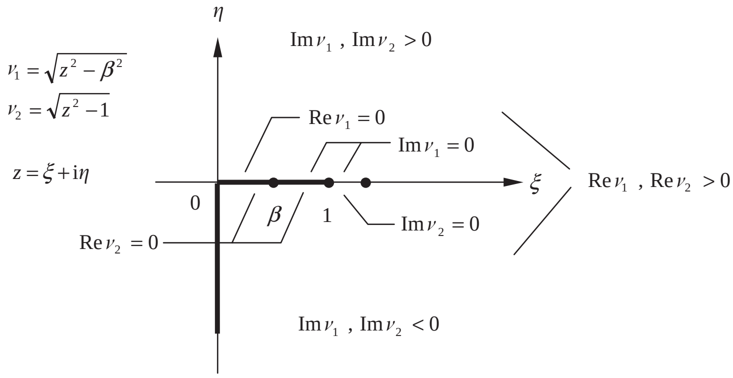


Fig. 4. Signs and characteristic values of multi-valued function $v_1 = \sqrt{z^2 - \beta^2}$, $v_2 = \sqrt{z^2 - 1}$ on complex plane.

After appropriate transformations using relation (20) and the formula

$$\mathrm{H}_v^{(1)}(\mathrm{e}^{\pi \mathrm{i}} z) = -\mathrm{e}^{-v\pi \mathrm{i}} \mathrm{H}_v^{(2)}(z) \tag{28}$$

results in

$$\begin{aligned} \varphi_{n,l} = 2\mathrm{i} \Bigg[& \int_0^\beta \frac{\sqrt{\beta^2 - \xi^2}}{(2\xi^2 - 1)^2 + 4\xi^2 \sqrt{\beta^2 - \xi^2}\sqrt{1 - \xi^2}} \mathrm{J}_l(\kappa\xi)\mathrm{H}_n^{(2)}(\kappa\xi)\,\mathrm{d}\xi \\ & + \int_\beta^1 \frac{4\xi^2(\xi^2 - \beta^2)\sqrt{1 - \xi^2}}{(2\xi^2 - 1)^4 + 16\xi^4(\xi^2 - \beta^2)(1 - \xi^2)} \mathrm{J}_l(\kappa\xi)\mathrm{H}_n^{(2)}(\kappa\xi)\,\mathrm{d}\xi \\ & - \pi \left. \frac{\sqrt{\xi^2 - \beta^2}}{[(2\xi^2 - 1)^2 - 4\xi^2\sqrt{\xi^2 - \beta^2}\sqrt{\xi^2 - 1}]'} \right|_{\xi = \xi_r} \mathrm{J}_l(\kappa\xi_r)\mathrm{H}_n^{(2)}(\kappa\xi_r) \Bigg], \\ & l - n = 2m, \quad l \geqslant n. \end{aligned} \tag{29}$$

When $l < n$, Bessel function products $\mathrm{J}_l(\,)\mathrm{H}_n^{(2)}(\,)$ which occur in formula (29) should be replaced by $\mathrm{J}_n(\,)\mathrm{H}_l^{(2)}(\,)$. The details of the presented method of calculating infinite integrals can be found in Ref. [17, pp. 132–134].

4. Solution of non-prismatic beam vibration problem

If a passive foundation pressure function (13) is substituted into Eq. (7), one obtains

$$\begin{aligned} & EJ(x)\frac{\partial^4 w}{\partial x^4} + \left(2\frac{\partial EJ(x)}{\partial x}\right)\frac{\partial^3 w}{\partial x^3} + \left(\frac{\partial^2 EJ(x)}{\partial x^2} - nN(x)\right)\frac{\partial^2 w}{\partial x^2} - n\frac{\partial N(x)}{\partial x}\frac{\partial w}{\partial x} - \omega^2 g\rho_B(x)w \\ & = n\left(p(x) - (1 - x^2)^{-1/2}{\sum_{l=0}^{\infty}}' r_l \mathrm{T}_l(x)\right). \end{aligned} \tag{30}$$

A solution of differential Eq. (30) in the form of the Chebyshev series

$$w(x) = \sum_{l=0}^{\infty}{}' a_l[w]\mathrm{T}_l(x) = \sum_{l=0}^{\infty}{}' w_l \mathrm{T}_l(x) \tag{31}$$

will be sought.

To solve Eq. (30), the theorem relating to ordinary differential equations ([15, p. 231]) is applied:

Theorem. *If function $f(x)$ satisfies the linear differential equation of order $n > 0$*

$$\sum_{m=0}^{n} \hat{P}_m(x) f^{(n-m)}(x) = \hat{P}(x) \tag{32}$$

and

$$Q_m(x) = \sum_{j=0}^{m} (-1)^{m+j} \binom{n-j}{m-j} \hat{P}_j^{(m-j)}(x), \quad m = 0, 1, \ldots, n, \tag{33}$$

where $\binom{n}{m} = n!/m!(n-m)!$ *and the Chebyshev series coefficients in functions* $(Q_0 f)^{(n)}, (Q_1 f)^{(n-1)}, \ldots, Q_n f, \hat{P}$ *are determinate, then for each integer k the following identity is true:*

$$\sum_{m=0}^{n} 2^{n-m} \sum_{j=0}^{m} b_{nmj}(k) a_{k-m+2j}[Q_m(x) f(x)] = \sum_{j=0}^{n} b_{nnj}(k) a_{k-n+2j}[\hat{P}(x)], \tag{34}$$

where $b_{nmj}(k)$ are polynomials of integer variable k

$$b_{nmj}(k) = (-1)^j \binom{m}{j} (k-n)_{n-m+j} (k-m+2j)(k+j+1)_{n-j} (k^2-n^2)^{-1},$$
$$m = 0, 1, \ldots, n; \quad j = 0, 1, \ldots, m, \tag{35}$$

$$(k)_n = \begin{cases} 1 & \text{for } n = 0, \\ k(k+1)(k+2)\ldots(k+n-1) & \text{for } n = 1, 2, 3, \ldots \end{cases} \tag{36}$$

and $a_k[h]$ is the kth coefficient of the Chebyshev series expansion of function $h(x)$ with respect to the Chebyshev polynomials of the first kind.

The proof of this theorem can be found in Ref. [15, pp. 231–234]. To apply the above theorem, it is necessary, among other things, to expand the right side of Eq. (30) into a Chebyshev series. Since function $(1-x^2)^{-1/2}$ relating to the foundation reaction has no such expansion, the left side and the right side of Eq. (30) are multiplied by $(1-x^2)$.

