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Journal of Sound and Vibration 275 (2004) 859–875
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Study on the equivalent circuit and coupled vibration for the longitudinally polarized piezoelectric ceramic hollow cylinders

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Received 31 October 2002; accepted 11 July 2003

Abstract

Longitudinally polarized piezoelectric ceramic disks with central holes have been the most commonly used elements in underwater acoustics and ultrasonics. For very thin piezoelectric ceramic disks, its vibration can be regarded as one-dimensional thickness extensional vibration, or plane radial vibration, and the coupling between these two different vibrational modes is neglected. However, for practical piezoelectric ceramic disks with central holes, the geometrical dimensions do not satisfy the requirements for the thin ring, and the coupling must be considered. In this paper, the coupled vibration of the longitudinally polarized piezoelectric ceramic disks with central holes is studied using an approximate analytical method. When the mechanical coupling coefficient is introduced and the shearing strain is ignored, the coupled vibration of the piezoelectric ceramic hollow cylinder is divided into two equivalent vibrations. One is the equivalent longitudinal extensional vibration, and the other is the equivalent radial vibration. These two equivalent vibrations are not independent; they are correlated together by the mechanical coupling coefficient. The equivalent circuit for the piezoelectric ceramic hollow cylinders in coupled vibration is derived, and the resonance frequency equations are obtained. Based on the coupled frequency equations, the longitudinal and radial resonance frequencies can be obtained when the dimensions and the material parameters are given. Compared with one-dimensional theory, the computed resonance frequencies in this paper are in good agreement with the measured results. Compared with the numerical methods, the analytic method presented in this paper is simple in computing the longitudinal and radial resonance frequencies and in analyzing the coupled vibrational modes of the piezoelectric ceramic disks with central holes.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.07.030

1. Introduction

Longitudinally polarized piezoelectric ceramic elements, such as disks, rings, cylinders with central holes are the important electromechanical transformation elements in sandwiched piezoelectric transducers used for underwater acoustics and ultrasonics. The vibration analysis theory of the piezoelectric ceramic thin disks or slender cylinders based on one-dimensional theory has been well established [1,2]. The coupled vibration of the piezoelectric ceramic disk, hollow cylinder whose wall thickness is much less than its average radius was also studied in previous works [3–7]. However, when the diameter and the thickness of the disk and the hollow cylinder become comparable with each other, the vibration of the elements is a complex coupled vibration. Numerical methods [8–11] have been used to study the coupled vibration of the piezoelectric ceramic elements. In our previous studies, the coupled vibration of the piezoelectric ceramic thick circular and the rectangular plates have been analyzed using an approximate analytical method [12–16]. In this paper, on the basis of the piezoelectric equations and wave equations, when the shearing strain is ignored, the three-dimensional coupled vibration of the longitudinally polarized piezoelectric ceramic hollow cylinder whose height and thickness are comparable with its radius is studied analytically. The equivalent circuit of the piezoelectric ceramic hollow cylinder in coupled vibration is obtained, and the resonance frequency equations of the piezoelectric ceramic hollow cylinder in axially symmetric vibration are derived that can be used to compute the longitudinal and radial resonance frequencies. The analytic method presented in this paper can be used to analyze the coupled vibration of other piezoelectric elements.

2. Equivalent circuit of the piezoelectric ceramic hollow cylinder in three-dimensional coupled vibration

The piezoelectric ceramic element with which we will be concerned is a hollow cylinder as shown in Fig. 1. In the figure, the height l is comparable with the outside radius a, and there will be no restriction on the inside radius b. In the figure, F_{z1}, F_{z2} are the longitudinal external forces, F_{ra}, F_{rb} are the external forces acting on radial surfaces of the cylinder; v_{z1}, v_{z2}, v_{ra}, v_{rb} are the velocities at the boundaries of the cylinder. The polarization direction is along the height of the hollow cylinder and the external exciting electric field is parallel to the polarization direction. In cylindrical co-ordinates, the three-dimensional motion equations for the cylinder in coupled

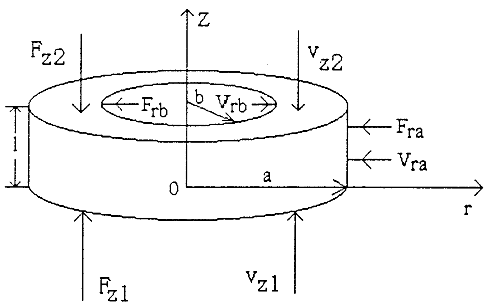

Fig. 1. Geometrical diagram of a piezoelectric ceramic hollow cylinder.

vibration are

$$\rho \frac{\partial^2 \xi_r}{\partial t^2} = \frac{\partial T_r}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_r - T_\theta}{r}, \tag{1}$$

$$\rho \frac{\partial^2 \xi_\theta}{\partial t^2} = \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_\theta}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2T_{r\theta}}{r}, \tag{2}$$

$$\rho \frac{\partial^2 \xi_z}{\partial t^2} = \frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_z}{\partial z} + \frac{T_{rz}}{r}, \tag{3}$$

where r, θ, z are the cylindrical co-ordinates; ξ_r, ξ_θ, ξ_z are the three displacement components; $T_r, T_\theta, T_z, T_{r\theta}, T_{rz}, T_{\theta z}$ are the stress components in the piezoelectric ceramic cylinder. The relationship between the strain and the displacement are as follows:

$$S_r = \frac{\partial \xi_r}{\partial r}, \quad S_\theta = \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\xi_r}{r}, \quad S_z = \frac{\partial \xi_z}{\partial z}, \tag{4}$$

$$S_{r\theta} = \frac{1}{r}\frac{\partial \xi_r}{\partial \theta} + \frac{\partial \xi_\theta}{\partial r} - \frac{\xi_\theta}{r}, \quad S_{\theta z} = \frac{1}{r}\frac{\partial \xi_z}{\partial \theta} + \frac{\partial \xi_\theta}{\partial z}, \quad S_{rz} = \frac{\partial \xi_r}{\partial z} + \frac{\partial \xi_z}{\partial r}. \tag{5}$$

Here, $S_r, S_\theta, S_z, S_{r\theta}, S_{\theta z}, S_{rz}$ are the strain components. It can be seen that for three-dimensional coupled vibration, the wave equations are complex, and the analytic solution is almost impossible if no assumptions are made. Although numerical methods can be used to analyze the coupled vibration of resonators, large special software must be used. In the following analysis, some assumptions are made to simplify the mathematical analysis of the coupled vibration of the piezoelectric ceramic hollow cylinder and an approximate analytic method is presented, which proves not only concise in physical concept, but also simple and time-saving in the calculation of the resonance frequency of the resonator in coupled vibration.

