Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Sound and Vibration 275 (2004) 991–1008

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

An asymmetric indirect Trefftz method for solving free-vibration problems

J.R. Chang[a,*], R.F. Liu[b]

[a] Department of System Engineering and Naval Architecture, National Taiwan Ocean University, 2 Pei-Ning Road, Keelung 202, Taiwan, ROC

[b] Department of Harbor and River Engineering, National Taiwan Ocean University, 2 Pei-Ning Road, Keelung 202, Taiwan, ROC

Received 15 April 2003; accepted 4 July 2003

Abstract

In this paper, a new asymmetric indirect Trefftz method (AITM) has been developed to solve free-vibration problems. The proposed method is categorized into a regular type of boundary element methods (BEMs) such that no singular or hypersingular integration is necessary. However, like other regular BEMs the proposed approach encounters the numerical instability as the number of elements increases. To deal with such an ill-posed behavior, Tikhonov's regularization method in conjunction with the generalized singular-value decomposition (GSVD) is adopted. It is proved that the degeneracy of the proposed indirect Trefftz method has the same mathematical structure as the direct Trefftz method. Thus, no special effort should be paid in programming. Besides, such an equivalency indicates that the current method does not have spurious eigensolutions. Furthermore, the proposed approach can easily treat a multiply connected domain of genus 1 as shown in Fig. 1. Due to its indirect nature, the present approach can also represent the mode shapes within its own mathematical formulations. Several numerical examples are given to show the validity of the proposed approach.

1. Introduction

For the solution of a free-vibration problem, many well-developed numerical techniques such as the finite element method (FEM), finite difference method (FDM) and boundary element method (BEM) can be adopted. The BEM requires discretization on boundary only so that it needs less mess effort than the other two. The corresponding fundamental solution for the

*Corresponding author. Tel.: +886-2-24622192 ext 6031; fax: +886-2-2462-5945.
E-mail address: cjr@sena.ntou.edu.tw (J.R. Chang).

0022-460X/$ - see front matter
doi:10.1016/S0022-460X(03)00813-7

Helmholtz equation, which is often used to describe the free-vibration problem for a finite domain, is the complex-valued Hankel function. It is then not surprising that a complex-valued computation should be considered in solving the Helmholtz equation by the BEM [1]. To avoid complex-valued computations, an augmented matrix is sometimes used which doubles the dimension of the influencing matrix. In order to reduce the numerical effort involving complex-valued computations, several incomplete BEM formulations have been proposed and we will give a brief review of those in the following, to our best knowledge.

Historically speaking, the first incomplete BEM for solving the eigenproblem was presented by De Mey [2]. He proposed a replacement of complex-valued fundamental solution by its corresponding real part. This BEM formulation is also referred to as the real-part BEM later by many researchers [3,4]. Hutchinson [5] pointed out that such a real-part formulation resulted in spurious eigensolutions. To avoid complicated computations in the domain of a complex number, Nowak and Brebbia [6], Kamiya and Andoh [7,8] developed the multiple reciprocity boundary element method (MR/BEM) to deal with the eigenproblem in the real number domain. Traditionally, the MR/BEM employed the fundamental solution of the Laplace operator as the zeroth order fundamental solution and used the reciprocity theorem to exploit a series of higher order fundamental solutions in the approximation of homogeneous terms. Chen and Wong [9] discovered the existence of a spurious eigensolution in MR/BEM by providing an analytical derivation for a one-dimensional problem. Later, Kamiya and Andoh [7] discovered the equivalence of the MR/BEM and the real-part BEM. Yeih et al. [10] further explained that the spurious eigensolution exists owing to only using the real-part kernels. Up to date, several techniques have been proposed and developed to filter out the spurious eigensolutoins, e.g., the singular-value decomposition (SVD) by Yeih et al. [11], the threshold method by Liou et al. [12], the domain partition technique by Chang et al. [3], the GSVD by Kuo et al. [13], and adding additional points outside the domain by Chen et al. [14]. In the above-mentioned approaches, although the BEM formulations are incomplete the kernels used are still singular.