Functions $\hat{P}_m, \hat{P}$ in the modified Eq. (30) are defined by the respective formulae

$$\hat{P}_0(x) = (1-x^2)EJ(x), \quad \hat{P}_1(x) = 2(1-x^2)\frac{\partial EJ(x)}{\partial x},$$
$$\hat{P}_2(x) = (1-x^2)\frac{\partial^2 EJ(x)}{\partial x^2} - nN(x), \quad P_3(x) = -(1-x^2)n\frac{\partial N(x)}{\partial x},$$
$$\hat{P}_4(x) = -(1-x^2)\omega^2 g\rho_B(x),$$
$$\hat{P}(x,t) = n\left((1-x^2)p(x) - (1-x^2)^{1/2}\sum_{l=0}^{\infty}{}' r_l \mathrm{T}_l(x)\right). \tag{37a–f}$$

If relation (33) is applied, then functions Q_m associated with $\hat{P}_m$ assume the form

$$Q_0(x) = (1-x^2)EJ(x), \quad Q_1(x) = -2(1-x^2)\frac{\partial EJ(x)}{\partial x} + 8xEJ(x),$$
$$Q_2(x) = (1-x^2)\frac{\partial^2 EJ(x)}{\partial x^2} - 12x\frac{\partial EJ(x)}{\partial x} - 12\,EJ(x) - (1-x^2)nN(x),$$
$$Q_3(x) = 4x\frac{\partial^2 EJ(x)}{\partial x^2} + 12\frac{\partial EJ(x)}{\partial x} + (1-x^2)\,n\frac{\partial N(x)}{\partial x} - 4xnN(x),$$
$$Q_4(x) = -2\frac{\partial^2 EJ(x)}{\partial x^2} + 2xn\frac{\partial N(x)}{\partial x} + 2nN(x) - (1-x^2)\omega^2 g\rho_B(x). \tag{38}$$

In further transformations, the relations (see [15, p. 128, (33), p. 124, (17), p. 123, (11)])

$$a_l[f(x)g(x)] = \frac{1}{2}\sum_{m=0}^{\infty}{}' a_m[f](a_{l-m}[g] + a_{l+m}[g]), \tag{39}$$

$$a_l = \frac{1}{2l}(a'_{l-1} - a'_{l+1}), \quad l \neq 0, \tag{40}$$

$$a_l[x^m f(x)] = 2^{-m}\sum_{j=0}^{m}\binom{m}{j} a_{l-m+2j}, \quad m = 0,1,2,\ldots \tag{41}$$

and this expansion of function $(1-x^2)^{1/2}$ ([15, p. 143, (4)])

$$(1-x^2)^{1/2} = -\frac{4}{\pi}\sum_{m=0}^{\infty}{}' \frac{1}{4m^2-1}T_{2m}(x), \tag{42}$$

where $a_l = a_l[f]$ and $a'_l = a_l[\partial f/\partial x]$, will be used. If formulae (37f), (38) are substituted into formula (34) and complex transformations are performed using formulae (39)–(41), this infinite system of algebraic equations for determining coefficients w_l of the expansion of displacement function $w(x)$ is obtained as

$$\sum_{l=0}^{\infty}{}'(E_{k,l} + nN_{k,l} + \omega^2 gG_{k,l})w_l$$
$$= n\left(F_k + \sum_{l=0}^{\infty}{}' r_l R_{k,l}\right), \quad k = 0,1,2,3,\ldots, \tag{43}$$

where

$$
\begin{aligned}
E_{k,l} = & -2(k+1)(k+2)(k+3)(k-4)(k-5)(l-1)le_{k-l-2} \\
& -2(k-1)(k-2)(k-3)(k+4)(k+5)(l-1)le_{k+l+2} \\
& +\big(4(k^2-9)(k^2-4)l(kl+11)-8(k+2)(k+3)(k-l)l(7k-23)\big)e_{k-l} \\
& +\big(4(k^2-9)(k^2-4)l(kl-11)+8(k-2)(k-3)(k+l)l(7k+23)\big)e_{k+l} \\
& +\big(2(k^2-9)(k^2-4)(k^2-1)(k-2l-4) \\
& -2(k^2-9)(k^2-4)(k-l+1)((k+1)(k-l+2)-12(k-1)) \\
& -8(k^2-9)(k-1)(k-2)(k-l)(k-l+2)-4(k-1)(k-2)(k-3)(k-l+2)(k-l+3)\big)e_{k-l+2} \\
& +\big(2(k^2-9)(k^2-4)(k^2-1)(k+2l+4) \\
& -2(k^2-9)(k^2-4)(k+l-1)((k-1)(k+l-2)+12(k+1)) \\
& +8(k^2-9)(k+1)(k+2)(k+l)(k+l-2)-4(k+1)(k+2)(k+3)(k+l-2)(k+l-3)\big)e_{k+l-2} \\
& +\left\{\begin{array}{ll}
-4(k^2-9)(k^2-4)(l-2)[(k-l+2)e_{k-l+2}+(k+l-2)e_{k+l-2}] & \text{for } l=0,1; \\
-4\sum_{j=1}^{l-1}\big[24l(2k^2-23)+(k^2-9)(k^2-4)(l-2)\big](k-l+2j)e_{k-l+2j} & \text{for } l=2,3; \\
-96l\sum_{j=1}^{l-1}(2k^2-23)(k-l+2j)e_{k-l+2j} & \\
-(k^2-9)(k^2-4)(l-2)[(k-l+2)e_{k-l+2}+(k+l-2)e_{k+l-2}] & \text{for } l\geqslant 4;
\end{array}\right\},
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
N_{k,l} = & \tfrac{1}{2}(k+1)(k+2)(k+3)(k-5)l(n_{k-l-4}-n_{k+l-4})-(k+2)(k+3)(2k^2-9k+13)l(n_{k-l-2}-n_{k+l-2}) \\
& +3(k^2-9)(k^2+2)l(n_{k-l}-n_{k+l})-(k-2)(k-3)(2k^2+9k+13)l(n_{k-l+2}-n_{k+l+2}) \\
& +\tfrac{1}{2}(k-1)(k-2)(k-3)(k+5)l(n_{k-l+4}-n_{k+l+4}),
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
G_{k,l} = & \tfrac{1}{8}((k+1)(k+2)(k+3)(g_{k-l-6}+g_{k+l-6})-6(k-1)(k+2)(k+3)(g_{k-l-4}+g_{k+l-4}) \\
& +15(k+1)(k-2)(k+3)(g_{k-l-2}+g_{k+l-2})-20k(k^2-7)(g_{k-l}+g_{k+l}) \\
& +15(k-1)(k+2)(k-3)(g_{k-l+2}+g_{k+l+2})-6(k+1)(k-2)(k-3)(g_{k-l+4}+g_{k+l+4}) \\
& +(k-1)(k-2)(k-3)(g_{k-l+6}+g_{k+l+6})),
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
F_k = & \tfrac{1}{4}(-(k+1)(k+2)(k+3)p_{k-6}+6(k-1)(k+2)(k+3)p_{k-4} \\
& -15(k+1)(k-2)(k+3)p_{k-2}+20k(k^2-7)p_k-15(k-1)(k+2)(k-3)p_{k+2} \\
& +6(k+1)(k-2)(k-3)p_{k+4}-(k-1)(k-2)(k-3)p_{k+6}),
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
R_{k,l} = & \frac{2}{\pi}\Bigg(\frac{(k+1)(k+2)(k+3)}{(k-l-4)^2-1}+\frac{(k+1)(k+2)(k+3)}{(k+l-4)^2-1} \\
& -\frac{4(k^2-4)(k+3)}{(k-l-2)^2-1}-\frac{4(k^2-4)(k+3)}{(k+l-2)^2-1}+\frac{6k(k^2-9)}{(k-l)^2-1}+\frac{6k(k^2-9)}{(k+l)^2-1}
\end{aligned}
$$

