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Analysis of axial vibration of compound bars by differential transformation method

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Abstract

Although there are solutions for the axial vibration of *compound* bars, i.e., bars of different cross-section connected in mechanical series, this note is to illustrate another solution. The primary objective is to analyze this problem by a relatively new technique originated in 1986 and known as the *differential transformation* (DT) method. The numerical solutions are compared with classical exact solution and other studies in the literature. The accuracy, simplicity, and effectiveness of the DT approach are demonstrated.

1. Introduction

In certain design applications, it is advantageous to use several bars, each having different properties (cross-sectional area, length, mass, and elastic modulus) and connected in *mechanical series*. The individual portions are denoted as segments and the entire system is called a *compound* bar (see Fig. 1). Such a system is analyzed here.

The first method of solution used here is the exact, classical solution of the set of governing differential equations. However, this method becomes very unwieldy if there are more than two segments. For instance, for a two-segment bar, the transcendental frequency equation consists of two terms containing products of sines and cosines, i.e., to the second degree in trigonometric functions. In contrast, the three-segment-bar frequency equation consists of sixteen terms containing trigonometric functions to the fifth degree.

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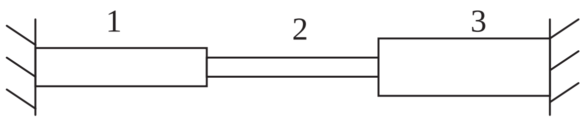

Fig. 1. A representative compound bar having three segments.

In view of the aforementioned difficulties, it is desirable to investigate alternative methods of solution. Here we use a relatively new method introduced by Zhou [1] to solve problems involving electrical circuits. This method, called the *differential transformation* (DT) method, is based on the Taylor series expansion and is exact in a series sense, i.e., it converges to the exact solution. This method was first applied to eigenvalue problems by Chen and Ho [2]. Since then, researchers started applying this DT method to solve many engineering problems [3–8]. These studies have demonstrated that the DT method is simple and effective.

2. Classical solution

The system considered is a compound bar consisting of an arbitrary number (n) of prismatic segments. The governing differential equations of motion are

$$a_i^2 u_{i,xx} = u_{i,tt}, \quad i = 1, \ldots, n, \tag{1}$$

where a_i is the acoustic wave velocity of a typical segment i, $u_i = u_i(x,t)$ is the axial displacement of segment i at position x and time t, and $(\ )_{,xx}$ denotes $\partial^2(\ )/\partial x^2$.

The general solutions of Eq. (1) are

$$u_i(x,t) = U_i(x)\cos\omega t, \tag{2}$$

where ω is the circular natural frequency and $U_i(x)$, the mode shape of segment i, is governed by

$$a_i^2 U_{i,xx} = -\omega^2 U_i. \tag{3}$$

The general solutions for the mode shapes are

$$U_i(x) = \alpha_i \cos(\omega x/a_i) + \beta_i \sin(\omega x/a_i), \tag{4}$$

where α_i and β_i are constants of integration.

The origin of the co-ordinate system is taken to be at the left end of segment 1. Considering the n-segment compound bar to be fixed at both ends, for instance, one can write the two boundary conditions and $2(n-1)$ junction conditions as follows

$$\begin{aligned} & U_1(0) = 0, \\ & U_i\left(\sum L_i\right) = U_{i+1}\left(\sum L_i\right), \\ & A_i E_i U_{i,x}\left(\sum L_i\right) = A_{i+1}E_{i+1}U_{i+1,x}\left(\sum L_i\right), \\ & \vdots \\ & U_n\left(\sum L_n\right) = 0, \end{aligned} \tag{5}$$

where $\Sigma L_i = \sum_{j=1}^{i} L_j$ and $\Sigma L_n = \sum_{j=1}^{n} L_j$.

