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The ensemble statistics of the band-averaged energy of a random system

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Abstract

This paper is concerned with the response statistics of a dynamic system that has random properties. The frequency-band-averaged energy of the system is considered, and a closed form expression is derived for the relative variance of this quantity. The expression depends upon three parameters: the modal overlap factor m, a bandwidth parameter B, and a parameter α that defines the nature of the loading (for example single point forcing or rain-on-the-roof loading). The result is applicable to any single structural component or acoustic volume, and a comparison is made here with simulation results for a mass loaded plate. Good agreement is found between the simulations and the theory.

1. Introduction

In many practical situations the response of a dynamic system can be sensitive to small variations introduced during the manufacturing process, so that successive items from a production line can display a significant random spread in performance. This is true, for example, in the automotive industry, where there can be a large difference in the interior noise levels arising in nominally identical vehicles [1]. It is of interest to determine the statistics of the performance over the ensemble of random products, so that some estimate can be made of the percentage of items that fail to meet the design requirements. Predicting the performance statistics is particularly difficult at high frequencies of excitation where many modes of vibration of the system can contribute to the response, and this situation arises frequently in the study of noise and vibration levels in automotive, marine, and aerospace structures. In this case the mean energy of the response can be estimated by using statistical energy analysis (SEA) [2], although the method does

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not presently enable a robust calculation of the variance of the energy. This provides the motivation for the present work, which is concerned with response variance prediction for the case of high-frequency excitation.

As a prelude to considering the response of a complex built-up system, the response of a single random structural component (or acoustic volume) to harmonic excitation has been considered in a recent paper [3]. Specifically, the mean and variance of the kinetic energy across an ensemble of realizations of the system were evaluated as a function of the harmonic forcing frequency. The analysis was based on the assumption that the natural frequencies form a random point process with statistics governed by the Gaussian orthogonal ensemble (GOE) [4]. This built on earlier work by Weaver [5], and ultimately Lyon [6] and Davy [7], who both considered Poisson natural frequency statistics rather than GOE statistics. In Ref. [3] a closed form solution was presented for the relative variance (the variance divided by the square of the mean) as a function of the modal overlap factor m, and a parameter α that is related to the nature of the loading (for example a single point force or rain-on-the-roof forcing). This solution was agreed with simulations and experiments for the case of a mass-loaded plate. The purpose of the present work is to extend the analysis of Ref. [3] to the ensemble average of the frequency-band-averaged energy, as opposed to the energy at a single excitation frequency, since often the practical concern is with band-averaged response levels (for example in interior noise assessments). As in Ref. [3], the present analysis concerned with a single structural component, or acoustic volume, rather then the more general problem of a built-up system.

The variance of the band-averaged energy of a single component has previously been considered by Lyon [6], who presented graphical data for the relative variance as a function of the averaging bandwidth. These results were derived on the assumption that the natural frequencies form a Poisson point process, although the effect of "level repulsion" between the frequencies was also briefly considered. It is now known that for most practical systems, the natural frequencies are more likely to have GOE, rather than the Poisson, statistics, and the present work derives a closed form expression for the relative variance under GOE statistics. The applicability of the GOE to practical systems is discussed more fully in Ref. [3]. The result is a function of three parameters, the modal overlap factor m, the previously mentioned "load type" parameter α and a bandwidth parameter B. The theory is compared with Monte-Carlo simulations for a mass-loaded plate, and good agreement is obtained for both point loading and rain-on-the-roof loading. Existing theory for harmonic excitation is summarized in Section 2, and this is extended to the band-averaged response in Section 3. A comparison with numerical simulations is then reported in Section 4.