$$- \frac{4(k^2-4)(k-3)}{(k-l+2)^2-1} - \frac{4(k^2-4)(k-3)}{(k+l+2)^2-1} + \frac{(k-1)(k-2)(k-3)}{(k-l+4)^2-1} + \left. \frac{(k-1)(k-2)(k-3)}{(k+l+4)^2-1} \right) ((k+l+1) \bmod 2). \tag{48}$$

Parameters e_l, n_l, g_l, p_l in formulae (44)–(47) are expansion coefficients of this function

$$EJ(x) = \sum_{l=0}^{\infty}{}' e_l \mathrm{T}_l(x), \quad N(x) = \sum_{l=0}^{\infty}{}' n_l \mathrm{T}_l(x),$$
$$\rho_B(x) = \sum_{l=0}^{\infty}{}' g_l \mathrm{T}_l(x), \quad p(x) = \sum_{l=0}^{\infty}{}' p_l \mathrm{T}_l(x). \tag{49}$$

Details on the transformation of Eq. (34) can be found in the author's papers [10,11].

For $k = 0, 1, 2, 3$, the first four equations in the infinite system of algebraic Eqs. (43) are satisfied as regards identity (the number of the equations is equal to the order of Eq. (32)). The equations are replaced by equations representing boundary conditions. In the considered case, the boundary conditions have this form (see formulae (8))

$$m(\mp 1) = -EJ(x) \left. \frac{\partial^2 w(x)}{\partial x^2} \right|_{x=\mp 1} = 0,$$
$$t(\mp 1) = -\left(\frac{\partial}{\partial x} EJ(x) \right) \frac{\partial^2 w(x)}{\partial x^2} - EJ(x) \frac{\partial^3 w(x)}{\partial x^3} + nN(x) \left. \frac{\partial w(x)}{\partial x} \right|_{x=\mp 1} = 0. \tag{50}$$

To calculate the values of the forces at the bar's ends (formula (50)), one uses the Chebyshev series expansions of functions $EJ(x), N(x)$ and the following relations ([15, p. 48, (14), (16)])

$$T_n(-1) = (-1)^n, \quad T_n(1) = 1,$$
$$T_n'(-1) = -(-1)^n n^2, \quad T_n'(1) = n^2,$$
$$T_n''(-1) = (-1)^n n^2(n^2-1)/3, \quad T_n''(1) = n^2(n^2-1)/3,$$
$$T_n'''(-1) = -(-1)^n n^2(n^2-1)(n^2-4)/15, \quad T_n'''(1) = n^2(n^2-1)(n^2-4)/15. \tag{51}$$

The values of polynomials $T_n(x)$ and their derivatives at points ∓ 1 are substituted into formulae (49) specifying the expansions of functions EJ, N, $\partial EJ/\partial x$, to obtain

$$EJ_- = EJ(-1) = \sum_{l=0}^{\infty}{}' e_l \mathrm{T}_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l e_l, \quad EJ_+ = EJ(+1) = \sum_{l=0}^{\infty}{}' e_l \mathrm{T}_l(1) = \sum_{l=0}^{\infty}{}' e_l,$$
$$EJ'_- = \left. \frac{\partial EJ}{\partial x} \right|_{x=-1} = \sum_{l=0}^{\infty}{}' e_l \mathrm{T}_l'(-1) = -\sum_{l=0}^{\infty}{}' (-1)^l l^2 e_l, \quad EJ'_+ = \left. \frac{\partial EJ}{\partial x} \right|_{x=+1} = \sum_{l=0}^{\infty}{}' e_l \mathrm{T}_l'(1) = \sum_{l=0}^{\infty}{}' l^2 e_l,$$
$$N_- = N(-1) = \sum_{l=0}^{\infty}{}' n_l T_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l n_l, \quad N_+ = N(+1) = \sum_{l=0}^{\infty}{}' n_l \mathrm{T}_l(1) = \sum_{l=0}^{\infty}{}' n_l. \tag{52}$$

Using the calculated values of polynomials $T_n^{(m)}(\mp 1)$ and the values of constants $EJ_-, EJ_+, EJ'_-, EJ'_+, N_-, N_+$, one gets four additional equations describing the boundary conditions

$$m(-1) = m_- = -EJ_-\tfrac{1}{3}\sum_{l=0}^{\infty}{}'(-1)^l l^2(l^2-1)w_l = \sum_{l=0}^{\infty}{}' B_{0,l}w_l = 0,$$

$$m(+1) = m_+ = -EJ_+\tfrac{1}{3}\sum_{l=0}^{\infty}{}' l^2(l^2-1)w_l = \sum_{l=0}^{\infty}{}' B_{1,l}w_l = 0,$$

$$t(-1) = t_- = \sum_{l=0}^{\infty}{}'(-1)^l l^2\left[-\tfrac{1}{3}(l^2-1)EJ'_- + \tfrac{1}{15}(l^2-1)(l^2-4)EJ_- - nN_-\right]w_l = \sum_{l=0}^{\infty}{}' B_{2,l}w_l = 0,$$

$$t(+1) = t_+ = -\sum_{l=0}^{\infty}{}' l^2\left[\tfrac{1}{3}(l^2-1)EJ'_+ + \tfrac{1}{15}(l^2-1)(l^2-4)EJ_+ - nN_+\right]w_l = \sum_{l=0}^{\infty}{}' B_{3,l}w_l = 0. \qquad (53)$$

The four Eqs. (53) and Eqs. (43) (for $k = 4, 5, 6, \ldots$) form an infinite system of algebraic equations allowing one to determine the unknown coefficients of the expansion of displacement function $w(x)$ (formula (31)). If this system is written in a simplified form

$$\sum_{l=0}^{\infty}{}' B_{k,l}w_l = 0, \quad k = 0, 1, 2, 3,$$
$$\sum_{l=0}^{\infty}{}'\left(E_{k,l} + nN_{k,l} + \omega^2 gG_{k,l}\right)w_l = n\left(F_k + \sum_{l=0}^{\infty}{}' r_l R_{k,l}\right), \quad k = 4, 5, 6, \ldots. \qquad (54)$$