Another trend to deal with the Helmholtz eigenproblems is to use the regular BEM. Unlike the singular type BEM, the regular BEM adopts non-singular kernels to construct the boundary integral equations. The first regular BEM was originated by De Mey [2] to calculate eigenvalues of the Helmholtz equation. He pointed out that one could replace the complex-valued fundamental solution by its imaginary part, which is a regular solution. Besides, he also proposed another source-free formulation, in which a solution satisfying the Helmholtz equation was adopted, as the auxiliary system. However, he could not reach the correct answer by means of this regular approach. From the mathematical point of view, no matter which regular formulations are used, the same situation should be encountered. It is then very puzzling why De Mey [2] claimed that one could obtain the solution using one regular formulation but have no definite result using another kind of the regular formulations. Kim and Kang [15] used the wave-type base functions, which are periodical along each element and propagating into the domain of interest, to construct the needed equations. They pointed out that some incorrect answers would appear and explained this phenomenon as due to incompleteness of base functions. Later, Kang et al. [16] proposed another regular formulation using the so-called non-dimensional dynamic influence functions. Simply speaking, their method took the response at any point inside the domain of interest as a linear combination of influences from all boundary source points. In fact, the wave-type base functions and the non-dimensional dynamic influence functions are the non-singular part of the complex fundamental solution of the Helmholtz equation and more specifically, called the general solution or the imaginary part of the fundamental solution. For the non-singular general solution as the radial basis function (RBF) and its usage, Chen [17], Chen and Tanaka [18] proposed the boundary particular method (BPM) and the boundary knot method (BKM), respectively. The BKM and the BPM use the general solution in conjunction respectively with the dual reciprocity principle and the multiple reciprocity principle. Both methods are mesh free boundary techniques and have been applied to the Helmholtz problems. Sound and convincing demonstrated results show that both of them, unlike the Kang and Kim's method [15], can handle inhomogeneous problems successfully. In addition, the BKM has a symmetric formulation for the Helmholtz problem [19]. Thus, the non-singular general solution formulation is not necessarily asymmetric. The symmetric BKM scheme will be useful in the efficient solution of the Helmholtz eigenvalue problems.

Recently, an important work on the regular BEM proposed by Kuo et al. [13] revealed several important issues for the incomplete BEMs, including both of the regular type and singular ones. The first is that the spurious eigensolution existing in the boundary integral equation is similar to the concept of an indefinite form of zero divided by zero. The second is that the regular BEM encounters the numerical instability when the number of elements increases. Meanwhile, a combined use of the Tikhonv's regularization method and the GSVD was also proposed to deal with the spurious eigensolutions and numerical instability at the same time. In their work, the two kinds of the regular formulations, the imaginary-part dual BEM and the plane wave method, were used. However, their methods encounter difficulties in dealing with a multiply connected domain of genus 1 and they cannot represent field quantities within their own mathematical structure.

Another candidate for the regular BEM is the Trefftz method. The Trefftz method can be viewed as an eigenfunction approach for the solution of the partial differential equations. For a specific problem, the so-called T-complete functions can be constructed based on its geometry and operator. By using the reciprocity theorem or the generalized weight residual method, a boundary type integral equation can be constructed either in the direct manner or indirect one. The Trefftz method has been adopted to solve many problems, such as the plane elasticity problem by Jin et al. [20], the Kirchhoff plate bending problem by Jin et al. [21], and the acoustic problem by Harari et al. [22]. In addition, two important review articles addressing on the Trefftz method [23] and various formulations with available boundary-type solution procedures [24] can be found. For the Helmholtz operator, Cheung et al. [25], Huang and Shaw [26] used the Trefftz method to solve the radiation problems. However, to our best knowledge, few attempts [27–29] addressed the use of the Trefftz method to deal with the free-vibration problems of a finite domain. This may arise from its numerical instability nature.

Following with our previous studies [28,29], the main goal of this paper is to construct the AITM to solve the free-vibration problem. In particular, we wish to develop a regular BEM that can represent field quantities and deal with a multiply connected domain of genus 1. It is found that the AITM has the same mathematical structure as the direct one does; therefore, there exists no spurious eigenvalue but only numerical instability in this proposed approach. A combined use of Tikhonov's regularization method and the GSVD is adopted to treat inherent numerical instability. Five numerical examples including two benchmark examples originally designed for the direct Trefftz method [28] are provided to show the validity of our proposed approach.

2. Mathematical backgrounds

2.1. Direct Trefftz formulation

Consider a finite domain as shown in Fig. 1, the governing equation, i.e., the Helmholtz equation, for a free-vibration problem is written as

$$(\nabla^2 + k^2)u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{1}$$

where ∇^2 is the Laplace operator, k is the wave number, and $u(\mathbf{x})$ is the physical quantity at $\mathbf{x}$.