2. Summary of existing results for harmonic excitation

The complex amplitude of the response of a harmonically excited proportionally damped system can be written in the form [8]

$$u(\omega, \mathbf{x}) = \sum_n \frac{P_n \phi_n(\mathbf{x})}{(\omega_n^2 - \omega^2 + \mathrm{i}\eta\omega\omega_n)}, \tag{1}$$

where ω is the excitation frequency, η is the loss factor, and the co-ordinates $\mathbf{x}$ range over the domain of the system. Furthermore ϕ_n, ω_n and P_n are, respectively, the mode shape (scaled to unit generalized mass), the natural frequency, and the generalized force associated with the nth mode of vibration. By making use of the orthogonality of the mode shapes, the time- and space-averaged kinetic energy density of the system follows from Eq. (1) as

$$T(\omega) = \frac{\omega^2}{4R}\int_R \rho(\mathbf{x})|u(\omega,\mathbf{x})|^2\,\mathrm{d}\mathbf{x} = \sum_n \frac{\omega^2 a_n}{[(\omega_n^2-\omega^2)^2+(\eta\omega\omega_n)^2]}, \tag{2}$$

where $\rho(\mathbf{x})$ is the density, R is the spatial extent of the system (length, area or volume, as appropriate), and

$$a_n = |P_n|^2/4R. \tag{3}$$

Although the following analysis is concerned with the statistics of T, it can be noted that expressions similar to Eq. (2) can be derived for both the potential energy and the total energy of the system, so that the derived results are also applicable to these quantities. By noting that the major contribution to Eq. (2) from mode n arises when $\omega_n \approx \omega$, Eq. (2) can be replaced by the following close approximation

$$T(\omega) = \sum_n a_n g(\omega_n,\omega), \quad g(\omega_n,\omega) = [4(\omega_n-\omega)^2+(\eta\omega)^2]^{-1}. \tag{4, 5}$$

The approximation represented by Eq. (5) is commonly adopted to simplify various integrals that appear in the subsequent analysis [3,6,7] and it has been found that the resulting loss of accuracy is negligible. Eq. (4) can also be written in the form

$$T(\omega) = \int_{-\infty}^{\infty} g(\omega',\omega)\xi(\omega')\,\mathrm{d}\omega', \quad \xi(\omega') = \sum_n a_n\delta(\omega'-\omega_n). \tag{6, 7}$$

The statistics of T over an ensemble of random structures can now be investigated by considering the system natural frequencies to form a random point process and the mode shapes to be random quantities [3]. With this approach the mean and variance of the energy can be written in the form

$$\mu_T(\omega) = E[a_n]\int_{-\infty}^{\infty} v g(\omega_n,\omega)\,\mathrm{d}\omega_n, \quad \sigma_T^2(\omega) = 2\int_0^{\infty} |F(\theta)|^2 S_\xi(\theta)\,\mathrm{d}\theta, \tag{8, 9}$$

where $E[\,]$ represents the ensemble average, v is the modal density of the subsystem, $S_\xi(\theta)$ is the spectrum of $\xi(\omega)$, omitting the delta function at the origin that accounts for the mean value, and $F(\theta)$ is the Fourier transform of $g(\omega_n,\omega)$, so that

$$F(\theta) = \int_{-\infty}^{\infty} g(\omega_n,\omega)\exp(-\mathrm{i}\theta\omega_n)\,\mathrm{d}\omega_n = \frac{\pi}{2\eta\omega}\exp[-(\eta\omega|\theta|/2+\mathrm{i}\omega\theta)]. \tag{10}$$

The form of the spectrum $S_\xi(\theta)$ of the random process $\xi(\omega)$ depends upon the statistical model that is adopted for the system natural frequencies. Several models were investigated in Ref. [3], and it was concluded that the GOE provides an extremely accurate model for random structural dynamic systems. With this approach it is assumed that the natural frequencies have the same statistics as the eigenvalues of a particular type of random matrix, a postulate that has been confirmed both numerically and experimentally for a wide range of systems (for example Refs. [9,10]). In this case the spectrum has form [3]

$$S_\xi(\theta) = (1/2\pi)\{E[a_n^2]v - E[a_n]^2 v b(\theta/2\pi v)\}, \tag{11}$$

where

$$b(\theta) = \begin{cases} 1 - 2|\theta| + |\theta|\ln(1 + 2|\theta|) & |\theta| \leqslant 1, \\ -1 + |\theta|\ln\left(\dfrac{2|\theta| + 1}{2|\theta| - 1}\right) & |\theta| \geqslant 1. \end{cases} \quad (12)$$