From Fig. 1, it can be seen that as the polarization direction is parallel to that of the electric field, the vibration of the hollow cylinder can be regarded as a coupled one of longitudinal and radial extensional vibrations approximately; therefore shearing and torsion stress and strain can be ignored. In cylindrical coordinates, when the edge effect of the electric field is ignored, we have

$$E_1 = E_2 = 0, \quad E_3 \neq 0, \; D_1 = D_2 = 0, \quad D_3 \neq 0, \; T_{r\theta} = T_{rz} = T_{\theta z} = 0, \; S_{r\theta} = S_{rz} = S_{\theta z} = 0. \tag{6}$$

Here, E_1, E_2, and E_3 are components of the electric field in the r, θ, and z directions, D_1, D_2, and D_3 are components of the electric displacement. Using Eq. (6), the piezoelectric constitutive equations can be reduced to the following form:

$$S_r = s_{11}^E T_r + s_{12}^E T_\theta + s_{13}^E T_z + d_{31} E_3, \tag{7}$$

$$S_\theta = s_{12}^E T_r + s_{11}^E T_\theta + s_{13}^E T_z + d_{31} E_3, \tag{8}$$

$$S_z = s_{13}^E T_r + s_{13}^E T_\theta + s_{33}^E T_z + d_{33} E_3, \tag{9}$$

$$D_3 = d_{31} T_r + d_{31} T_\theta + d_{33} T_z + \varepsilon_{33}^T E_3. \tag{10}$$

In these equations, s_{ij}^E is the elastic compliance constants measured at constant electric field, d_{31} and d_{33} are the piezoelectric strain constants, and ε_{33}^T is the dielectric constant measured at constant stress. For axis-symmetrical coupled vibration of the piezoelectric ceramic cylinder, we have $\xi_\theta = 0, \partial\xi_\theta/\partial\theta = 0$, $\partial T_\theta/\partial\theta = 0$. The equations of motion can be reduced to the following form:

$$\rho\partial^2\xi_r/\partial t^2 = \partial T_r/\partial r + (T_r - T_\theta)/r, \tag{11}$$

$$\rho\partial^2\xi_z/\partial t^2 = \partial T_z/\partial z, \tag{12}$$

$$S_r = \frac{\partial\xi_r}{\partial r}, \quad S_\theta = \frac{\xi_r}{r}, \quad S_z = \frac{\partial\xi_z}{\partial z}. \tag{13}$$

2.1. Equivalent circuit for the equivalent radial vibration of the piezoelectric hollow cylinder in coupled vibration

From Eqs. (7) and (8), we have

$$S_r - S_\theta = (s_{11}^E - s_{12}^E)\,(T_r - T_\theta), \tag{14}$$

$$S_r + S_\theta = (s_{11}^E + s_{12}^E)\,(T_r + T_\theta) + 2d_{31}E_3 + 2s_{13}^E T_z. \tag{15}$$

Let $n = T_z/(T_r + T_\theta)$, n is defined as the mechanical coupling coefficient. From Eqs. (14) and (15), we have

$$T_r - T_\theta = \frac{S_r - S_\theta}{s_{11}^E - s_{12}^E}, \tag{16}$$

$$T_r + T_\theta = \frac{S_r + S_\theta - 2d_{31}E_3}{s_{11}^E + s_{12}^E + 2s_{13}^E n}. \tag{17}$$

Using Eqs. (16) and (17), we can obtain the radial stress T_r as follows:

$$T_r = \left(\frac{S_r - S_\theta}{s_{11}^E - s_{12}^E} + \frac{S_r + S_\theta - 2d_{31}E_3}{s_{11}^E + s_{12}^E + 2s_{13}^E n}\right)\Big/ 2. \tag{18}$$

Substituting Eqs. (13), (16), and (18) into wave equation (11) yields the radial equivalent wave equation of motion

$$\rho(\partial^2\xi_r/\partial t^2)/E_r = \partial^2\xi_r/\partial r^2 + (\partial\xi_r/\partial r)/r - \xi_r/r^2. \tag{19}$$

In Eq. (19),

$$E_r = \frac{s_{11}^E + s_{13}^E n}{(s_{11}^E - s_{12}^E)(s_{11}^E + s_{12}^E + 2s_{13}^E n)} = \frac{1 - v_{13}n}{s_{11}^E(1 + v_{12})(1 - v_{12} - 2v_{13}n)},$$

and E_r is defined as the radial equivalent elastic constant of equivalent radial vibration of the piezoelectric ceramic hollow cylinder. $v_{12} = -s_{12}^E/s_{11}^E$, $v_{13} = -s_{13}^E/s_{11}^E$. In the above transformations, $\partial E_3/\partial r = 0$ is used. For simple harmonic motion, $\xi_r = \xi_{ra}\exp(\mathrm{j}\omega t)$; the radial wave equation becomes

$$\mathrm{d}^2\xi_{ra}/\mathrm{d}r^2 + (\mathrm{d}\xi_{ra}/\mathrm{d}r)/r - \xi_{ra}/r^2 + k_r^2\xi_{ra} = 0, \tag{20}$$

where $k_r = \omega/V_r$, $V_r = (E_r/\rho)^{1/2}$, k_r and V_r are called the radial equivalent wave number and the radial equivalent sound speed, and ω is the angular frequency. Eq. (20) is a Bessel's equation of the first order that has the solution

$$\xi_{ra} = A_r \mathrm{J}_1(k_r r) + B_r \mathrm{Y}_1(k_r r), \tag{21}$$

where $\mathrm{J}_1(k_r r)$ and $\mathrm{Y}_1(k_r r)$ are Bessel functions of the first and second kind. A_r and B_r are two constants that can be determined by the boundary conditions. From Eq. (21), the equivalent radial velocity amplitude can be obtained as

$$v_r = \mathrm{j}\omega[A_r \mathrm{J}_1(k_r r) + B_r \mathrm{Y}_1(k_r r)]. \tag{22}$$

From Fig. 1, we have

$$v_r|_{r=a} = -v_{ra}, \quad v_r|_{r=b} = v_{rb}. \tag{23}$$

Substituting Eq. (22) into Eq. (23), after some transformations, the constants A_r and B_r can be obtained as

$$B_r = \frac{1}{\mathrm{j}\omega} \frac{v_{ra}\mathrm{J}_1(k_r b) + v_{rb}\mathrm{J}_1(k_r a)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)}, \tag{24}$$

$$A_r = -\frac{1}{\mathrm{j}\omega} \frac{v_{ra}\mathrm{Y}_1(k_r b) + v_{rb}\mathrm{Y}_1(k_r a)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)}. \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (21) and then into Eqs. (13) and (18) yields the expression of the equivalent radial stress

$$\begin{aligned} T_r = {} & v_{ra} \frac{\begin{array}{c}\{\mathrm{J}_1(k_r b)[k_r \mathrm{Y}_0(k_r r)(s_{11}^E + s_{13}^E n) - \mathrm{Y}_1(k_r r)(s_{11}^E + s_{12}^E + 2s_{13}^E n)/r] \\ -\mathrm{Y}_1(k_r b)[k_r \mathrm{J}_0(k_r r)(s_{11}^E + s_{13}^E n) - \mathrm{J}_1(k_r r)(s_{11}^E + s_{12}^E + 2s_{13}^E n)/r]\}\end{array}}{\mathrm{j}\omega[\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)](s_{11}^E - s_{12}^E)(s_{11}^E + s_{12}^E + 2s_{13}^E n)} \\ & + v_{rb} \frac{\begin{array}{c}\{\mathrm{J}_1(k_r a)[k_r \mathrm{Y}_0(k_r r)(s_{11}^E + s_{13}^E n) - \mathrm{Y}_1(k_r r)(s_{11}^E + s_{12}^E + 2s_{13}^E n)/r] \\ -\mathrm{Y}_1(k_r a)[k_r \mathrm{J}_0(k_r r)(s_{11}^E + s_{13}^E n) - \mathrm{J}_1(k_r r)(s_{11}^E + s_{12}^E + 2s_{13}^E n)/r]\}\end{array}}{\mathrm{j}\omega[\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)](s_{11}^E - s_{12}^E)(s_{11}^E + s_{12}^E + 2s_{13}^E n)} \\ & - \frac{d_{31}E_3}{s_{11}^E + s_{12}^E + 2s_{13}^E n}. \end{aligned} \tag{26}$$