For the example of a two-segment bar, Eqs. (5) reduce to

$$U_1(0) = 0, \quad U_1(L_1) = U_2(L_1), \quad A_1E_1U_{1,x}(L_1) = A_2E_2U_{2,x}(L_1), \quad U_2(L_1 + L_2) = 0. \tag{6}$$

Substituting Eq. (4) into Eqs. (5) leads to a set of $2n$ homogeneous algebraic equations in the coefficients α_i and β_i ($i = 1, \ldots, n$). The determinant of this set of equations must be set equal to zero to guarantee a non-trivial solution. This frequency determinant is transcendental in the frequency ω because its coefficients are of the form of both sine and cosine functions. For a two-segment bar, for instance, the frequency equation is

$$\tan(\omega L_1/a_1)\cot(\omega L_2/a_2) = -(a_2A_1E_1)/(a_1A_2E_2). \tag{7}$$

3. DT method

The problem can also be solved by use of the DT method. An arbitrary function $f(x)$ can be expanded in a Taylor series about a point $x = 0$ as

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k f}{dx^k}\right]_{x=0}. \tag{8}$$

The kth order differential transform of a function $f(x)$ about a point $x = 0$ is defined as

$$F(k) = \frac{1}{k!}\left[\frac{d^k f}{dx^k}\right]_{x=0}, \tag{9}$$

and the inverse differential transform is

$$f(x) = \sum_{k=0}^{\infty} x^k F(k). \tag{10}$$

Taking the differential transform of the governing equations, Eq. (3), yields

$$F_i(k+2) = -(\omega/a_i)^2 \frac{F_i(k)}{(k+1)(k+2)}, \tag{11}$$

where F_i is the differential transform of $U_i(x)$. Using the inverse differential transform relationship of Eq. (10), one can express the two boundary conditions and $2(n-1)$ junction conditions, Eqs. (5), in terms of $F_i(0)$, $F_i(1)$, and natural frequency ω. To avoid a trivial solution, the determinant of the coefficients of α_i and β_i must vanish and thus the natural frequency ω can be found.

For the example of a two-segment bar fixed at both ends, the differential transforms of the governing equations are

$$F_1(k+2) = -(\omega/a_1)^2 \frac{F_1(k)}{(k+1)(k+2)}, \quad F_2(k+2) = -(\omega/a_2)^2 \frac{F_2(k)}{(k+1)(k+2)}. \tag{12}$$

Also, the differential transforms of the boundary and junction conditions (6) are

$$F_1(0) = 0, \quad \sum_{k=0}^{\infty} (L_1)^k F_1(k) = \sum_{k=0}^{\infty} (L_1)^k F_2(k),$$
$$A_1 E_1 \sum_{k=0}^{\infty} k(L_1)^{k-1} F_1(k) = A_2 E_2 \sum_{k=0}^{\infty} k(L_1)^{k-1} F_2(k),$$
$$\sum_{k=0}^{\infty} (L_1 + L_2)^k F_2(k) = 0. \tag{13}$$

Substituting Eq. (11) into Eqs. (13) yields a set of four simultaneous algebraic equations with undetermined differential transforms $F_1(0)$, $F_1(1)$, $F_2(0)$, and $F_2(1)$. Setting the determinant equal to zero results in a polynomial equation in frequency squared (ω^2) of order six. For instance, if three terms of the series are used and $k_2 = 1$, $k_1/k_2 = \delta$, $L_1 = L_2 = 1$, and $m_1 = m_2 = 1$, the polynomial frequency equation is

$$\left(\frac{1}{6} - \frac{1}{18\delta}\right)(\omega^2)^3 + \left(\frac{1}{4} - \frac{\delta}{3} - \frac{1}{12\delta}\right)(\omega^2)^2 + \left(1 + \frac{1}{6\delta}\right)(\omega^2) - (\delta + 1) = 0, \tag{14}$$

where k_1 and k_2 are the axial stiffnesses ($= A_1E_1/L_1$ and A_2E_2/L_2) of segments 1 and 2.

4. Numerical results

As a first step toward evaluating the DT method, the case of a two-segment bar is studied for various values of the ratio k_1/k_2. The following parameters are held fixed: $k_2 = 1$, $L_1 = L_2 = 1$, and $m_1 = m_2 = 1$. The convergence of the DT method for the same two-segment bar is illustrated in Table 1. The exact solution is the iterative solution of Eq. (7). It is noted that for k_1/k_2 values up to 1.02, convergence requires only 15 terms, while for k_1/k_2 values of 4.00 and 9.00, 20 terms are needed.