Eqs. (8)–(11) yield

$$\mu_T(\omega) = \frac{E[a_n]\pi\nu}{2\eta\omega}, \quad r_T^2(\omega) = \frac{\sigma_T^2}{\mu_T^2} = \frac{1}{\pi\nu}\int_0^\infty \{\alpha - b(\theta/2\pi\nu)\}\exp(-\eta\omega\theta)\,\mathrm{d}\theta, \quad (13, 14)$$

where r_T^2 is termed the relative variance, and

$$\alpha = E[a_n^2]/E[a_n]^2. \quad (15)$$

Furthermore, Eq. (14) can be evaluated to yield [3]

$$r_T^2 = \frac{1}{\pi m}\left\{\alpha - 1 + \frac{1}{2\pi m}[1 - \exp(-2\pi m)] + \mathrm{E}_1(\pi m)\left[\cosh(\pi m) - \frac{1}{\pi m}\sinh(\pi m)\right]\right\}, \quad (16)$$

where E_1 is the exponential integral and $m = \omega\eta\nu$ is the modal overlap factor. It is shown in Ref. [3] that for single point loading $\alpha = 2.75$ provides a very good fit to empirical data (note that Gaussian mode shapes would yield $\alpha = 3$). For rain-on-the-roof forcing in which the response is averaged over the loading statistics prior to considering the statistics over the system ensemble, the value is $\alpha = 1$. This is also the value of α obtained when the response to single point forcing is averaged over all possible locations of the point load, prior to considering the ensemble statistics. For many multiple point loads whose phase varies over the system ensemble, the appropriate value is $\alpha = 2$. These results are fully explained in Ref. [3].

Eq. (16) is a cumbersome result, and an approximation that is correct to second order in $1/(\pi m)$ is

$$r_T^2(\omega) \approx (\alpha - 1)/\pi m + 1/(\pi m)^2. \quad (17)$$

Eq. (17) provides an accurate approximation to the relative variance over most of the range of m of practical interest [3]; for reference, the next term in the series is $-1/(\pi m)^3$, so that the accuracy of the equation can be judged for particular values of α and m. In the following section the foregoing analysis is extended to cover the statistics of the frequency-band-averaged kinetic energy.

3. Mean and variance of the band-averaged energy

3.1. The relative variance

If the kinetic energy is averaged over a frequency band $R_\omega = [\omega - \Delta/2,\ \omega + \Delta/2]$ then Eq. (4) can be replaced by

$$T(\omega, \Delta) = \sum_n a_n(1/\Delta)\int_{R_\omega} g(\omega_n, \omega')\,\mathrm{d}\omega', \quad (18)$$

where it has been assumed that the generalized force, and hence the coefficient a_n, is constant over the frequency band R_ω. This in turn assumes that the applied physical loading, or the spectrum of the loading, is approximately constant over the averaging band; if the loading actually varies rapidly with frequency then the width of an averaging band will need to be chosen to match the stated assumptions. In what follows the case $\Delta \ll \omega$ will be considered, so that the averaging bandwidth is small relative to the centre frequency of the excitation; this reasonable for most types of averaging employed in structural dynamics, since even for one-third octave band averaging the result is $\Delta/\omega \approx 0.23$. The ensemble mean of the band-averaged energy is given by Eq. (8) with the function $g(\omega_n, \omega)$ replaced by its band-averaged value, so that Eq. (13) becomes

$$\mu_T(\omega, \Delta) = (1/\Delta) \int_{R_\omega} \frac{E[a_n]\pi v}{2\eta\omega'} \, d\omega' \approx \frac{E[a_n]\pi v}{2\eta\omega}. \tag{19}$$