From Fig. 1, we have

$$F_{ra} = -T_r|_{r=a} S_a, \quad F_{rb} = -T_r|_{r=b} S_b. \tag{27}$$

Here, $S_a = 2\pi al$, $S_b = 2\pi bl$. S_a and S_b are the outer and inner areas of side surface of the hollow cylinder. Substituting Eq. (26) into Eq. (27) yields

$$\begin{aligned} -F_{ra} = {} & \frac{\rho V_r S_a}{\mathrm{j}} \frac{\mathrm{J}_1(k_r b)\mathrm{Y}_0(k_r a) - \mathrm{Y}_1(k_r b)\mathrm{J}_0(k_r a)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} v_{ra} + \frac{\rho V_r S_a}{\mathrm{j}} \frac{s_{11}^E + s_{12}^E + 2s_{13}^E n}{k_r a(s_{11}^E + s_{13}^E n)} v_{ra} \\ & + \frac{\rho V_r S_a}{\mathrm{j}} \frac{\mathrm{J}_1(k_r a)\mathrm{Y}_0(k_r a) - \mathrm{Y}_1(k_r a)\mathrm{J}_0(k_r a)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} v_{rb} - \frac{2\pi a d_{31} E_3 l}{s_{11}^E + s_{12}^E + 2s_{13}^E n}, \end{aligned} \tag{28}$$

$$-F_{rb} = \frac{\rho V_r S_b}{\mathrm{j}} \frac{\mathrm{J}_1(k_r a)\mathrm{Y}_0(k_r b) - \mathrm{Y}_1(k_r a)\mathrm{J}_0(k_r b)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} v_{rb} - \frac{\rho V_r S_b}{\mathrm{j}} \frac{s_{11}^E + s_{12}^E + 2s_{13}^E n}{k_r b(s_{11}^E + s_{13}^E n)} v_{rb} + \frac{\rho V_r S_b}{\mathrm{j}} \frac{\mathrm{J}_1(k_r b)\mathrm{Y}_0(k_r b) - \mathrm{Y}_1(k_r b)\mathrm{J}_0(k_r b)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} v_{ra} - \frac{2\pi b d_{31} E_3 l}{s_{11}^E + s_{12}^E + 2s_{13}^E n}. \tag{29}$$

Let $V_3 = E_3 l$, $Z_{ra} = \rho V_r S_a$, $Z_{rb} = \rho V_r S_b$,

$$F'_{ra} = \frac{F_{ra}}{2\pi a} + \frac{n d_{33} V_3}{s_{11}^E + s_{12}^E + 2s_{13}^E n}, \quad F'_{rb} = \frac{F_{rb}}{2\pi b} + \frac{n d_{33} V_3}{s_{11}^E + s_{12}^E + 2s_{13}^E n}.$$

Eqs. (28) and (29) can be rewritten as the following forms:

$$F'_{ra} = \frac{Z_{ra}}{\mathrm{j}} \frac{v_{ra} a}{2\pi a^2} \left[\frac{\mathrm{Y}_1(k_r b)\mathrm{J}_0(k_r a) - \mathrm{J}_1(k_r b)\mathrm{Y}_0(k_r a)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} - \frac{s_{11}^E + s_{12}^E + 2s_{13}^E n}{k_r a(s_{11}^E + s_{13}^E n)} \right] + \frac{Z_{ra}}{\mathrm{j}} \frac{v_{rb} b}{2\pi ab} \frac{\mathrm{Y}_1(k_r a)\mathrm{J}_0(k_r a) - \mathrm{J}_1(k_r a)\mathrm{Y}_0(k_r a)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} + \frac{d_{31} + n d_{33}}{s_{11}^E + s_{12}^E + 2s_{13}^E n} V_3, \tag{30}$$

$$F'_{rb} = \frac{Z_{rb}}{\mathrm{j}} \frac{v_{rb} b}{2\pi b^2} \left[\frac{\mathrm{Y}_1(k_r a)\mathrm{J}_0(k_r b) - \mathrm{J}_1(k_r a)\mathrm{Y}_0(k_r b)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} + \frac{s_{11}^E + s_{12}^E + 2s_{13}^E n}{k_r b(s_{11}^E + s_{13}^E n)} \right] + \frac{Z_{rb}}{\mathrm{j}} \frac{v_{ra} a}{2\pi ab} \frac{\mathrm{Y}_1(k_r b)\mathrm{J}_0(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_0(k_r b)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} + \frac{d_{31} + n d_{33}}{s_{11}^E + s_{12}^E + 2s_{13}^E n} V_3. \tag{31}$$

Let

$$v'_{ra} = -v_{ra}[\mathrm{J}_0(k_r b)\mathrm{Y}_1(k_r b) - \mathrm{Y}_0(k_r b)\mathrm{J}_1(k_r b)],$$
$$v'_{rb} = -v_{rb}[\mathrm{J}_0(k_r a)\mathrm{Y}_1(k_r a) - \mathrm{Y}_0(k_r a)\mathrm{J}_1(k_r a)].$$

Using the relationship of $\mathrm{J}_{n+1}(x)\mathrm{Y}_n(x) - \mathrm{Y}_{n+1}(x)\mathrm{J}_n(x) = 2/(\pi x)$, we have

$$v'_{ra} = \frac{2}{\pi k_r ab} v_{ra} a, \quad v'_{rb} = \frac{2}{\pi k_r ab} v_{rb} b. \tag{32}$$

Substituting Eq. (32) into Eqs. (30) and (31), after some transformations, we have

$$F''_{ra} = \frac{\pi^2 (k_r b)^2 Z_{ra}}{4\mathrm{j}} \left[\frac{\mathrm{Y}_1(k_r b)\mathrm{J}_0(k_r a) - \mathrm{J}_1(k_r b)\mathrm{Y}_0(k_r a)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} - \frac{1 - \nu_{12} - 2\nu_{13} n}{k_r a(1 - \nu_{13} n)} \right] v'_{ra} + \mathrm{j} \frac{Z_{ra} \pi k_r b}{2} \frac{1}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} v'_{rb} + N_{31} V_3, \tag{33}$$

$$F''_{rb} = \frac{\pi^2 (k_r a)^2 Z_{rb}}{4\mathrm{j}} \left[\frac{\mathrm{Y}_1(k_r a)\mathrm{J}_0(k_r b) - \mathrm{J}_1(k_r a)\mathrm{Y}_0(k_r b)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} + \frac{1 - \nu_{12} - 2\nu_{13} n}{k_r b(1 - \nu_{13} n)} \right] v'_{rb} + \mathrm{j} \frac{Z_{rb} \pi k_r a}{2} \frac{1}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} v'_{ra} + N_{31} V_3. \tag{34}$$