A question arises about the effect of symmetry on the rate of convergence of the DT solution. To study this, two three-segment bars are considered. Both have the same material and

Table 1
Convergence of the DT solution for the fundamental natural frequency ω (rad/s) for a bar system consisting two segments and fixed at both ends (Q is the number of terms)

k_1/k_2	Exact	$Q = 8$	$Q = 9$	$Q = 10$	$Q = 15$	$Q = 20$	$Q = 25$
0.25	1.1503	1.1445	1.1509	1.1506	**1.1503**	1.1503	1.1503
0.49	1.3329	1.3254	1.3346	1.3335	**1.3329**	1.3329	1.3329
0.50	1.3388	1.3309	1.3404	1.3393	**1.3388**	1.3388	1.3388
0.98	1.5629	1.5327	1.5664	1.5662	**1.5629**	1.5629	1.5629
1.00	1.5708	1.5393	1.5743	1.5743	**1.5708**	1.5708	1.5708
1.02	1.5786	1.5458	1.5822	1.5824	**1.5786**	1.5786	1.5786
4.00	2.3005	2.0228	2.0826	4.6114	2.3016	**2.3005**	2.3005
9.00	2.7352	4.3903	2.1756	2.5302	2.7594	**2.7352**	2.7352

Table 2
Values of fundamental natural frequency ω (rad/s) for two different three-segment bars each being fixed at both ends

Case	Exact	DT	DT converged at
1 (symmetric)	1.0696	1.0696	$Q = 14$
2 (nonsymmetric)	1.4881	1.4881	$Q = 12$

Table 3
The three lowest natural frequencies (rad/s) for the four-segment shaft which is free at both ends

Mode	Beddoe [9]	Exact
1	5537	5551.1
2	8395	8415.5
3	12047	12080.1

Table 4
Input data for two multi-story buildings modeled by Li [10][1]

20-Story building				16-Story building			
Segment	AE (N)	L (m)	m (kg/m)	Segment	AE (N)	L (m)	m (kg/m)
1 (base)	3.46 e11	12.9	3.49 e5	1	2.56 e11	13.0	2.91 e5
2	3.35 e11	11.6	3.29 e5	2	2.32 e11	12.0	2.97 e5
3	3.21 e11	11.6	3.18 e5	3	2.20 e11	12.0	2.90 e5
4	2.94 e11	11.6	3.09 e5	4	2.10 e11	12.0	2.93 e5
5 (top)	2.74 e11	11.6	2.91 e5				

[1] Boundary conditions: fixed at the base, free at the top.

$L_1 = L_2 = L_3 = 1$ and $m_1 = m_2 = m_3 = 1$. Case 1 (symmetric): $k_1 = 1$, $k_2 = 4$, $k_3 = 1$. Case 2 (nonsymmetric): $k_1 = 1$, $k_2 = 2$, $k_3 = 4$.
The results are presented in Table 2. The DT results for Case 2 converge slightly faster than those for Case 1.

The next example considered is a four-segment circular-section shaft undergoing free torsional vibration, which was studied by Beddoe [9]. Its boundary condition is considered free–free. The input data are: material properties (steel): $G = 79.3$ GPa, specific weight: 75.4 $\mathrm{kN/m^3}$, diameters: $d_1 = d_4 = 0.0508$ m, $d_2 = 0.0762$ m, $d_3 = 0.0635$ m, lengths: $L_1 = 1.310$ m, $L_2 = L_3 = 0.457$ m, $L_4 = 0.762$ m.

The first three natural frequencies are listed in Table 3. The changes necessary in our equations to model the free–free boundary conditions are presented in Appendix A. As can be seen, the agreement is very good.