Thus, under the stated assumptions, the ensemble mean of the kinetic energy is not affected by frequency-band averaging. With regard to the variance, Eq. (9) remains valid providing $F(\theta)$ is replaced by the following band-averaged value

$$\begin{aligned} F(\theta, \Delta) &= \int_{-\infty}^{\infty} \left\{ (1/\Delta) \int_{R_\omega} g(\omega_n, \omega') \, d\omega' \right\} \exp(-i\theta\omega_n) \, d\omega_n \\ &= (1/\Delta) \int_{R_\omega} \frac{\pi}{2\eta\omega'} \exp[-(\eta\omega'|\theta|/2 + i\omega'\theta)] \, d\omega'. \end{aligned} \tag{20}$$

Now under the assumption $\Delta \ll \omega$ the term ω' in the denominator of the integrand, and the term $\eta\omega'$ in the exponent, can be approximated as constant over the integration range with $\omega' \approx \omega$. The final term in the exponent leads to oscillations of the integrand however, and it cannot be taken to be constant; averaging over this term alone yields

$$F(\theta, \Delta) = \left[\frac{\sin(\theta\Delta/2)}{\theta\Delta/2} \right] F(\theta). \tag{21}$$

Eq. (14) for the relative variance then becomes

$$r_T^2(\omega, \Delta) = \frac{1}{\pi v} \int_0^{\infty} \left\{ \alpha - b\left(\frac{\theta}{2\pi v}\right) \right\} \exp(-\eta\omega\theta) \left[\frac{\sin(\theta\Delta/2)}{\theta\Delta/2} \right]^2 d\theta. \tag{22}$$

This is a daunting integral, given to the complexity of Eq. (12). Progress can be made however by rewriting Eq. (22) in the form

$$r_T^2(\omega, \Delta) = \frac{1}{\pi v} \int_0^{\infty} \left\{ \alpha - b\left(\frac{\theta}{2\pi v}\right) \right\} \exp(-\eta\omega\theta) \left[\frac{2 - e^{i\theta\Delta} - e^{-i\theta\Delta}}{(\theta\Delta)^2} \right] d\theta. \tag{23}$$

Now by comparing Eq. (23) with Eq. (14) it can be seen that

$$\frac{\partial^2}{\partial\eta^2} r_T^2(\omega, \Delta) = \left(\frac{\omega}{\Delta}\right)^2 \{2r_T^2(\omega)|_\eta - r_T^2(\omega)|_{\eta - i(\Delta/\omega)} - r_T^2(\omega)|_{\eta + i(\Delta/\omega)}\}, \tag{24}$$

where $r_T^2(\omega)|_x$ represents $r_T^2(\omega)$ evaluated with $\eta = x$. The right hand side of this result can be evaluated by using Eq. (17) for $r_T^2(\omega)$, so that

$$\frac{\partial^2}{\partial \eta^2} r_T^2(\omega, \Delta) = \left(\frac{\omega}{\Delta}\right)^2 \left(\frac{\alpha - 1}{\pi \omega v}\right) \left\{\frac{2}{\eta} - \frac{1}{[\eta - \mathrm{i}(\Delta/\omega)]} - \frac{1}{[\eta + \mathrm{i}(\Delta/\omega)]}\right\} + \left(\frac{\omega}{\Delta}\right)^2 \left(\frac{1}{\pi \omega v}\right)^2 \left\{\frac{2}{\eta^2} - \frac{1}{[\eta - \mathrm{i}(\Delta/\omega)]^2} - \frac{1}{[\eta + \mathrm{i}(\Delta/\omega)]^2}\right\}. \tag{25}$$