In Eqs. (33) and (34), $F''_{ra} = F'_{ra}(\pi^2 k_r ab)$, $F''_{rb} = F'_{rb}(\pi^2 k_r ab)$, $\nu_{12} = -s_{12}^E/s_{11}^E$, $\nu_{13} = -s_{13}^E/s_{11}^E$.

$$N_{31} = \pi^2 k_r ab \frac{d_{31} + n d_{33}}{s_{11}^E + s_{12}^E + 2s_{13}^E n},$$

where N_{31} is defined as the electro-mechanical transformation coefficient of the piezoelectric ceramic hollow cylinder in radial vibration. Eqs. (33) and (34) can be rewritten as the following simplified forms:

$$F''_{ra} = (Z_2 + Z_3)v'_{ra} + Z_3 v'_{rb} + N_{31} V_3, \tag{35}$$

$$F''_{rb} = (Z_1 + Z_3)v'_{rb} + Z_3 v'_{ra} + N_{31} V_3. \tag{36}$$

In these two equations, Z_1, Z_2, Z_3 are three mechanical impedances, their expressions are as follows:

$$\begin{aligned} Z_1 = & \frac{\pi^2 (k_r a)^2 Z_{rb}}{4\mathrm{j}} \left[\frac{\mathrm{Y}_1(k_r a)\mathrm{J}_0(k_r b) - \mathrm{J}_1(k_r a)\mathrm{Y}_0(k_r b)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} + \frac{1 - \nu_{12} - 2\nu_{13} n}{k_r b(1 - \nu_{13} n)} \right] \\ & - \mathrm{j}\frac{Z_{rb}}{2} \frac{\pi k_r a}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)}, \end{aligned} \tag{37}$$

$$\begin{aligned} Z_2 = & \frac{\pi^2 (k_r b)^2 Z_{ra}}{4\mathrm{j}} \left[\frac{\mathrm{Y}_1(k_r b)\mathrm{J}_0(k_r a) - \mathrm{J}_1(k_r b)\mathrm{Y}_0(k_r a)}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} - \frac{1 - \nu_{12} - 2\nu_{13} n}{k_r a(1 - \nu_{13} n)} \right] \\ & - \mathrm{j}\frac{Z_{ra}}{2} \frac{\pi k_r b}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)}, \end{aligned} \tag{38}$$

$$Z_3 = \mathrm{j}\frac{Z_{rb}}{2} \frac{\pi k_r a}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)} = \mathrm{j}\frac{Z_{ra}}{2} \frac{\pi k_r b}{\mathrm{J}_1(k_r a)\mathrm{Y}_1(k_r b) - \mathrm{J}_1(k_r b)\mathrm{Y}_1(k_r a)}. \tag{39}$$

For the electrical characteristics of piezoelectric ceramic hollow cylinder in coupled radial vibration, from Eq. (10) the electric displacement can be expressed as

$$D_3 = \frac{d_{31} + n d_{33}}{s^E_{11} + s^E_{12} + 2 s^E_{13} n}(S_r + S_\theta) + \varepsilon^T_{33} E_3 - \frac{2 d_{31} E_3 (d_{31} + n d_{33})}{s^E_{11} + s^E_{12} + 2 s^E_{13} n}. \tag{40}$$

Substituting Eqs. (13) and (21) into Eq. (40) yields

$$D_3 = \frac{d_{31} + n d_{33}}{s^E_{11} + s^E_{12} + 2 s^E_{13} n} k_r [A_r \mathrm{J}_0(k_r r) + B_r \mathrm{Y}_0(k_r r)] + \varepsilon^T_{33} E_3 - \frac{2 d_{31} E_3 (d_{31} + n d_{33})}{s^E_{11} + s^E_{12} + 2 s^E_{13} n}. \tag{41}$$

Let the current flowing into the hollow cylinder be I_{31}; then for a harmonic motion $I_{31} = \mathrm{d}Q/\mathrm{d}t$, where Q is the surface charge. Since the value of D_3 at the surface is equal to the surface charge density, we can find Q by performing the integration

$$Q = 2\pi \int D_3 \, r \, \mathrm{d}r. \tag{42}$$

Evaluating this integral yields

$$Q = \frac{2\pi(d_{31} + n d_{33})}{s^E_{11} + s^E_{12} + 2 s^E_{13} n}[A_r k_r C(1) + B_r k_r C(2)] + \pi(a^2 - b^2)\left[\varepsilon^T_{33} E_3 - \frac{2 d_{31} E_3 (d_{31} + n d_{33})}{s^E_{11} + s^E_{12} + 2 s^E_{13} n}\right], \tag{43}$$

where $C(1) = (1/k_r^2)[k_r a \mathrm{J}_1(k_r a) - k_r b \mathrm{J}_1(k_r b)]$, $C(2) = (1/k_r^2)[k_r a \mathrm{Y}_1(k_r a) - k_r b \mathrm{Y}_1(k_r b)]$. Substituting the expressions of A_r and B_r into Eq. (43), using $I_{31} = \mathrm{j}\omega Q$, we have

$$I_{31} = \mathrm{j}\omega C_{0r} V_3 - \frac{2\pi(d_{31} + n d_{33})}{s^E_{11} + s^E_{12} + 2 s^E_{13} n}(a v_{ra} + b v_{rb}). \tag{44}$$

In Eq. (44)

$$C_{0r} = \frac{\varepsilon_{33}^T S}{l}\left[1 - \frac{2d_{31}(d_{31}+nd_{33})}{\varepsilon_{33}^T(s_{11}^E + s_{12}^E + 2s_{13}^E n)}\right]$$

and C_{0r} is the equivalent clamped capacitance of radial vibration of the piezoelectric ceramic hollow cylinder in coupled vibration. $S = \pi(a^2 - b^2)$, S is the cross-sectional area of the piezoelectric ceramic hollow cylinder. Using the above relations, Eq. (44) can be rewritten as

$$I_{31} = \mathrm{j}\omega C_{0r} V_3 - N_{31}(v'_{ra} + v'_{rb}). \tag{45}$$

Combining Eqs. (35), (36) and (45), the equivalent circuit of the radial vibration of the piezoelectric ceramic hollow cylinder in coupled vibration can be obtained as shown in Fig. 2.

2.2. The equivalent circuit for the equivalent longitudinal vibration of the piezoelectric hollow cylinder in coupled vibration

In the above analysis, the equivalent radial vibration for the cylinder in coupled vibration is studied. In this section, the equivalent longitudinal vibration for the piezoelectric ceramic hollow cylinder in coupled vibration will be analyzed. From Eqs. (9) and (10), we have

$$S_z = (s_{33}^E + s_{13}^E/n)T_z + d_{33}E_3, \tag{46}$$

$$E_3 = [D_3 - (d_{33} + d_{31}/n)T_z]/\varepsilon_{33}^T. \tag{47}$$

Substituting Eq. (47) into Eq. (46) yields

$$S_z = \left(s_{33}^E + \frac{s_{13}^E}{n}\right)T_z + d_{33}\left[D_3 - \left(d_{33} + \frac{d_{31}}{n}\right)T_z\right]\Big/\varepsilon_{33}^T. \tag{48}$$

From Eq. (48), we can get

$$T_z = E_z(S_z - d_{33}D_3/\varepsilon_{33}^T), \tag{49}$$

where

$$E_z = \left[s_{33}^E + \frac{s_{13}^E}{n} - \frac{d_{33}}{\varepsilon_{33}^T}\left(d_{33} + \frac{d_{31}}{n}\right)\right]^{-1} = \left\{s_{33}^E\left[1 - \frac{v_{31}}{n} - k_{33}^2\left(1 - \frac{\lambda_{31}}{n}\right)\right]\right\}^{-1},$$

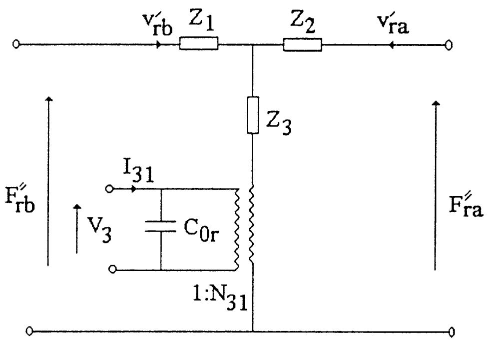

Fig. 2. Equivalent circuit of the radial vibration of the piezoelectric ceramic hollow cylinder in coupled vibration.