Li [10] presented different ways to model the axial vibration of a multi-story building: ‘‘tapered bar’’ model, ‘‘multi-segment bar’’ model, and the finite element method. Of course, the boundary conditions used were fixed at the base and free at the top. Li [10] also reported experimental

Table 5
The three lowest natural frequencies (rad/s) for the two multi-story buildings specified in Table 4

Mode	Li analyt.	Li exp.	DT	No. of terms req'd. for DT convergence
20-Story building				
1	26.3	27.3	27.6	15
2	70.4	—	79.1	21
3	110.3	—	131.2	25
16-Story building				
1	29.7	29.9	29.1	15
2	77.8	—	85.1	15
3	124.5	—	141.8	25

results for the fundamental frequency. His results by the three different methods agreed well with the experimental results for two different buildings, a 20-story and a 16-story one. Again, the changes necessary in our equations to model the boundary condition of free at the top are explained in Appendix A. Input data from Li's Model 2 are listed in Table 4, and results are presented in Table 5. The agreement between the present results and the other investigators' results is good but not as good as in Table 3.

The addition of lumped masses and/or springs can easily be accommodated by incorporating them in the "junction" or "end" boundary conditions as desired. However, in the interest of brevity, no numerical results are presented for this situation.

5. Concluding remarks

Another exact method for free vibration analysis of compound axial or torsional bars has been presented and evaluated. This method is a relatively new technique, known as the differential transformation method. Analyses were applied to various example problems with up to five segments, including classical solutions and some taken from the literature. Agreement among the present exact methods and those of others was very good. Agreement between the exact classical method and the DT method was perfect, shown in Tables 1–3, since DT is also an exact method. Exact results by the classical method were not calculated. However, the present results were compared with those of Li and the agreement was fair.

It should be pointed out that the DT method was more convenient to program than the classical method.

Appendix A. Modifications necessary to handle other boundary conditions and tapered compound bars

Some of the problems treated in the body of the paper have differential boundary conditions at the ends of the compound bar than the main case of fixed at both ends. These changes in boundary conditions change certain equations in the body of the paper as described here.

A.1. Free–free boundary conditions

In the case of torsional vibration of a circular-section shaft, all of the U's, A's, and E's in Eqs. (5) are replaced by θ's, J's, and G's, respectively. For the case of both ends free, the first and last equations in Eqs. (5) become

$$\theta_{1,x}(0) = 0, \quad \theta_{n,x}(\Sigma L_n) = 0.$$

A.2. Fixed–free boundary conditions

In the case of a building model, the base is fixed and the top is free. This necessitates only one change in Eqs. (5); the last equation in the set becomes

$$U_{n,x}(\Sigma L_n) = 0.$$

A.3. Tapered compound bars

By combining the present methodology for compound bars with the methodology of Ref. [7] for tapered bars, one can handle tapered compound bars.

References

[1] J.K. Zhou, *Differential Transformation and its Application for Electric Circuits*, Huazhong University Press, Wuhan, P. R. China, 1986 (in Chinese).
[2] C.K. Chen, S.H. Ho, Application of differential transformation to eigenvalue problems, *Applied Mathematics and Computation* 79 (1996) 173–188.
[3] M.J. Jang, C.L. Chen, Analysis of the response of a strongly nonlinear damped system using a differential transformation technique, *Applied Mathematics and Computation* 88 (1997) 137–151.
[4] M. Malik, H.H. Dang, Vibration analysis of continuous systems by differential transformation, *Applied Mathematics and Computation* 96 (1998) 17–26.
[5] C.K. Chen, S.H. Ho, Transverse vibration of a rotating twisted Timoshenko beam under axial loading using differential transform, *International Journal of Mechanical Sciences* 41 (1999) 1339–1356.
[6] M. Malik, M. Allali, Characteristic equations of rectangular plates by differential transformation, *Journal of Sound and Vibration* 233 (2) (2000) 359–366.
[7] H. Zeng, C.W. Bert, Vibration analysis of a tapered bar by differential transformation, *Journal of Sound and Vibration* 242 (2) (2000) 737–739.
[8] M. Malik, C.W. Bert, Solution of eigenvalue problems for rectangular plates by differential transform, in: P. Schiavone, C. Costanda, A. Mioduchowski (Eds.), *Integral Methods in Science and Engineering*, Birkhauser, Boston, 2002, pp. 173–178.
[9] B. Beddoe, Wave theory of free torsional vibration of composite systems of shafts, *Journal of Mechanical Engineering Science* 7 (1965) 48–56.
[10] Q.S. Li, Exact solutions for free longitudinal vibrations of non-uniform rods, *Journal of Sound and Vibration* 234 (2000) 1–19.