The direct Trefftz formulation is constructed as follows [28]. Let $W(\mathbf{x})$ be a field satisfying the Helmholtz equation

$$(\nabla^2 + k^2)W(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{2}$$

then by the reciprocity theorem one can have

$$\int_{\Gamma} W(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \, \mathrm{d}\Gamma(\mathbf{x}) = \int_{\Gamma} u(\mathbf{x}) \frac{\partial W(\mathbf{x})}{\partial n} \, \mathrm{d}\Gamma(\mathbf{x}), \tag{3}$$

where n is the out-normal direction at the boundary point $\mathbf{x}$ and Γ denotes the boundary. The choice of $W(\mathbf{x})$ depends on the problem itself. A complete set of $W(\mathbf{x})$, written as $\{W_i(\mathbf{x})\}$, is chosen to give enough bases to represent physical quantities. This complete set is called the T-complete function set, which provides the complete function bases to represent physical fields. For example, a simply connected domain shown in Fig. 1(a) and having the origin located inside

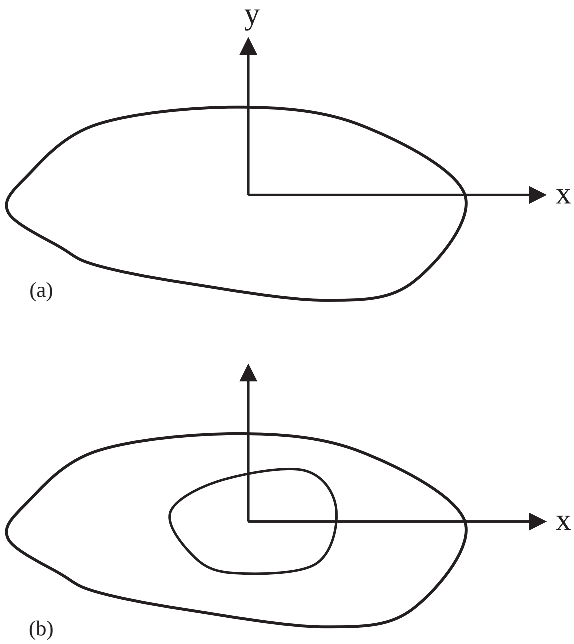

Fig. 1. (a) A simply connected domain. (b) A doubly connected domain of genus 1.

the interested domain, it is convenient to have the T-complete set as

$$\{J_0(kr), J_m(kr)\cos(m\theta), J_m(kr)\sin(m\theta)\} \quad \text{for } m = 1, 2, 3\ldots, \tag{4}$$

in which J_m is the first kind Bessel function of mth order, r is the Euclidean distance from the origin to a domain point, and θ is the angle between the x-axis and the radial vector from the origin to the domain point. For a multiply connected domain of genus 1 (i.e., with one hole) and locating the origin inside the hole as shown in Fig. 1(b), the T-complete set is

$$\{J_0(kr),\ Y_0(kr), J_m(kr)\cos(m\theta), J_m(kr)\sin(m\theta),\ Y_m(kr)\cos(m\theta),\ Y_m(kr)\sin(m\theta)\}$$
$$\text{for } m = 1, 2, 3\ldots, \tag{5}$$

where Y_m is the second kind Bessel function of mth order.

For the Robin (radiation) boundary condition, $\alpha_1 u + \beta_1 t = 0$ where $t(\mathbf{x}) \equiv \partial u(\mathbf{x})/\partial n$ one can assign

$$u = \beta_1 \psi, \quad t = -\alpha_1 \psi \tag{6}$$

then substituting them into Eq. (3) yields

$$\int_\Gamma \left[\alpha_1 W(\mathbf{x}) + \beta_1 \frac{\partial W(\mathbf{x})}{\partial n}\right] \psi(\mathbf{x})\, \mathrm{d}\Gamma(\mathbf{x}) = 0. \tag{7}$$

Changing the base functions, $W(\mathbf{x})$, and adopting constant piecewise discretization for boundary unknowns Ψ can yield

$$\{\alpha_1[\tilde{\mathbf{U}}] + \beta_1[\tilde{\mathbf{T}}]\}[\psi] = 0, \tag{8}$$

where the components of the matrices $[\tilde{\mathbf{U}}]$ and $[\tilde{\mathbf{T}}]$ are

$$\tilde{U}_{ij} \equiv \int_{\Gamma_j} W_i(\mathbf{x})\, \mathrm{d}\Gamma(\mathbf{x}), \tag{9a}$$

$$\tilde{T}_{ij} \equiv \int_{\Gamma_j} \frac{\partial W_i(\mathbf{x})}{\partial n}\, \mathrm{d}\Gamma(\mathbf{x}), \tag{9b}$$

in which Γ_j is the jth element on the boundary and $W_i(\mathbf{x})$ is the ith base function.