E_z is called the longitudinal equivalent elastic constant of the piezoelectric ceramic cylinder in coupled vibration. $k_{33}^2 = d_{33}^2/\varepsilon_{33}^T s_{33}^E$, $v_{31} = -s_{13}^E/s_{33}^E$, $\lambda_{31} = -d_{31}/d_{33}$. Substituting Eq. (49) into Eq. (12) yields

$$\partial^2\xi_z/\partial t^2 = V_z^2(\partial^2\xi_z/\partial z^2), \tag{50}$$

where $V_z = (E_z/\rho)^{1/2}$, V_z is called the longitudinal equivalent sound speed. Here $\partial D_3/\partial z = 0$ was used in deriving Eq. (50). For a harmonic motion, $\xi_z = \xi_{za}\ \exp(\mathrm{j}\omega t)$, Eq. (50) can be reduced to

$$\mathrm{d}^2\xi_{za}/\mathrm{d}z^2 + k_z^2\xi_{za} = 0, \tag{51}$$

where $k_z = \omega/V_z$, k_z is defined as the longitudinal equivalent wave number. The solution of Eq. (51) is

$$\xi_{za} = A_z\sin(k_z z) + B_z\cos(k_z z). \tag{52}$$

The equivalent longitudinal velocity for the hollow cylinder in coupled vibration is

$$v_z = \mathrm{j}\omega[A_z\sin(k_z z) + B_z\cos(k_z z)]. \tag{53}$$

From Fig. 1, using Eq. (53), we have

$$v_{z1} = v_z(z=0) = \mathrm{j}\omega B_z, \tag{54}$$

$$v_{z2} = -v_z(z=l) = -\mathrm{j}\omega[A_z\sin(k_z l) + B_z\cos(k_z l)]. \tag{55}$$

From the above equations, the constants A_z and B_z can be obtained:

$$A_z = -\frac{1}{\mathrm{j}\omega}\times\left[\frac{v_{z1}}{\tan(k_z l)} + \frac{v_{z2}}{\sin(k_z l)}\right], \tag{56}$$

$$B_z = \frac{v_{z1}}{\mathrm{j}\omega}. \tag{57}$$

Using the relation of $S_z = \partial\xi_z/\partial z$ and Eq. (52), substituting Eqs. (56) and (57) into Eq. (49) yields

$$T_z = -\frac{k_z E_z}{\mathrm{j}\omega}\times\left[\left(\frac{v_{z1}}{\tan k_z l} + \frac{v_{z2}}{\sin k_z l}\right)\cos(k_z z) + v_{z1}\sin(k_z z)\right] - \frac{d_{33}D_3E_z}{\varepsilon_{33}^T}. \tag{58}$$

From Fig. 1, the external forces can be expressed as

$$F_{z1} = -ST_z|_{z=0} = \frac{k_z E_z S}{\mathrm{j}\omega}\times\left(\frac{v_{z1}}{\tan k_z l} + \frac{v_{z2}}{\sin k_z l}\right) + \frac{d_{33}D_3E_zS}{\varepsilon_{33}^T}, \tag{59}$$

$$F_{z2} = -ST_z|_{z=1} = \frac{k_z E_z S}{\mathrm{j}\omega}\times\left(\frac{v_{z1}}{\sin k_z l} + \frac{v_{z2}}{\tan k_z l}\right) + \frac{d_{33}D_3E_zS}{\varepsilon_{33}^T}. \tag{60}$$

The voltage and current of the piezoelectric ceramic hollow cylinder resonator in longitudinal vibration can be obtained:

$$I_{33} = \mathrm{j}\omega D_3\pi a^2, \tag{61}$$

$$V_3 = \int_0^1 E_3\,\mathrm{d}z. \tag{62}$$

Substituting Eq. (58) into Eq. (47) and then into Eq. (62) yields

$$V_3 = \frac{1}{\varepsilon_{33}^T} \times \left[1 + \left(d_{33} + \frac{d_{31}}{n}\right)\frac{d_{33}E_z}{\varepsilon_{33}^T}\right] D_3 l + \frac{(d_{33} + d_{31}/n)E_z}{\mathrm{j}\omega\varepsilon_{33}^T}(v_{z1} + v_{z2}). \tag{63}$$

Substituting Eq. (61) into Eq. (63) yields

$$I_{33} = \mathrm{j}\omega C_{0z} V_3 - N_{33}(v_{z1} + v_{z2}). \tag{64}$$

Here,

$$C_{0z} = \frac{S\varepsilon_{33}^T}{l[1 + (d_{33} + d_{31}/n)d_{33}E_z/\varepsilon_{33}^T]},$$

where C_{0z} is called the equivalent clamped capacitance of the equivalent longitudinal vibration of the hollow cylinder in coupled vibration,

$$N_{33} = \frac{(d_{33} + d_{31}/n)E_z S}{l[1 + (d_{33} + d_{31}/n)d_{33}E_z/\varepsilon_{33}^T]} = \frac{(d_{33} + d_{31}/n)S}{l(s_{33}^E + s_{13}^E/n)},$$

where N_{33} is defined as the equivalent electro-mechanical conversion coefficient of the ceramic cylinder resonator in coupled vibration. From the above analysis, Eqs. (59) and (60) can be rewritten as

$$F'_{z1} = \left(\frac{\rho S V_z}{\mathrm{j}\sin k_z l} - \frac{N_{33}^2}{\mathrm{j}\omega C_{0z}} + \frac{N_{33}^2}{\mathrm{j}\omega C_p}\right)(v_{z1} + v_{z2}) + \mathrm{j}\rho S V_z \tan\left(\frac{k_z l}{2}\right) v_{z1} + N_{33} V_3, \tag{65}$$

$$F'_{z2} = \left(\frac{\rho S V_z}{\mathrm{j}\sin k_z l} - \frac{N_{33}^2}{\mathrm{j}\omega C_{0z}} + \frac{N_{33}^2}{\mathrm{j}\omega C_p}\right)(v_{z1} + v_{z2}) + \mathrm{j}\rho S V_z \tan\left(\frac{k_z l}{2}\right) v_{z2} + N_{33} V_3. \tag{66}$$