5. Expansion of beam—elastic half-plane interaction problem

In the considered problem, the following conditions of consistency between the half-plane and the beam (formula (5) with dimensionless quantities)

$$v(x, 0) = w(x) \quad \text{for } -1 \leqslant x \leqslant +1 \qquad (55)$$

corresponding to the relation

$$w_l = v_l, \quad l = 0, 1, 2, \ldots, \qquad (56)$$

where $v(x, 0)$ is the amplitude of displacement of the half-plane's boundary and $w(x)$ is the amplitude of vibration of the beam's axis, were assumed. After substituting relation (56) into system of Eqs. (54), using Eqs. (21), (22) and (27) and performing appropriate transformations, one gets the infinite system of algebraic equations

$$
\begin{aligned}
&-\frac{P_0}{a\mu}\left[{\sum_{m=0}^{\infty}}' r_{2m}\left({\sum_{l=0}^{\infty}}'(-1)^{m+l}B_{k,2l}\quad \varphi_{2l,2m}\right)\right.\\
&\left.+\sum_{m=0}^{\infty} r_{2m+1}\left(\sum_{l=0}^{\infty}(-1)^{m+l}B_{k,2l+1}\quad \varphi_{2l+1,2m+1}\right)\right]=0,\quad k=0,1,2,3,\\
&-{\sum_{m=0}^{\infty}}' r_{2m}\left(\frac{P_0}{a\mu}\left({\sum_{l=0}^{\infty}}'(-1)^{m+l}C_{k,2l}\quad \varphi_{2l,2m}\right)+nR_{k,2m}\right)\\
&-\sum_{m=0}^{\infty} r_{2m+1}\left(\frac{P_0}{a\mu}\left(\sum_{l=0}^{\infty}(-1)^{m+l}C_{k,2l+1}\quad \varphi_{2l+1,2m+1}\right)+nR_{k,2m+1}\right)\\
&=nF_k,\quad k=4,5,6,\ldots, \qquad (57)
\end{aligned}
$$

where

$$C_{k,l}=E_{k,l}+nN_{k,l}+\omega^2 gG_{k,l}. \qquad (58)$$

This system of equations allows one to determine unknown coefficients r_m. If coefficients r_m are substituted into formulae (21), the terms of the expansion of sought half-plane displacement function (17) and, if relation (55) is taken into account, beam displacement function (31) can be determined.

6. Numerical example

To illustrate the method, it will be applied to the problem of the harmonically excited vibration of the beam. The beam's cross-section is symmetrical relative to axis X and the function describing the beam's height has the form (Fig. 5)

$$H(X)=-a\big(\big((x/a)^2-1\big)-1\big)/50. \qquad (59)$$

After a change to dimensionless quantities (6) the function is expressed by the formula

$$h(x)=-((x^2-1)-1)/50. \qquad (60)$$

The other parameters for the problem are: for the beam $E=2.8\times10^{10}\,\mathrm{N/m^2}$; $\rho_{BV}=2400\,\mathrm{kg/m^3}$ (ρ_{BV} is mass per unit volume): $a=1\,\mathrm{m}$ and for the half-plane $\mu=1.5\times10^{8}\,\mathrm{N/m^2}$;

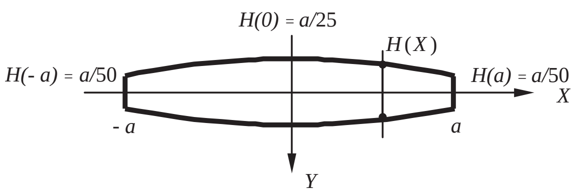


Fig. 5. Characteristic values of function $H(X)$ describing beam's height.

$\rho = 2000\,\mathrm{kg/m^3}$; $v = 0.3$. Two kinds of loads: a uniformly distributed load (Fig. 6) and a linearly variable antisymmetrical load (Fig. 7) act on the beam. The spatial distributions of the loads are expressed by these functions

$$P_1(X) = p_1 = \frac{P_0}{2a}, \quad P_2(X) = p_2\,\frac{X}{a} = \frac{3P_0}{2a^2}\,X. \tag{61}$$

The functions in the dimensionless form are described by the formulae

$$p_1(x) = \tfrac{1}{2}, \quad p_2(x) = \tfrac{3}{2}\,x. \tag{62, 63}$$

The beam's dimensionless geometric characteristics after the expansion into Chebyshev series assume the form

$$\begin{aligned} \mathrm{J}(x) &= \tfrac{1}{48}\big(\tfrac{1}{2} \times 252\mathrm{T}_0(x) - 111\mathrm{T}_2(x) + 18\mathrm{T}_4(x) - \mathrm{T}_6(x)\big) \times 10^{-6}, \\ A(x) &= \big(\tfrac{1}{2} \times 6\mathrm{T}_0(x) - \mathrm{T}_2(x)\big) \times 10^{-2}, \end{aligned} \tag{64}$$

where $A(x)$ is the beam's cross-sectional area. Hence after the substitution of the other numerical values one gets $EJ(x)$ and $\rho_B = \rho_{BV} A(x)$. Calculations were performed for four excitation frequencies: $\omega_1 = 25\pi\,\mathrm{rad/s}$, $\omega_2 = 50\pi\,\mathrm{rad/s}$, $\omega_3 = 100\pi\,\mathrm{rad/s}$, $\omega_4 = 200\pi\,\mathrm{rad/s}$. In order to solve an infinite system of algebraic Eqs. (57), it is limited to a finite system. Then the displacement function and the passive foundation pressure function are expressed by the finite sums of Chebyshev series

$$\mathbf{w}(x) = {\sum_{l=0}^{lw}}' w_l \mathrm{T}_l(x), \quad \mathbf{r}(x) = (1 - x^2)^{-1/2} {\sum_{l=0}^{lw}}' r_l \mathrm{T}_l(x). \tag{65}$$

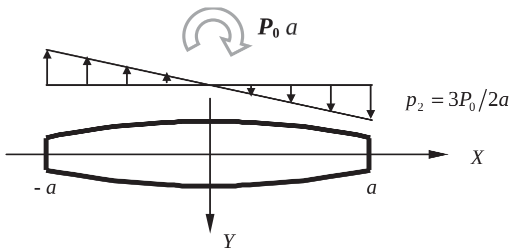


Fig. 6. Spatial distribution of load acting on system—symmetrical load.

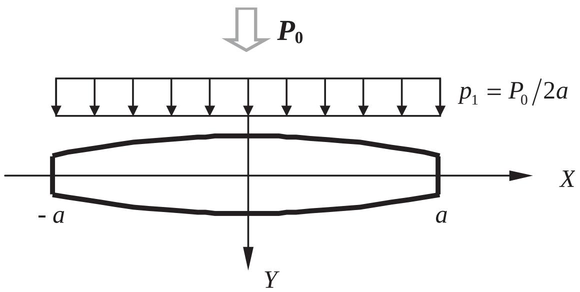


Fig. 7. Spatial distribution of load acting on system—asymmetrical load.