This result can now be integrated twice with respect to η, and the constants of integration selected to ensure that $r_T^2(\omega, \Delta) \to r_T^2(\omega)$ as $\Delta \to 0$. This yields

$$r_T^2(\omega, \Delta) = \frac{\alpha - 1}{\pi m}\left(\frac{1}{B^2}\right)\left\{2B\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{B}\right)\right] - \ln(1 + B^2)\right\} + \frac{1}{(\pi m)^2}\left(\frac{1}{B^2}\right)\ln(1 + B^2), \tag{26}$$

where the bandwidth parameter B is defined as

$$B = \Delta/\omega\eta. \tag{27}$$

Eq. (26) is based on the approximate result for the pure tone relative variance, Eq. (17), and this equation is recovered when $B \to 0$. For reference, including the next term in the expansion represented by Eq. (17) would lead to an additional term $-(\pi m)^{-3}(1 + B^2)^{-1}$ in Eq. (26).

Eq. (26) is the main result of the present work. It is informative to investigate the behaviour of this result for large averaging bandwidth B, and this issue is considered in the following section.

3.2. Large bandwidth limits

Eq. (26) has been derived under the conditions $\Delta \ll \omega$ (so that Eqs. (19) and (21) apply) and $\pi m \gg 1$ (so that Eq. (17) applies). Within these constraints a large bandwidth B must be associated with the condition $\Delta v \gg 1$, so that there are a large number of natural frequencies within the averaging band. For the case of rain-on-the-roof excitation ($\alpha = 1$) and large B, Eq. (26) yields

$$r_T^2(\omega, \Delta) \approx \frac{1}{(\pi m)^2}\left(\frac{1}{B^2}\right)\ln(B^2) = \frac{2}{(\pi N)^2}\{\ln N - \ln m\}, \tag{28}$$

where $N = \Delta v$ is the mean number of natural frequencies that lie within the averaging band. For rain-on-the-roof forcing the magnitude of the generalized force is the same for every mode, so that the term a_n that appears in Eqs. (2) and (18) is independent of n. In this case it might be expected that the kinetic energy given by Eq. (18) will be approximately proportional to the number of modes that lie within R_ω, since most of the contribution to the integral of $g(\omega_n, \omega)$ arises from the region $\omega \approx \omega_n$. Thus the relative variance of $T(\omega, \Delta)$ might be expected to be approximately the same as the relative variance of the number of modes in R_ω. The mean number of modes in R_ω is N and the variance is referred to in the literature as the *number variance*. The number variance for the GOE is well known [4], and this leads to the following result for the relative variance

$$r^2 \approx \frac{2}{(\pi N)^2}\{\ln N + 2.18\}. \tag{29}$$

Eq. (29) has a similar form to Eq. (28), and the two results agree at fixed m for $N \to \infty$. For finite N and realistic m, the relative variance yielded by Eq. (28) is less than that given by the number variance; for example, with $N = 10$ and $m = 2$, Eq. (28) yields $r_T = 0.057$ for the relative standard deviation, while the number variance yields $r = 0.095$. The number variance is therefore a fairly inaccurate measure of the response variance.

If the excitation is not rain-on-the-roof, then $\alpha \neq 1$ and for large B the leading order term in Eq. (26) is

$$r_T^2(\omega, \Delta) \approx (\alpha - 1)/N. \tag{30}$$

Now the relative number variance for a Poisson point process is $1/N$, and it might thus be thought that Eq. (30) is indicative of Poisson, rather than GOE, natural frequency statistics. However, the assumption of Poisson natural frequencies actually leads to the asymptotic result $r_T^2 \approx \alpha/N$, which clearly differs from Eq. (30).

Various forms of asymptotic analysis can be applied to Eqs. (16) and (24) to yield results in other limiting cases. For example, if Eq. (16) is expanded for *small* πm, then the result for rain-on-the-roof excitation is $r_T^2(\omega) \approx 1/(\pi m)$. If this expression is then used in Eq. (24) the result for large B is $r_T^2(\omega, \Delta) \approx 1/N$. This is the relative number variance for a Poisson process, but of more relevance, it is also the relative number variance for a GOE process with small N [4]. A detailed study shows that the asymptotic analysis based on the assumption of small πm and large B is valid only when N is small, and hence the mathematical result is consistent with physical expectations. Other analyses of this type can be performed to yield results of academic interest, but the main result of practical significance is Eq. (26).