Here,

$$C_p = \frac{S\varepsilon_{33}^T}{l[1 + (d_{33} + d_{31}/n)d_{33}E_z/\varepsilon_{33}^T]} \times \frac{(d_{33} + d_{31}/n)}{d_{31}/n} = C_{0z}\left(1 + \frac{d_{33}}{d_{31}} n\right), \quad F'_{z1} = F_{z1} + F_{3c},$$

$$F'_{z2} = F_{z2} + F_{3c}, F_{3c} = \frac{S d_{31} E_z/n}{l[1 + (d_{33} + d_{31}/n)d_{33}E_z/\varepsilon_{33}^T]} \times V_3, \quad S = \pi(a^2 - b^2).$$

C_p and F_{3c} are caused by the coupling between the radial and longitudinal vibrations in the hollow cylinder. From Eqs. (64)–(66), the equivalent circuit of the equivalent longitudinal vibration of the piezoelectric ceramic hollow cylinder in coupled vibration can be obtained as shown in Fig. 3. In

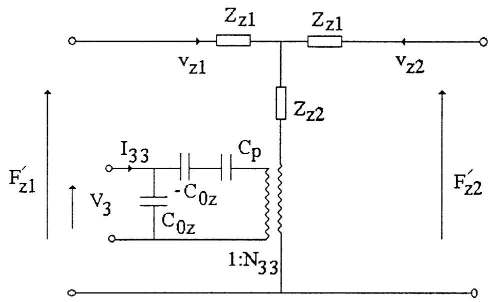

Fig. 3. Equivalent circuit of the longitudinal vibration of the piezoelectric ceramic hollow cylinder in coupled vibration.

the figure, $Z_{z1} = \mathrm{j}\rho S V_z \tan(k_z l/2), Z_{z2} = \rho S V_z/\mathrm{j}\sin(k_z l)$. It can be seen that it is different from the traditional one-dimensional equivalent circuit of a slender piezoelectric ceramic rod that an additional capacitance is created. The additional capacitance is resulted from the coupling between the longitudinal and the radial vibrations in the hollow cylinder.

2.3. Equivalent circuit of the piezoelectric ceramic hollow cylinder in coupled vibration

In practical cases, the piezoelectric ceramic hollow cylinder is excited by an external alternating electric signal that is in the direction of the longitudinal axis. As is discussed above, since the longitudinal and the radial dimensions of the cylinder are comparable, when the piezoelectric ceramic hollow cylinder is excited electrically, it will experience a complex coupled vibration, and this coupled vibration is composed of the equivalent radial and longitudinal vibrations. Let the current of the hollow cylinder resonator in coupled vibration be I_3. From the above analysis, following relation can be obtained:

$$I_3 = I_{31} + I_{33}. \tag{67}$$

From the above analysis and Eq. (67), the equivalent circuit of the piezoelectric ceramic hollow cylinder in coupled vibration can be derived as shown in Fig. 4.

It can be seen that the vibration of a piezoelectric ceramic hollow cylinder with comparable radial and longitudinal dimensions is a very complex coupled vibration. However, when its dimensions satisfy certain conditions, the coupled vibration can be simplified to some simple vibrations. For example, when the piezoelectric ceramic hollow cylinder becomes a slender rod, that is to say, when the longitudinal length is far larger than its radial radius, we have $l \gg a$. In this case, the mechanical coupling coefficient n becomes infinity. And therefore, we have the following equations:

$$E_z = \left[s_{33}^E + \frac{s_{13}^E}{n} - \frac{d_{33}}{\varepsilon_{33}^T}\left(d_{33} + \frac{d_{31}}{n}\right)\right]^{-1} = \frac{1}{s_{33}^E(1 - k_{33}^2)} = \frac{1}{s_{33}^D}, \tag{68}$$

$$V_z = (E_z/\rho)^{1/2} = [1/(s_{33}^D \rho)]^{1/2}. \tag{69}$$

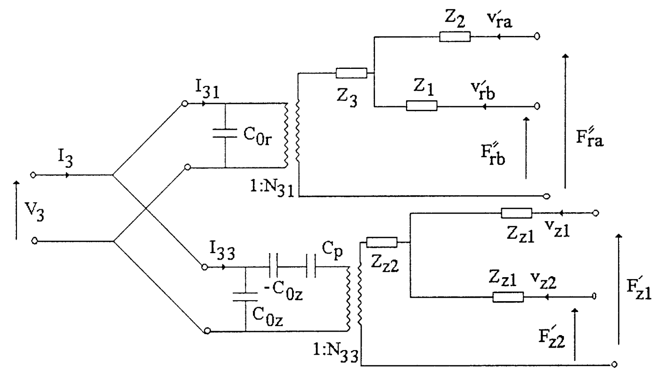

Fig. 4. Equivalent circuit of the piezoelectric ceramic hollow cylinder resonator in coupled vibration.

In Eq. (68), $k_{33}^2 = d_{33}^2/(s_{33}^E \varepsilon_{33}^T)$, k_{33} is just the electro-mechanical coupling coefficient of the piezoelectric ceramic slender rod polarized in the longitudinal vibration. In another case, when the thickness of the cylinder is much less than its radius, we have $l \ll a$. In this case, the mechanical coupling coefficient n becomes zero; the following equations can be obtained:

$$E_r = \frac{s_{11}^E + s_{13}^E n}{(s_{11}^E - s_{12}^E)(s_{11}^E + s_{12}^E + 2s_{13}^E n)} = \frac{1}{s_{11}^E(1 - \nu_{12}^2)}, \tag{70}$$

$$V_r = (E_r/\rho)^{1/2} = \left[\frac{1}{s_{11}^E \rho (1 - \nu_{12}^2)}\right]^{1/2}. \tag{71}$$

It can be seen that in this case the coupled vibration of the hollow cylinder becomes the plane radial vibration of a thin piezoelectric ceramic ring.

3. The resonance frequency equations of the piezoelectric ceramic hollow cylinder in coupled vibration

To analyze the frequency characteristics of the piezoelectric ceramic hollow cylinder resonator in coupled vibration, the frequency equation of the resonator must be derived. In general cases, the frequency characteristics are studied when the vibrator is free from external forces. From Fig. 4, when the cylinder vibrates freely, the external forces can be ignored. In this case, we have

$$F_{r1} = F_{r2} = 0, \quad F_{z1} = F_{z2} = 0. \tag{72}$$

Using Eq. (72), the admittance of the piezoelectric ceramic hollow cylinder resonator in coupled vibration can be obtained as the following form:

$$Y_3 = \frac{I_3}{V_3} = \frac{I_{31} + I_{33}}{V_3} = Y_{31} + Y_{33}. \tag{73}$$