2.2. No spurious eigensolution in the direct Trefftz method

It has been proven that there exists no spurious eigensolution in the direct Trefftz method [28]. For the readers' convenience, we briefly introduce the proof strategy below. Consider the original problem having boundary condition $\alpha_1 u + \beta_1 t = 0$ on the boundary, and the corresponding influencing matrix $\mathbf{A}_1$ is

$$\mathbf{A}_1 = \alpha_1 \tilde{\mathbf{U}} + \beta_1 \tilde{\mathbf{T}}. \tag{10a}$$

Let us pick another complementary problem with the boundary condition $\alpha_2 u + \beta_2 t = 0$ on the boundary such that $\det\begin{Vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{Vmatrix} \neq 0$, the influencing matrix $\mathbf{A}_2$ for this complementary problem is

$$\mathbf{A}_2 = \alpha_2 \tilde{\mathbf{U}} + \beta_2 \tilde{\mathbf{T}}. \tag{10b}$$

These two systems cannot have the same eigensolution. That is, at a specific wave number k, it is impossible to have the same non-trivial boundary eigensolution $\Psi(\mathbf{x})$ for both systems [13].

Lemma 1. *Given that the governing equation is a Helmholtz equation,* $(\nabla^2 + k^2)u(\mathbf{x}) = 0$, *for a domain* Ω *enclosed by the boundary* Γ, *and that the overspecified homogeneous boundary conditions are* $u(\mathbf{x}) = 0$ *and* $t(\mathbf{x}) = 0$ *for* $\mathbf{x}$ *on a sub-boundary* $\Gamma_1 \subset \Gamma$, *there exists a unique solution,* $u(\mathbf{x}) = 0$ *for* $\mathbf{x} \in \Omega \cup \Gamma$.

Definition 1. Two sets of boundary conditions, $\alpha_1(\mathbf{x})u(\mathbf{x}) + \beta_1(\mathbf{x})t(\mathbf{x}) = 0$ and $\alpha_2(\mathbf{x})u(\mathbf{x}) + \beta_2(\mathbf{x})t(\mathbf{x}) = 0$, where $\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x}), \beta_1(\mathbf{x})$ and $\beta_2(\mathbf{x})$ are given functions, are said to be homogeneous linearly independent boundary conditions if and only if $\det\begin{Vmatrix} \alpha_1(\mathbf{x}) & \beta_1(\mathbf{x}) \\ \alpha_2(\mathbf{x}) & \beta_2(\mathbf{x}) \end{Vmatrix} \neq 0$ for any $\mathbf{x}$ on the boundary.

Theorem 1. *For the Helmholtz equation, given two systems having homogenous linear independent boundary conditions on part of the boundary denoted as* Γ_1, *it is impossible for both systems to have the same eigensolution.*

Theorem 1 supports the conclusion we mentioned above and also hints that if there exists an ‘eigensolution’ to make two systems with homogeneous linear independent boundary conditions, degenerated at the same time, it must be a spurious eigensolution. Following this, we can have the following theorem and its proof is given in Ref. [28].

Theorem 2. *For the Helmholtz equation, with a boundary condition* $\alpha_1 u + \beta_1 t = 0$ *the direct Trefftz formulation* $\mathbf{A}_1(k)\Psi = 0$ *cannot have a spurious eignesolution.*

The Trefftz method adopts non-singular base functions and can thus be categorized into the regular BEM formulations. However, the regular formulation leads to the ill-posed behaviors as the number of elements increase. Kuo et al. [13] pointed out the culprit and proposed a combined use of Tikhonov’s regularization method and GSVD to fix it. In the following we simply introduce their approach since we will use the same technique later on.

2.3. Techniques to treat numerical instability for a regular BEM

From Theorem 1, it can be seen that the spurious eigensolution will appear in two systems having homogeneous linearly independent boundary conditions simultaneously. That is, we have a system as $[\mathbf{A}_1]_{n\times n}\Psi_{n\times 1} = [\mathbf{A}_2]_{n\times n}\Psi_{n\times 1} = 0$. Since both problems can have common spurious eigensolutions, we can intuitively decompose both matrices into the following form

$$\mathbf{PW}_1\mathbf{x} = \mathbf{PW}_2\mathbf{x} = 0, \tag{11}$$

where $\mathbf{PW}_1 = \mathbf{A}_1$ and $\mathbf{PW}_2 = \mathbf{A}_2$. Then the spurious eigenvalues will result in the rank deficiency of matrix $\mathbf{P}$ and true eigenvalues will result in the rank deficiency of matrix $\mathbf{W}_1$ for the original problem. When the spurious eigenvalues are encountered, basically we want to extract them out by finding matrix $\mathbf{P}$. This concept is quite similar to perform a numerical operation of L'Hospital rule on an indefinite 0/0 expression. The above-mentioned technique can be achieved using the QR factorization, which is the first step of the generalized singular value decomposition.