Testing the convergence of the solutions, the system was solved for an ever larger approximating base size $lw = 12$, 24, 36, 48 and $lw = 13$, 25, 37, 49 for the symmetrical load and the asymmetrical load, respectively. The functions yielded by the calculations are shown in Figs. 8–11. Since functions $\mathbf{r}(x)$, $\mathbf{w}(x)$ assume complex values, their real parts Re(z) and imaginary parts Im(z) are shown in the figures which also include functions abs(z), arg(z) representing, respectively, the modulus and the argument of complex number $\mathbf{z}$. Graphs of function $\mathbf{r}(x)\sqrt{1-x^2}$, determined for the loads described by respectively formula (62) and (63), are shown in Figs. 8 and 10. Graphs of displacement function $\mathbf{w}(x)$ are shown in Figs. 9 and 11. The calculations were performed for excitation frequency $\omega = 100\pi$ rad/s. The graphs shown in Figs. 12–15 represent the relationships between functions $\mathbf{r}(x)\sqrt{1-x^2}$, $\mathbf{w}(x)$ calculated for different excitation frequencies $\omega = 25\pi, 50\pi, 100\pi, 200\pi$ rad/s In this case, parameter lw equals 48 or 49. Complex displacement functions $\mathbf{w}(0)$, $\mathbf{w}(1)$ (for the beam's middle and end) versus dimensionless frequency parameter $\omega a/c_2$ for a case when the respective load shown in Figs. 6 and 7 acts on the system are plotted in Figs. 16 and 17. Frequencies $\omega = 25\pi, 50\pi, 100\pi, 200\pi$ rad/s assumed for the calculations correspond to the following dimensionless frequency parameter values $\omega a/c_2 = 0.287, 0.573, 1.147, 2.294$.

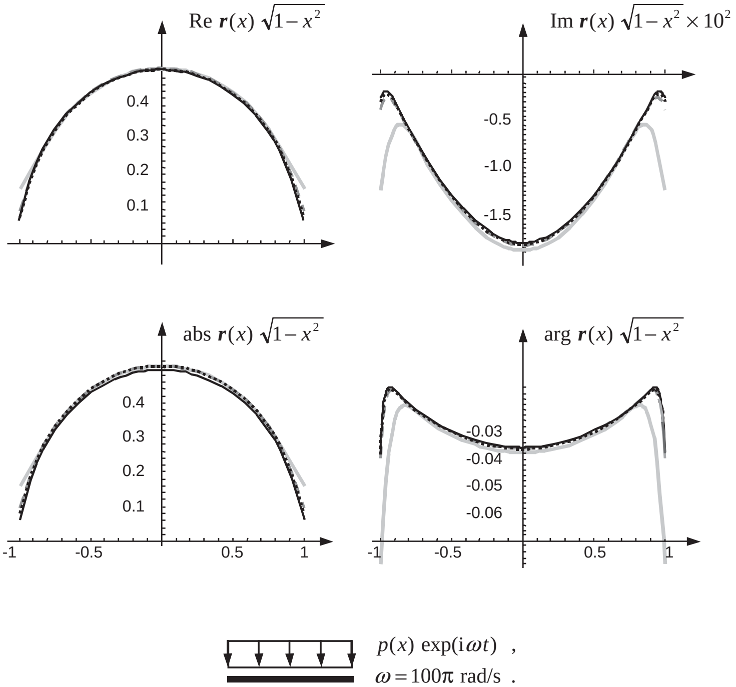


Fig. 8. Graphs of complex function $\mathbf{r}(x)\sqrt{1-x^2}$ ($\mathbf{r}(x)$—passive foundation pressure function) for different approximating base sizes $lw = 12$ (——), 24 (- - -), 36 (········), 48 (——) in case when load shown in Fig. 6 acts on system.

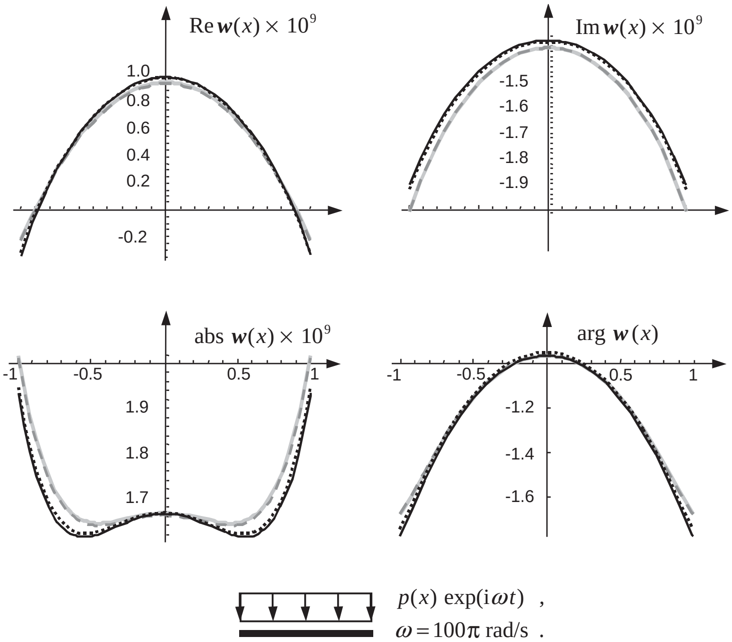


Fig. 9. Graphs of complex function $\mathbf{w}(x)$ for different approximating base sizes $lw = 12$ (——), 24 (- - -), 36 (········), 48 (——) in case when load shown in Fig. 6 acts on system.

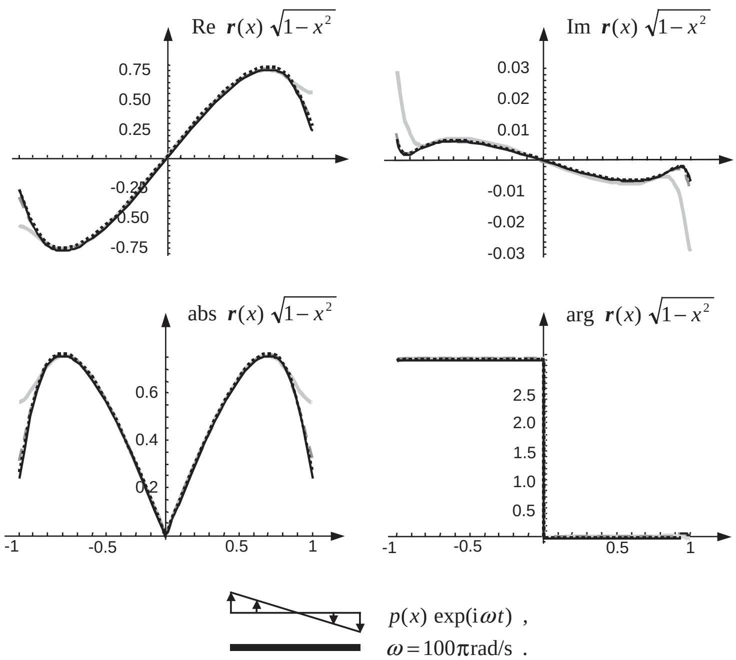


Fig. 10. Graphs of complex function $\mathbf{r}(x)\sqrt{1-x^2}$ ($\mathbf{r}(x)$—passive foundation pressure function) for different approximating base sizes $lw = 13$ (——), 25 (- - -), 37 (········), 49 (——) in case when load shown in Fig. 7 acts on system.