Here, Y_{31} and Y_{33} are the equivalent electric admittances of the piezoelectric ceramic hollow cylinder resonator in equivalent radial and longitudinal vibrations, respectively. For the equivalent radial vibration, its equivalent admittance Y_{31} can be derived as

$$\begin{aligned} Y_{31} = & \frac{\mathrm{j}\omega\varepsilon_{33}^T S}{l} \times \left\{ 1 - (k_p')^2 + (k_p')^2 \frac{2}{a^2 - b^2} \right. \\ & \left. \times \frac{[a\mathrm{J}_1(k_r a) - b\mathrm{J}_1(k_r b)][Y(b) - Y(a)] + [a\mathrm{Y}_1(k_r a) - b\mathrm{Y}_1(k_r b)][J(a) - J(b)]}{J(a)Y(b) - J(b)Y(a)} \right\}, \end{aligned} \tag{74}$$

where $J(a) = J(x)|_{x=a}$, $J(b) = J(x)|_{x=b}$, $Y(a) = Y(x)|_{x=a}$, $Y(b) = Y(x)|_{x=b}$, $J(x)$ and $Y(x)$ are two introduced functions, their expressions are

$$J(x) = [k_r \mathrm{J}_0(k_r x) - 2\mathrm{J}_1(k_r x)/x](1 - \nu_{12} - 2\nu_{13} n)/(1 + \nu_{12}) + k_r \mathrm{J}_0(k_r x), \tag{75}$$

$$Y(x) = [k_r \mathrm{Y}_0(k_r x) - 2\mathrm{Y}_1(k_r x)/x](1 - \nu_{12} - 2\nu_{13} n)/(1 + \nu_{12}) + k_r \mathrm{Y}_0(k_r x). \tag{76}$$

In Eq. (74),

$$(k_p')^2 = \frac{2d_{31}(d_{31} + nd_{33})}{\varepsilon_{33}^T(s_{11}^E + s_{12}^E + 2s_{13}^E n)},$$

k_p' is defined as the electro-mechanical coupling coefficient of the equivalent radial vibration of the cylinder in coupled vibration. It can be expressed as the following form:

$$(k_p')^2 = k_p^2 \times \frac{1 - n/\lambda_{31}}{1 - 2v_{13}n/(1 - v_{12})}. \tag{77}$$

In this equation, $k_p^2 = 2d_{31}^2/[\varepsilon_{33}^T(s_{11}^E + s_{12}^E)], k_p$ is the electro-mechanical coupling coefficient of the piezoelectric ceramic thin ring or plate in plane radial vibration. $\lambda_{31} = -d_{31}/d_{33}, v_{12} = -s_{12}^E/s_{11}^E, v_{13} = -s_{13}^E/s_{11}^E$. For the equivalent longitudinal vibration, its equivalent admittance Y_{33} is

$$Y_{33} = \frac{I_{33}}{V_3} = \frac{\mathrm{j}\omega S\varepsilon_{33}^T}{l} \times \frac{1 - (k_{33}')^2}{1 - (k_{33}')^2 \tan(k_z l/2)/(k_z l/2)}. \tag{78}$$

In Eq. (78),

$$(k_{33}')^2 = \frac{d_{33}}{\varepsilon_{33}^T}\frac{(d_{33} + d_{31}/n)}{(s_{33}^E + s_{13}^E/n)},$$

and k_{33}' is the equivalent longitudinal electro-mechanical coupling coefficient of the hollow cylinder resonator in coupled vibration. It can be rewritten as

$$(k_{33}')^2 = k_{33}^2 \times \frac{1 - \lambda_{31}/n}{1 - v_{31}/n}, \tag{79}$$

where $k_{33}^2 = d_{33}^2/\varepsilon_{33}^T s_{33}^E, k_{33}$ is the electro-mechanical coupling coefficient of the slender piezoelectric ceramic element in longitudinal vibration, and $v_{31} = -s_{13}^E/s_{33}^E$. When the admittance of the hollow cylinder resonator has a maximal value, the resonator will resonate. Therefore, the resonance frequency equations for the piezoelectric ceramic hollow cylinder resonator in coupled vibration can be obtained from Eqs. (74) and (78):

$$1 - (k_{33}')^2 \tan(k_z l/2)/(k_z l/2) = 0, \tag{80}$$

$$J(a)Y(b) - J(b)Y(a). \tag{81}$$

Using the expressions of $J(x)$ and $Y(x)$, we can rewrite Eq. (81) as follows:

$$\begin{aligned} &\frac{k_r a \mathrm{J}_0(k_r a) - (1 - v_{12} - 2v_{13}n)/(1 - v_{13}n)\mathrm{J}_1(k_r a)}{k_r b \mathrm{J}_0(k_r b) - (1 - v_{12} - 2v_{13}n)/(1 - v_{13}n)\mathrm{J}_1(k_r b)} \\ &\quad = \frac{k_r a \mathrm{Y}_0(k_r a) - (1 - v_{12} - 2v_{13}n)/(1 - v_{13}n)\mathrm{Y}_1(k_r a)}{k_r b \mathrm{Y}_0(k_r b) - (1 - v_{12} - 2v_{13}n)/(1 - v_{13}n)\mathrm{Y}_1(k_r b)}. \end{aligned} \tag{82}$$

Eqs. (80) and (82) are the resonance frequency equations for the piezoelectric ceramic hollow cylinder resonator in coupled vibration. It seems that these two frequency equations are similar to those for the longitudinal vibration of a slender ceramic rod and the plane radial vibration of a thin ceramic plate. However, they are different. The difference is that for Eqs. (80) and (82), they are not independent of each other, but are correlated by the mechanical coupling coefficient.

In Eqs. (80) and (82), when the geometrical dimensions and the material parameters of the longitudinally polarized piezoelectric ceramic hollow cylinder are given, the unknown quantities are the angular frequency and the mechanical coupling coefficient. Therefore, from these two equations we can get the resonance frequencies of the hollow cylinder in coupled vibration. Since the resonance frequency equations are transcendental equations with two unknowns, it is difficult to find the solutions by an analytic method. Therefore, numerical methods are used. The procedures to solve the frequency equations are as follows: First, by assigning a value of the mechanical coupling coefficient, from Eqs. (80) and (82), two frequencies f_1' and f_2' can be obtained. Repeat this procedure by varying the value of the mechanical coupling coefficient until f_1' and f_2' become equal. In this case, the mechanical coupling coefficient and the frequency are the solutions to the frequency equations (80) and (82). In practical cases, the frequency equations are solved using mathematical softwares, such Mathematica or Matlab.

From the computed results it can be seen that when the geometrical dimensions and the material parameters of the piezoelectric ceramic hollow cylinder are given, for a certain vibrational mode, such as the fundamental mode, there exists two groups of solutions that are noted as f_r, n_r and f_z, n_z. From the above analysis, considering the practical vibrational characteristics, it can be seen that these two groups of solutions represent the equivalent longitudinal vibration and the plane radial vibration of the hollow cylinder in coupled vibration, respectively. The frequencies f_r and f_z are the resonance frequencies of the equivalent plane radial vibration mode and the longitudinal vibration mode of the hollow cylinder when the mechanical coupling between the radial and the longitudinal vibrations and the piezoelectric effect are considered. From the analysis and the computed results, it can also be seen that when the height is much less or larger than the radius of the hollow cylinder, the computed frequencies f_r and f_z are far away from each other; this is consistent with the measured results for a longitudinally polarized piezoelectric ceramic slender rod ($l \gg a$) or a thin disk ($l \ll a$). In these cases, the mechanical coupling between the radial and the longitudinal vibrations is weak; the vibration of the resonator can be regarded as decoupling, such as the one-dimensional longitudinal vibration of a slender ceramic rod or the plane radial vibration of a thin ceramic plate. However, if the height is comparable with the radius of the hollow cylinder, the resonance frequencies f_r and f_z are also comparable; in this case, the mechanical coupling is intense, and the vibration of the hollow cylinder is a complex coupled vibration.