Now the serious problem we encounter for the regular BEM, is not spurious eigensolution but numerical instability. To treat this, we need to add some small quantities into the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ to make the numerically tiny singular values occurring in both matrices become 'numerical spurious eigenvalues' such that the QR factorization can extract them. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ have the following singular value decompositions:

$$\mathbf{A}_1 = \mathbf{P}\Sigma_1\mathbf{V}_1^*, \tag{12a}$$

$$\mathbf{A}_2 = \mathbf{P}\Sigma_2\mathbf{V}_2^*, \tag{12b}$$

where $\mathbf{V}_i$ is the right unitary matrix of system i, superscript '*' means take the transpose and complex-conjugate of the matrix, and Σ_i is a singular value matrix of system i with singular values allocated in the diagonal line. When one of the singular values is numerically very small at a specific wave number, it can be said that the system degenerates, i.e., that the wave number is an eigenvalue. However, when a non-singular BEM is adopted, there exist many numerical tiny values in the singular values, which are not true zeros. This phenomenon becomes very severe when the number of elements increases and/or a direct eigenvalue search is used at a low wave number. Now let us add two small quantities in the singular value matrices to construct new influencing matrices as

$$\hat{\mathbf{A}}_1 = \mathbf{P}(\Sigma_1 + \varepsilon_1\mathbf{I})\mathbf{V}_1^*, \tag{13a}$$

$$\hat{\mathbf{A}}_2 = \mathbf{P}(\Sigma_2 + \varepsilon_2\mathbf{I})\mathbf{V}_2^*, \tag{13b}$$

where ε_i is the small value added to system i. The choice of ε_i is dependent on the problem itself; however, if they are larger than the unreasonable tiny values of singular values in the original two systems, but still small enough not to overcoat the true eigenvalue, one can then successfully extract the contaminated tiny value. If one takes the QR factorization of $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$, the unreasonable ones can be extracted. Adding such a small value in the singular value cannot change the facts of true degenerated singular value. That is, at the true eigenvalue, the singular value of system one should approach zero but its corresponding part in system two will not be close to zero. Using this method, we can successfully treat the ill-posed behaviors of the problem [13,28] and the numerical examples will be given in the next section.

2.4. Asymmetric indirect Trefftz method

Since a direct type BEM cannot represent mode shapes within its own mathematical structure, to develop an indirect Trefftz method, unlike the direct method in which the integral equation about physical quantities is constructed directly, the indirect method represents physical quantities by a linear superposition of sources. Following the idea of the Trefftz method, any physical quantities should be represented by a linear superposition of the base functions in

T-complete set; therefore, we can have

$$u(\mathbf{x}) = \sum_k a_k W_k(\mathbf{x}), \tag{14a}$$

$$t(\mathbf{x}) = \frac{\partial u(\mathbf{x})}{\partial n} = \sum_k a_k \frac{\partial W_k(\mathbf{x})}{\partial n}, \tag{14b}$$

where a_k is the undetermined coefficient. Consider a boundary value problem, $\alpha_1 u + \beta_1 t = 0$, one can directly yield the following boundary integral equation:

$$\sum_k \int_{\Gamma_j} \left\{ \alpha_1 W_k + \beta_1 \frac{\partial W_k}{\partial n} \right\} [a_k] \, \mathrm{d}\Gamma = 0. \tag{15}$$

In the above equation, the integral equation is not constructed on the whole boundary but on an element along the boundary. The reason why we can do this is because that the homogeneous boundary condition is always true on any boundary point. Another approach from a generalized weight residual method can also be used to derive this equation, and the readers can refer to the work of Mikhlin and Smoltskiy [30]. Eq. (15) can be rewritten in a matrix form as

$$\{\alpha_1 [\tilde{\mathbf{U}}]^{\mathrm{T}} + \beta_1 [\tilde{\mathbf{T}}]^{\mathrm{T}}\}[\mathbf{a}] = 0. \tag{16}$$

Now let us take a look at Eqs. (16) and (8), it can be found that the influence matrices of the direct Trefftz method and the proposed AITM are mutually transposed. From the knowledge of linear algebra, it can be said that the degeneracy of these two influencing matrices occurs at the same wave number. Following this, one can conclude that no spurious eigensolution exists in the AITM but numerical instability still occurs in the AITM. In order to treat the numerical instability, the method mentioned in the previous subsection is adopted.