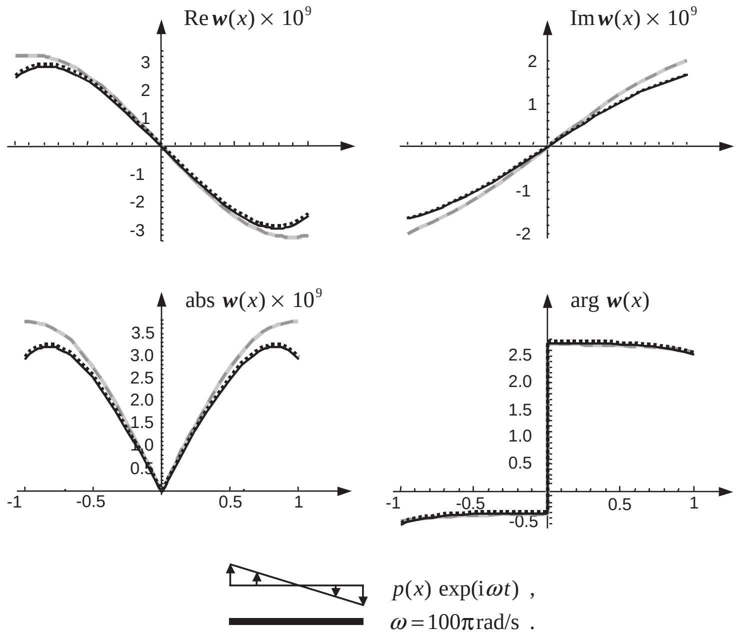


Fig. 11. Graphs of complex function $\mathbf{w}(x)$ for different approximating base sizes $lw = 13$ (——), 25 (- - -), 37 (········), 49 (——) in case when load shown in Fig. 7 acts on system.

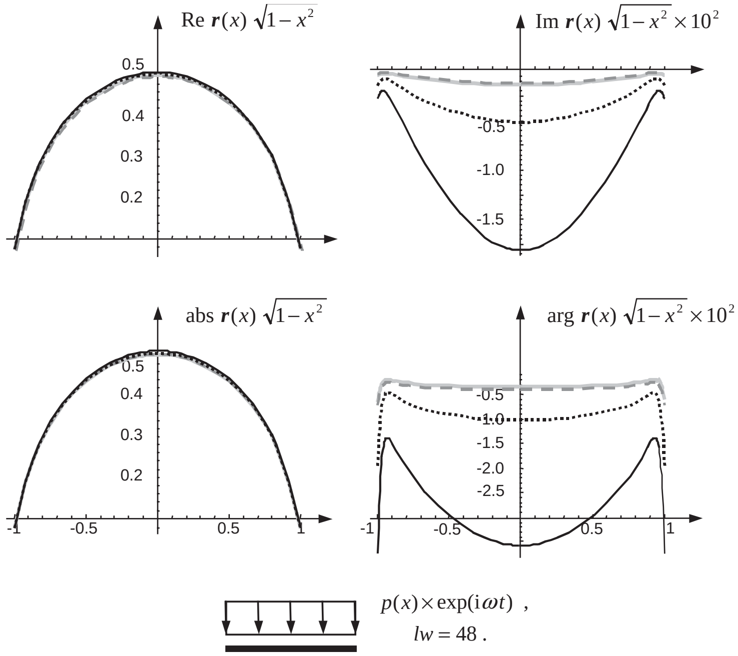


Fig. 12. Graphs of complex function $\mathbf{r}(x)\sqrt{1-x^2}$ for different excitation frequencies $\omega = 25\pi$ (——), 50π (- - -), 100π (········), 200π (——) in case when load shown in Fig. 6 acts on system.

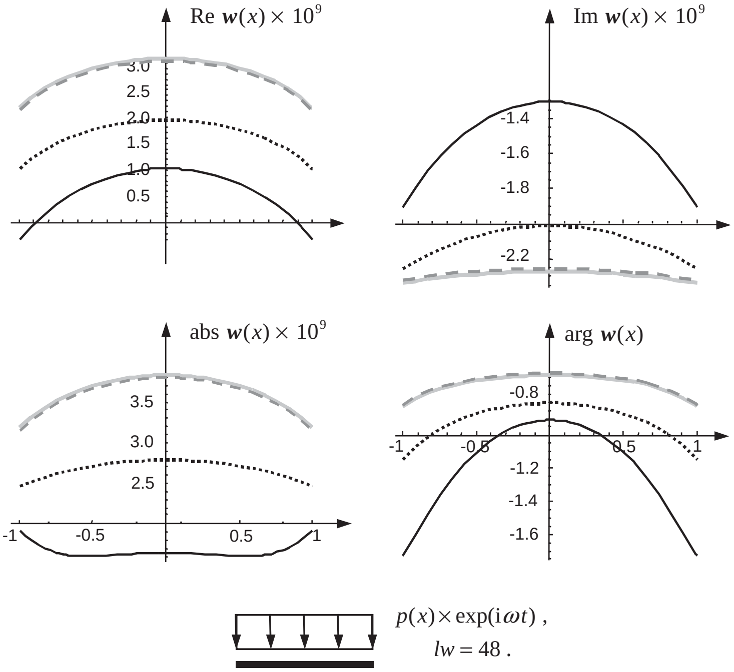


Fig. 13. Graphs of complex function $\mathbf{w}(x)$ for different excitation frequencies $\omega = 25\pi$ (——), 50π (- - -), 100π (········), 200π (——) in case when load shown in Fig. 6 acts on system.