4. Comparison with numerical simulations

An ensemble of random structures has been generated for validation purposes by considering a simply supported rectangular plate to which are attached a number of randomly placed point masses and point springs. The plate has planform dimensions 1.1985 m × 1.35 m, thickness 5 mm, density 8000 kg/m^3, Young's modulus 1.89×10^{11} N/m^2 and Poisson ratio 0.3. The modal density of the plate is 0.0176 modes/rad/s, and the frequency range considered is 0–12000 rad/s, which covers approximately 211 modes. Ten point springs of stiffness 5×10^6 N/m are attached to the plate at random locations and, unless otherwise stated, ten point masses, each having 1.5% of the mass of the bare plate are also added. Each point spring has approximately the same (modulus) impedance as a point mass at $\omega = 2000$ rad/s; the springs are added to promote randomization of the system modes at low frequencies. An ensemble of structures can be generated by varying independently the position of each of the masses and the springs, and the results reported here have been generated by using an ensemble size of 500, with the spring and mass locations generated from a uniform distribution. For any particular configuration of masses and springs, the dynamic response of the system can be computed by using the Lagrange–Rayleigh–Ritz method [8], with the modes of the bare plate used as basis functions. By altering the loss factor η of the plate, the frequency range 0–12000 rad/s can be made to cover different ranges of the modal overlap factor $m = \omega\eta\nu$; the two values $\eta = 0.008$ and 0.03 have been used here, corresponding respectively to $m = 1.7$ and $m = 6.3$ at the highest frequency. The results obtained are plotted in what follows as a function of modal overlap rather than frequency.

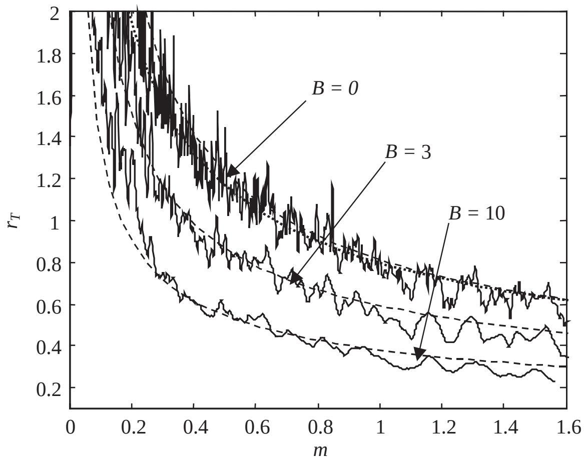

Fig. 1. Relative standard deviation of the energy density, r_T, against modal overlap m for an ensemble of plates (with randomly placed mass and stiffness) subjected to forcing at a single fixed point, for various values of the bandwidth parameter B, and $\eta = 0.008$: —, simulation result; - - -, approximate GOE prediction (Eq. (26)) with $\alpha = 2.75$; ......, exact GOE prediction (Eq. (16)) with $\alpha = 2.75$.