To obtain the non-trivial vector [**a**], we can pick up the corresponding singular vector in the SVD method as mentioned in Chen's work [14] (remember that a GSVD contains two steps: QR factorization and the SVD). After obtaining non-trivial vector [**a**], one can use Eq. (14a) to construct the corresponding mode shape easily. The multiple roots can also be detected using the SVD as mentioned in Huang's work [31].

The matrices in the AITM are asymmetric. A symmetric indirect Trefftz method is possible and has been worked out by the authors and others [29]. Although this AITM requires larger memory to store matrices, it has some merits that a symmetric indirect method cannot compete with. The first one is that the AITM can directly use the computer program of the direct Trefftz method. The second is that when a multiply connected domain is treated, the AITM is less sensitive to the numerical errors. Besides, when the geometry of the multiply connected domain becomes more complicate, it is expected that higher order Bessel functions are required. Nevertheless, these higher order Bessel functions tend to zero or infinity at a low kr value. The order of the Bessel basis functions in the symmetric indirect Trefftz method [29] is two times that of the current method. A higher order of Bessel basis functions definitely makes the influence matrix more inaccurate.

To deal with the problem of a multiply connected domain of genus 1, a domain partitioning technique, in which an artificial boundary should be introduced, has been suggested by Kang and Lee [32]. Although such an artificial boundary can overcome the difficulty in treating a multiply connected domain, the choice of this artificial boundary is arbitrary and thus it loses robusticity and is not easy to implement for inexperienced users. In addition, their method unavoidably enlarges the dimensions of matrices due to introducing this artificial boundary. However, our method can overcome the difficulty in treating a multiply connected domain of genus 1 without introducing any artificial boundary. The only thing we should do is simply to place the origin inside the hole and pick the T-complete function set as shown in Eq. (5).

To develop a regular method that can represent mode shape within its own mathematical structure, let us first explain why the direct-type regular BEM cannot represent physical quantities within its own structure. For the direct type, the physical quantities on the boundary are used directly in the formulation. If the BEM is a direct regular formulation, it leads to the following expression symbolically:

$$[\mathbf{U}][\mathbf{t}] - [\mathbf{T}][\mathbf{u}] = 0. \tag{17}$$

From Eq. (17), one can obtain the boundary unknown data. However, Eq. (17) does not tell us any information about inner points. The influencing matrices, $[\mathbf{U}]$ and $[\mathbf{T}]$, are built by placing observation point and source point on the boundary. Even we change the observation point to an inner point; Eq. (17) becomes a trivial equation. It means that the direct-type regular formulation cannot construct the physical quantities inside the domain within its own formulation and thus requires help from other formulations.

On the other hand, the indirect type BEM represents physical quantities inside the domain by a superposition of some sources. The unknowns in the resulting equation are strengths of these sources. After obtaining the strengths of sources, the solution can then be constructed easily. This merit also keeps for the indirect regular BEM. In our opinion, the indirect type regular BEM is more practical than a direct one because it can represent mode shapes easily. As for the computing cost of the AITM, apart from mesh efforts, we have to recognize that it will take more computation time as compared with the traditional BEM for eigenvaluse searching, since the user usually combines the latter with the LU decomposition to implement and the former the GSVD. However, the latter can be used to treat the well-posed problem only. In contrast, the former can be adopted to deal with the well-posed and ill-posed problems simultaneously. Moreover, unlike the LU decomposition only available for eigenvalue searching but not eigenmodes, the GSVD can also be used to obtain the eigenvalue and the mode shape within its own numerical schemes simultaneously.

3. Numerical examples

Example 1. A semi circular domain with the radius $R_0 = 1.4$ and the Dirichlet boundary condition, $u = 0$, are given.

Forty-one constant elements are used, and the Neumann condition problem, $t = 0$, is chosen as the auxiliary problem. By using Tikhonov's regularization method and the GSVD, eigenvalues are found successfully as shown in Fig. 2. In this figure, the value in the bracket is the analytical solution by using the dual complex BEM. The first three modes are obtained and illustrated in Figs. 3(a)–(c).