To obtain passive foundation pressure–time t relations one should use relations (6) and perform the transformations

$$\begin{aligned} r(x,t) &= r_c(x,t) + \mathrm{i} r_s(x,t) = \mathbf{r}(x)\exp(\mathrm{i}\omega t) = (\operatorname{Re}\mathbf{r}(x) + \mathrm{i}\operatorname{Im}\mathbf{r}(x))\exp(\mathrm{i}\omega t) \\ &= \operatorname{abs}(\mathbf{r}(x))\exp(\mathrm{i}(\omega t + \arg(\mathbf{r}(x)))) \\ &= \operatorname{abs}(\mathbf{r}(x))\cos(\omega t + \arg(\mathbf{r}(x))) + \mathrm{i}\operatorname{abs}(\mathbf{r}(x))\sin(\omega t + \arg(\mathbf{r}(x))). \end{aligned} \tag{66}$$

Similarly as for the displacement function, one gets

$$\begin{aligned} w(x,t) &= \mathbf{w}(x)\exp(\mathrm{i}\omega t) = w_c(x,t) + \mathrm{i} w_s(x,t) \\ &= \operatorname{abs}(\mathbf{w}(x))\cos(\omega t + \arg(\mathbf{w}(x))) + \mathrm{i}\operatorname{abs}(\mathbf{w}(x))\sin(\omega t + \arg(\mathbf{w}(x))). \end{aligned} \tag{67}$$

Graphs of functions $r_c(x,t)$, $r_s(x,t)$ and $w_c(x,t)$, $w_s(x,t)$ for different values of time t are shown in Figs. 18–21. Similarly as in the previous cases, the action of loads specified by formulae (62) and (63) was analyzed.

It follows from the obtained results that a sufficiently good approximation of the solutions is obtained for $lw = 36$ (37). The differences between the results for $lw = 36$ (37) and $lw = 48$ (49) are slight. Considering an influence of the dimensionless frequency parameter $\omega a/c_2$ on the

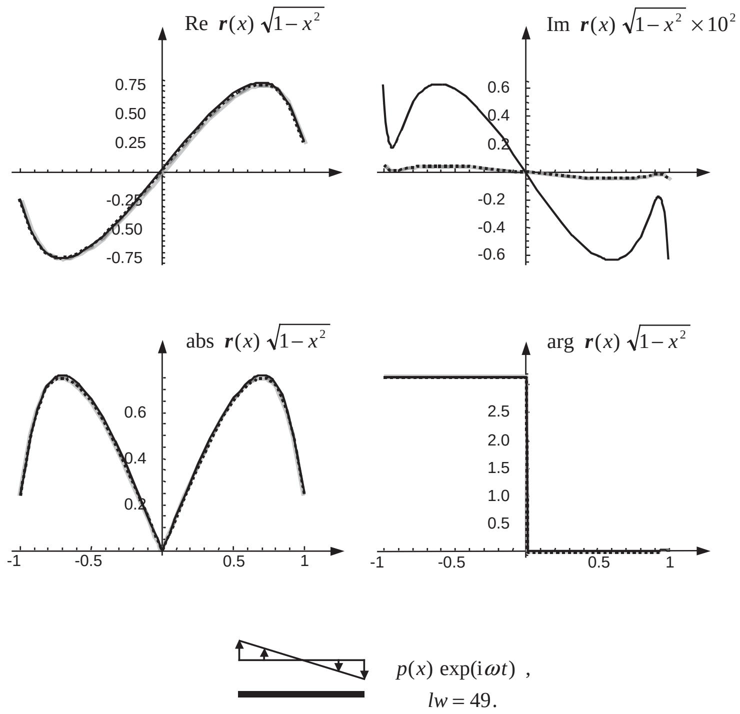


Fig. 14. Graphs of complex function $\mathbf{r}(x)\sqrt{1-x^2}$ for different excitation frequencies $\omega = 25\pi$ (——), 50π (- - -), 100π (········), 200π (——) in case when load shown in Fig. 7 acts on system.

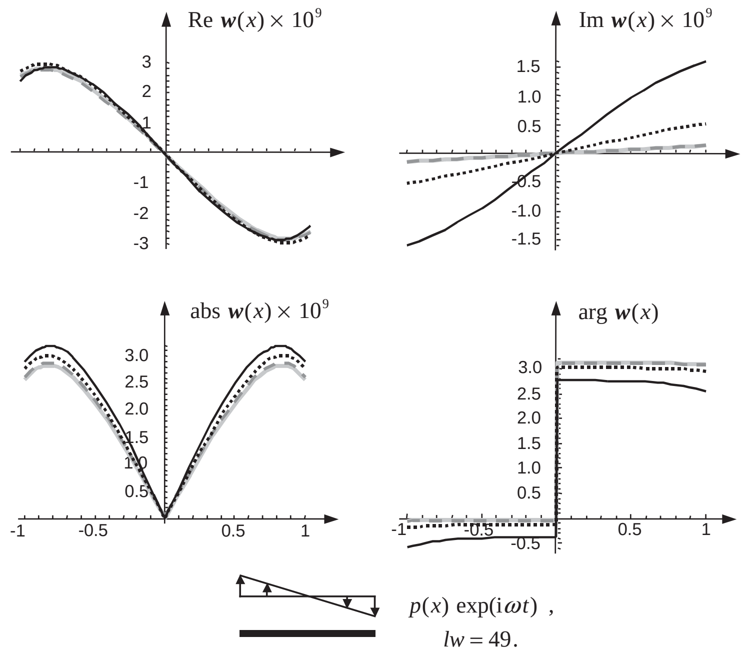


Fig. 15. Graphs of complex displacement function $w(x)$ for different excitation frequencies $\omega = 25\pi$ (——), 50π (- - -), 100π (········), 200π (——) in case when load shown in Fig. 7 acts on system.

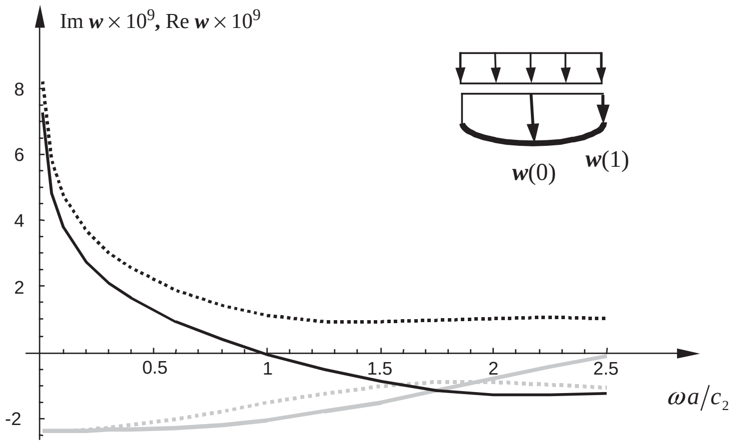


Fig. 16. Graphs of complex displacement functions **w**(1) (——, Re; ——, Im), **w**(0) (·······, Re; ·······, Im) for different values of dimensionless frequency parameter $\omega a/c_2$when system is subjected to load shown in Fig. 6.