Results for the relative standard deviation of the response are shown in Fig. 1 for the case $\eta = 0.008$ with a single point load applied to the plate. Three averaging bandwidths B are shown, and it can be noted that the average number of modes in a band is given by Bm, so that, for example, $B = 10$ and $m = 1.6$ corresponds to approximately 16 modes in the band. In general, the theoretical predictions shown in the figure have been made by using Eq. (26), with $\alpha = 2.75$ to represent single point loading [3], although a second curve corresponding to Eq. (16) is shown for $B = 0$. The difference in the two theoretical predictions for $B = 0$ arises from the approximation involved in moving from Eq. (16) to Eq. (17), since it can be noted that Eq. (26) reduces to Eq. (17) when $B = 0$. Eq. (17) is an approximation to Eq. (16) that is strictly valid for large πm, but it can be seen that in practice this yields an accurate prediction for values of m as low as 0.4. Since the approximation involved in Eq. (17) also underlies Eq. (26), it can be expected that this accounts for much of the inaccuracy in the prediction obtained for $B \neq 0$ with $m < 0.4$. Unfortunately it has not been possible to derive an equivalent to Eq. (26) that is based on Eq. (16) rather than Eq. (17); in principle this could be done via Eqs. (16) and (24), but the required integrations are very unwieldy. Results for single point loading in the higher damping case ($\eta = 0.03$) are shown in Fig. 2. In both Figs. 1 and 2 there is convincing agreement between the analytical results and the numerical simulations, although the simulation results shown in Fig. 2 are slightly erratic.

Results for rain-on-the-roof excitation ($\alpha = 1$) are shown in Fig. 3 for the case of low damping, $\eta = 0.008$. Again, two predictions are shown for the case $B = 0$, and it can be seen that Eq. (16) is in noticeably closer agreement with the simulations; in this case the more approximate result provides a good estimate for $m > 1$. It can be seen that the relative standard deviation becomes quite low when the number of modes in the band is high; for example, with $B = 10$ and $m = 1.6$, the predicted relative standard deviation is around 0.04 while that yielded by the simulations is

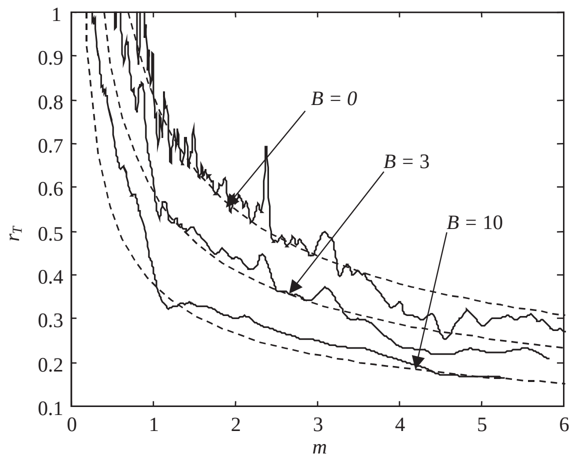

Fig. 2. Relative standard deviation of the energy density, r_T, against modal overlap m for an ensemble of plates (with randomly placed mass and stiffness) subjected to forcing at a single fixed point, for various values of the bandwidth parameter B, and $\eta = 0.03$: —, simulation result; - - -, GOE prediction (Eq. (26)) with $\alpha = 2.75$.

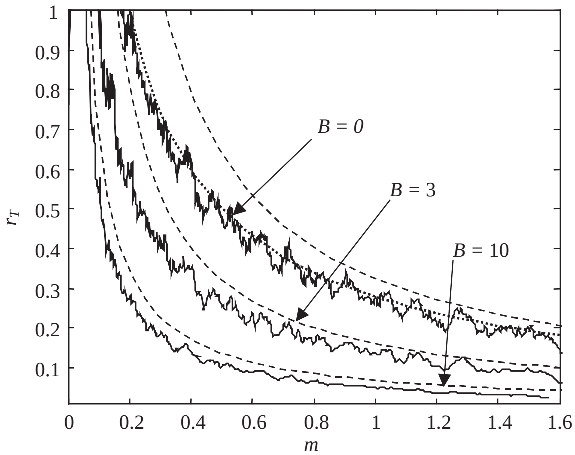

Fig. 3. Relative standard deviation of the energy density, r_T, against modal overlap m for an ensemble of plates (with randomly placed mass and stiffness) subjected to spatially averaged forcing, for various values of the bandwidth parameter B, and $\eta = 0.008$: —, simulation result; - - -, approximate GOE prediction (Eq. (26)); ...., exact GOE prediction (Eq. (16)).

around 0.025. To put this into context, the 60% error in the predicted relative standard deviation is equivalent to a 0.1% error in the mean squared response at these levels.