4. Experiments

The resonance frequencies of the piezoelectric ceramic hollow cylinder resonators are measured to test and verify the proposed theory for the analysis of the hollow cylinder resonator in coupled vibration. The piezoelectric ceramic material used here is an equivalent of PZT-4 that is made in China. The standard material parameters are used in the design and calculation. The material parameters are: $\rho = 7500\ \mathrm{kg/m^3}$, $k_p = 0.58$, $k_{33} = 0.70$, $s_{11}^E = 12.3 \times 10^{-12}\ \mathrm{m^2/N}$, $s_{12}^E = -4.05 \times 10^{-12}\ \mathrm{m^2/N}$, $s_{13}^E = -5.31 \times 10^{-12}\ \mathrm{m^2/N}$, $s_{33}^E = 15.5 \times 10^{-12}\ \mathrm{m^2/N}$, $d_{31} = -123 \times 10^{-12}\ \mathrm{C/N}$, $d_{33} = 496 \times 10^{-12}\ \mathrm{C/N}$, $\varepsilon_{33}^T/\varepsilon_0 = 1300$. The resonance frequencies of the piezoelectric ceramic hollow cylinder in coupled vibration are measured using HP 4294A precision impedance analyzer. The geometrical dimensions, the computed resonance frequencies, and the measured results are shown in Table 1, where f_r and f_z are the computed radial and longitudinal frequencies of the hollow cylinder in coupled vibration and f_{rm} and f_{zm} are the measured results. For comparison, the resonance frequencies f_{1r} and f_{1z} that are computed from one-dimensional theory are also given in the table.

Table 1
Calculated and measured resonance frequencies for the piezoelectric ceramic hollow cylinder in coupled vibration

l (mm)	a (mm)	b (mm)	f_{1r} (kHz)	f_{1z} (kHz)	f_r (kHz)	f_z (kHz)	f_{rm} (kHz)	f_{zm} (kHz)
5.0	19.0	7.5	42.38	307.47	42.30	314.66	41.57	325.73
6.0	20.0	8.0	40.05	256.23	40.04	256.39	40.17	258.81
6.0	25.0	8.5	34.13	256.23	34.06	263.66	35.05	264.83
5.0	25.0	11.0	30.80	307.47	30.75	315.50	30.36	321.73
8.0	30.0	7.0	32.11	192.17	31.96	196.92	31.43	209.59
8.0	30.0	8.0	30.89	192.17	30.78	197.91	×	×
8.0	30.0	9.0	29.73	192.17	29.64	198.93	×	×

In this paper, the fundamental vibrational mode of the piezoelectric ceramic hollow cylinder in coupled vibration is studied. The reason is that in most cases the fundamental vibration mode of piezoelectric vibrators is widely used. It has high electro-mechanical coupling coefficient, sensitivity, and low mechanical and dielectric loss. As for the higher vibrational modes, the analysis is similar to that described in the above sections. For example, for the second vibrational mode, the second roots of Eqs. (80) and (82) must first be found. Then the resonance frequency for the second vibrational mode of the cylinder resonator can be computed. However, the analysis for the high vibrational modes is complex. The reason is that the modal interaction must be considered. For example, for the second vibrational mode, the interaction between the radial vibration of order one and the longitudinal vibration of order two, the interaction between the thickness vibration of order one and the radial vibration of order two, and the interaction between the radial vibration of order two and the longitudinal vibration of order two must be analyzed and considered at the same time.

As for the frequency error, it is thought that two kinds of errors should be considered. One is the systematic error; the other is the random error. The systematic error is caused by the appropriate analysis method, while the random error is determined by many uncertain factors. Sometimes the uncertain factors are more important, they affect the measured results. Considering the above facts, the following factors should be taken into account for the frequency error analysis: (1) The standard material parameters are different from the practical values. An error of 3%–5% can be caused by this factor. (2) In this method, to simplify the analysis, the mechanical coupling coefficient is considered as a constant. However, the mechanical coupling coefficient is different at different positions in the cylinder resonator. (3) The longitudinal and the radial extensional vibrations in the resonator are supposed. However, for the coupled vibration of the piezoelectric ceramic hollow cylinder, the shearing and other strains may exist in the resonator. (4) The analytical method presented in this paper is an approximate one. It can be used to analyze the resonance frequency of resonators in coupled vibration; however, it cannot be used to calculate the vibrational displacement distribution.

In order to understand the effect of geometrical dimensions of the piezoelectric ceramic tubes on the resonance frequency, the effect of the inner radius of the ceramic tube is studied. Using the coupled vibration theory, the resonance frequency of the piezoelectric ceramic tubes with different inner radius is calculated by changing the value of the inner radius while fixing the outer radius and the thickness. The theoretical results are shown in Table 1. It can be seen that when the inner radius is increased while fixing the outer radius and the thickness, the one-dimensional radial resonance frequency is decreased, while the one-dimensional longitudinal resonance frequency is unchanged. The radial resonance frequency by the coupled vibrational theory is decreased, while the longitudinal resonance frequency by the coupled vibrational theory is increased. On the other hand, it can also be seen that the radial resonance frequency by the coupled vibrational theory is lower than that by one-dimensional theory, while the longitudinal resonance frequency by the coupled vibrational theory is higher than that by one-dimensional theory.

From the above analysis, it can be seen that the resonance frequency for a piezoelectric ceramic resonator in coupled vibration can be calculated by the developed approximate analytical method. Compared with numerical methods, this method is concise in physical concept, the calculation is time-saving, and the analysis of the results is simple. However, this method cannot be used to analyze the vibrational mode function, and therefore, it is difficult to obtain the vibrational displacement distribution in the resonator in coupled vibration.

5. Conclusions

In this paper, the coupled vibration of a longitudinally polarized piezoelectric ceramic hollow cylinder is studied. An approximate analytic method is developed to analyze the complex coupled vibration, and the equivalent circuit for the ceramic cylinder in coupled vibration is obtained. To sum up the above analysis, the following conclusions can be drawn:

(1) When the mechanical coupling coefficient is introduced, the complex coupled vibration of the piezoelectric ceramic hollow cylinder can be divided into two equivalent extensional vibrations: one is the longitudinal vibration, and the other is the plane radial vibration. However, these two vibrations are not independent of each other. They are correlated by the mechanical coupling coefficient.

(2) The equivalent circuit for the piezoelectric ceramic hollow cylinder in coupled vibration is derived. It can be used in the frequency analysis of the hollow cylinder in coupled vibration.

(3) There are two kinds of resonance frequencies for the coupled vibration of the piezoelectric ceramic hollow cylinder; one is the radial resonance frequency and the other is the longitudinal resonance frequency. These two frequencies are different from those from one-dimensional theory.

(4) In general cases, the vibration of the hollow cylinder with comparable dimensions is a complex coupled vibration. However, when the geometrical dimensions satisfy certain conditions, the vibration can be regarded as one-dimensional vibration, such as the longitudinal vibration of a slender ceramic rod, or the radial vibration of a thin piezoelectric ceramic plate.

(5) In this paper, the shearing strains and torsion are ignored and the vibration of the hollow cylinder is assumed to be a coupled one of two equivalent extensional vibrations.

(6) The method is simple and the resonance frequencies obtained are in good agreement with the measured results.

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