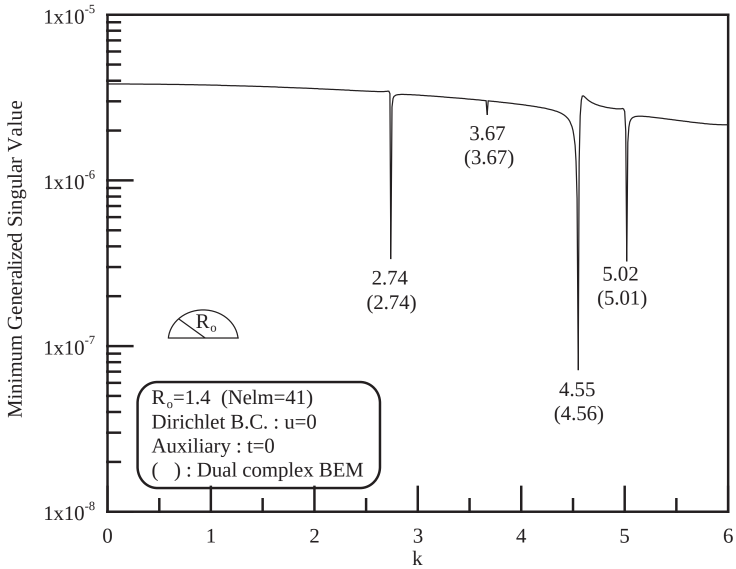

Fig. 2. Eigenvalue searching for a semi circular domain subjected to the Dirichlet boundary condition.

Example 2. A circular domain with the radius $R_0 = 1.0$ and the Neumman boundary condition, $t = 0$, are given.

In this example, we can see that the AITM is valid for all kinds of boundary conditions. Fifty-one constant elements are used and the Dirichlet condition problem, $u = 0$, is used as the auxiliary problem. By using the proposed method, eigenvalues are successfully found as in the direct Trefftz method [28] and are very close to the analytical values, as shown in Fig. 4.

Example 3. A rectangular membrane with two edges of lengths $L_a = 1.5$ and $L_b = 1.0$ is given, respectively and the Neumann boundary condition, $t = 0$, is prescribed.

In this example, a domain without radial symmetry is illustrated. Forty-one constant elements are used and the Dirichlet boundary problem, $u = 0$, is chosen as the auxiliary problem. It can be found in Fig. 5 that the numerical results match the analytical solutions very well. The first three modes can be found easily by the current method as shown in Fig. 6(a)–(c).

We have claimed that any problem having a linearly independent boundary condition to the original problem can be used as an auxiliary problem. In this example, we use another auxiliary problem $2u + 3t = 0$ to support our argument. The results are shown in Fig. 7, and our approach works very well as expected.

Example 4. An annular region of the outer radius $R_0 = 1.0$ and inner radius $R_i = 0.2$, is given, respectively and a Dirichlet boundary condition, $u = 0$, is prescribed on the boundary.