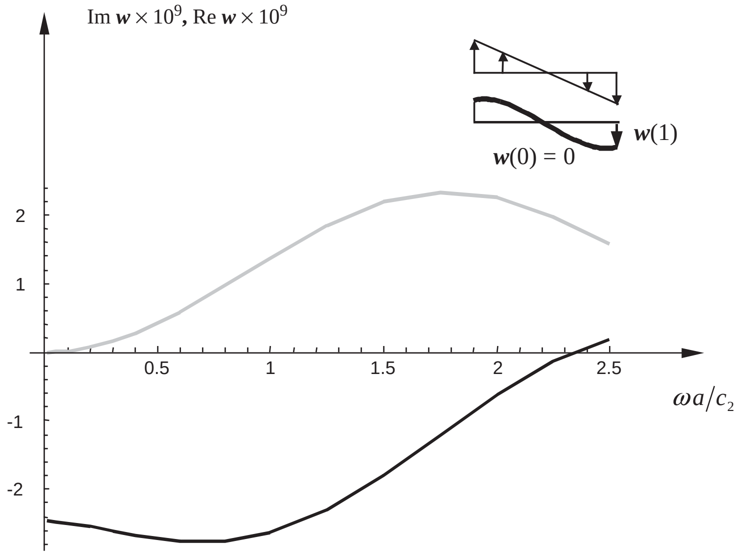
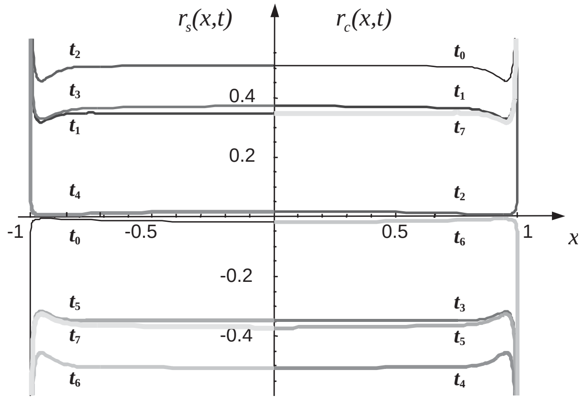


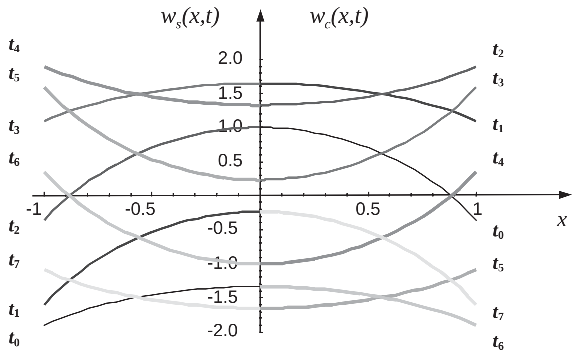
Fig. 17. Graphs of complex displacement function **w**(1) (——, Re; ——, Im) for different values of dimensionless frequency parameter $\omega a/c_2$when system is subjected to load shown in Fig. 7.

displacement function one notes that it is significant for the lower dimensionless frequencies ($\omega a/c_2 < 1$) and small for the higher ones ($\omega a/c_2 > 1$), when the load is symmetric (Fig. 16). It the case of asymmetrical load the influence in question is significant in the whole given range of parameters (Fig. 17).



$t_j = 0.25 \times 10^{-2}\ j$ (s)

Fig. 18. Graphs of passive foundation pressure functions $r_s(x,t) = \mathrm{abs}(\mathbf{r}(x))\sin(100\pi t + \arg(\mathbf{r}(x)))$ and $r_c(x,t) = \mathrm{abs}(\mathbf{r}(x))\cos(100\pi t + \arg(\mathbf{r}(x)))$ for different values of time t in case when load shown in Fig. 6 acts on system.



$t_j = 0.25 \times 10^{-2}\ j$ (s)

Fig. 19. Graphs of displacement functions $w_s(x,t) = \mathrm{abs}(\mathbf{w}(x))\sin(100\pi t + \arg(\mathbf{w}(x)))$ and $\mathbf{w}_c(x,t) = \mathrm{abs}(\mathbf{w}(x))\cos(100\pi t + \arg(\mathbf{w}(x)))$ for different values of time t in case when load shown in Fig. 6 acts on system.

7. Conclusion

The obtained theoretical results and the provided numerical example validate the proposed method of solving the problem of vibration of non-prismatic beams resting on an inertial elastic foundation and demonstrate its usefulness. Because of the very good approximating properties of Chebyshev polynomials, it seems that the proposed method will be particularly useful for solving the problem of vibration of beams with complex geometry and an arbitrary distribution of mass and strength parameters. The formulae derived in this paper are readily applicable to complex cases of harmonically excited vibration.

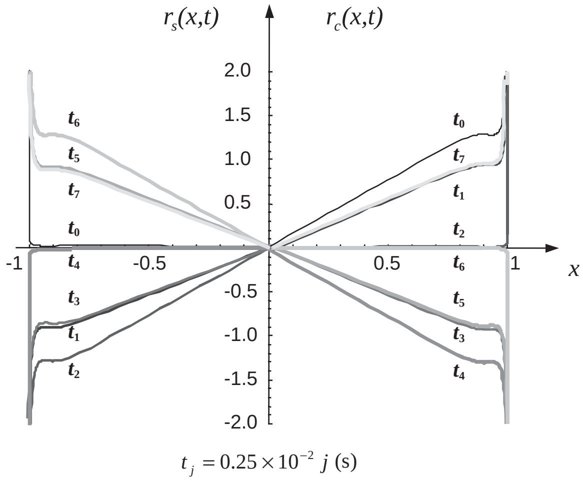


Fig. 20. Graphs of passive foundation pressure functions $r_s(x,t) = \text{abs}(\mathbf{r}(x))\sin(100\pi t + \arg(\mathbf{r}(x)))$ and $r_c(x,t) = \text{abs}(\mathbf{r}(x))\cos(100\pi t + \arg(\mathbf{r}(x)))$ for different values of time t in case when load shown in Fig. 7 acts on system.

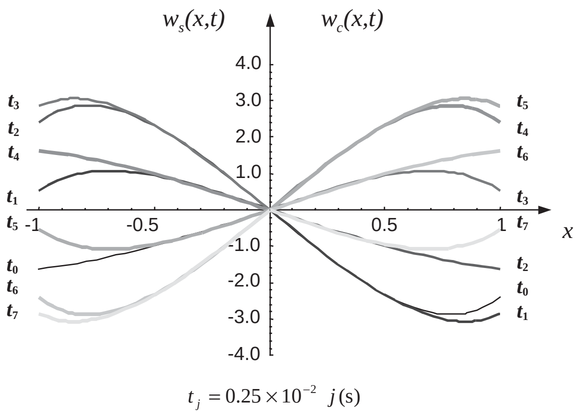


Fig. 21. Graphs of displacement functions $w_s(x,t) = \text{abs}(\mathbf{w}(x))\sin(100\pi t + \arg(\mathbf{w}(x)))$ and $w_c(x,t) = \text{abs}(\mathbf{w}(x))\cos(100\pi t + \arg(\mathbf{w}(x)))$ for different values of time t in case when load shown in Fig. 7 acts on system.

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