Results for rain-on-the-roof excitation with the higher damping level, $\eta = 0.03$, are shown in Fig. 4, where a log scale has been used to emphasize the differences between the theory and simulations at high values of m. The simulation results for the default number of random masses (10) are shown together with those obtained for 20 masses and 40 masses. Considering initially the results for 10 masses, the apparently large errors shown for the relative standard deviation at the

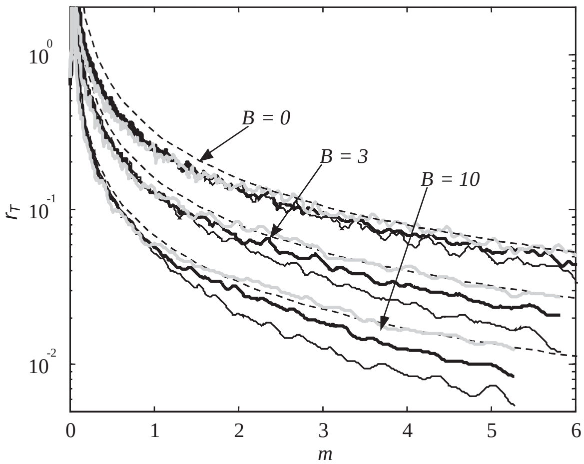

Fig. 4. Relative standard deviation of the energy density, r_T, against modal overlap m for an ensemble of plates (with randomly placed mass and stiffness) subjected to spatially averaged forcing, for various values of the bandwidth parameter B, and $\eta = 0.03$: —, simulation, 10 masses; ——, simulation, 20 masses; ——, simulation, 40 masses; - - -, GOE prediction (Eq. (26)).

highest frequency translate to small errors in the mean squared response: approximately 0.1%, 0.045% and 0.006% for $B = 0$, 3 and 10, respectively. Nonetheless, the reason for the discrepancy has been investigated, and this would seem to lie in the fundamental assumption of GOE natural frequency statistics in the analysis of Sections 2 and 3. It was shown in Ref. [3] that the natural frequencies show many of the characteristics of the GOE for the type of mass loading considered here, including the fact that the natural frequency spacings have a Rayleigh distribution and a correlation function that closely matches the GOE prediction. For the results shown in Fig. 4, where up to 50 natural frequencies are included in the averaging band, it is conceivable that the resulting small standard deviation will be sensitive to deviations from the GOE statistics. To explore this issue, additional simulations have been performed using 20 and 40 added masses, and it can be seen from Fig. 4 that as the randomness of the system is increased in this way the simulation results converge to the predicted values. Taken as a whole, Figs. 1–4 indicate that Eq. (26) yields a good estimate of the relative variance of the response of a random system, and leads to a very accurate estimate of the mean squared response.

5. Conclusions

The analysis presented in Ref. [3] for the relative standard deviation of the harmonic response of a random system has been extended to the frequency-band-averaged response. The main result is Eq. (26), and this reduces to the established result, Eq. (17), when the averaging bandwidth is zero. In the derivation of Eq. (26) it has been assumed that the natural frequencies of the random system have GOE statistics, and that the modal overlap factor is high, so that $\pi m \gg 1$. The last condition is required to ensure that Eq. (17), which forms the background to Eq. (26), is a good approximation to the more exact result, Eq. (16). The numerical results indicate that Eq. (26)

provides a good estimate of the relative standard deviation if $m > 0.4$ for point loading and if $m > 1$ for rain-on-the-roof loading. At large values of m the relative standard deviation becomes small, and the simulation results were found to be sensitive to small deviations away from ideal GOE natural frequency statistics. However in this range the error in the predicted mean squared response is very small (typically less than 0.1%), even if the error in the relative standard deviation is as much as 100%.

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