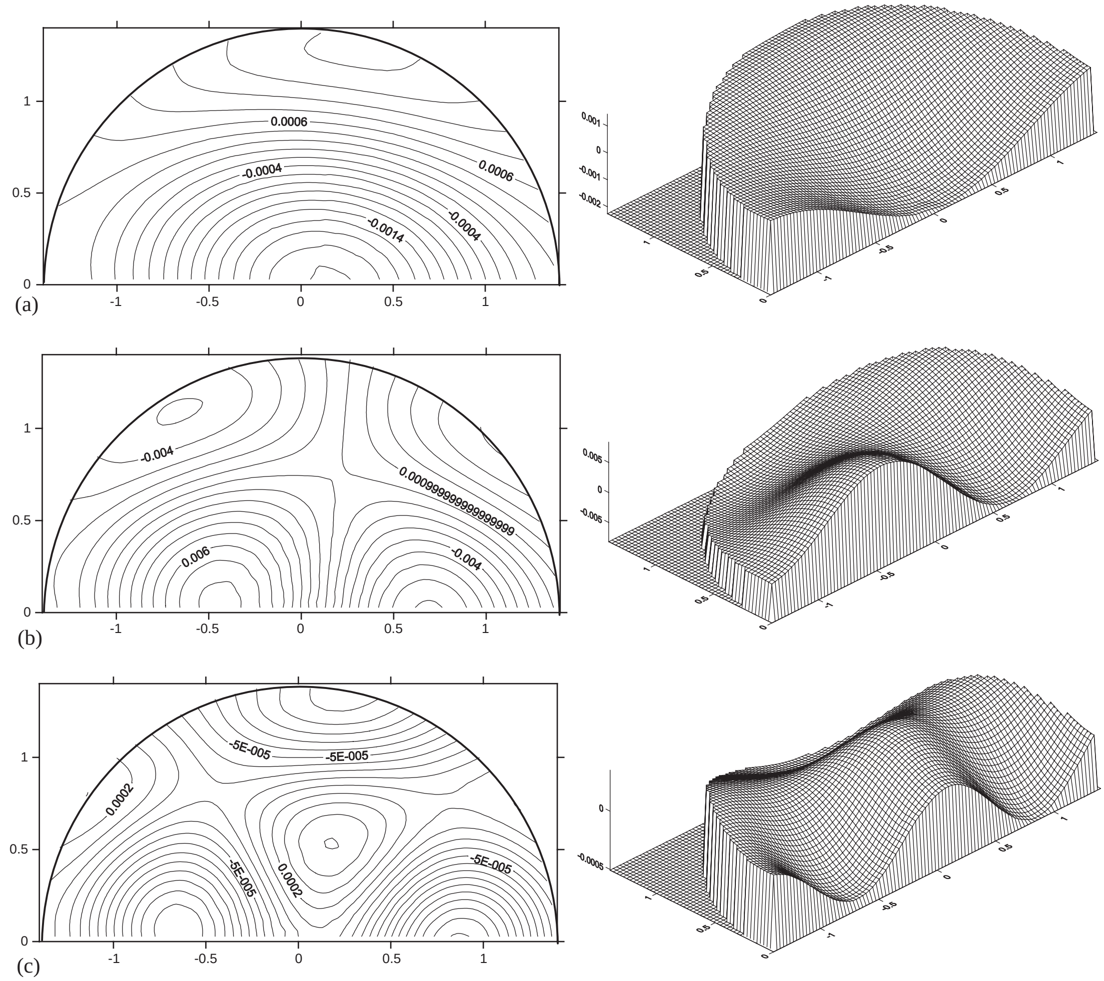

Fig. 3. First three modes of a semi circle: (a) the first mode; (b) the second mode; and (c) the third mode.

The domain is a multiply connected domain. This example shows the superiority of the current approach over Kuo's method [13]. His method fails to handle the multiply connected problems. In contrast, the proposed AITM can deal with this problem easily by putting the origin inside the hole. In this example, twenty-nine elements are used for the outer and inner boundaries. The auxiliary problem is the Neumann problem, $t = 0$. As sown in Fig. 8, by using the proposed method, eiegnvalues can be calculated as accurate as the direct Trefftz method does [28]. The analytical solutions are obtained using the eigenequation [33]

$$[J_m(kR_0)Y_m(kR_i) - Y_m(kR_0)J_m(kR_i)] = 0. \tag{18}$$

The first three modes are shown in Figs. 9(a)–(c) and the numbers inside the domain of those figures represent the contour lines of displacements, u.

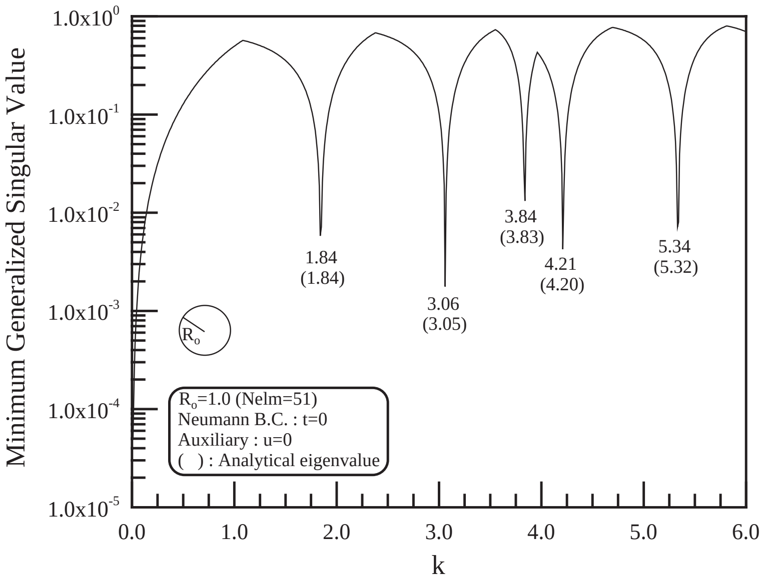

Fig. 4. Eigenvalue searching for a circular domain subjected to the Neumann boundary condition.

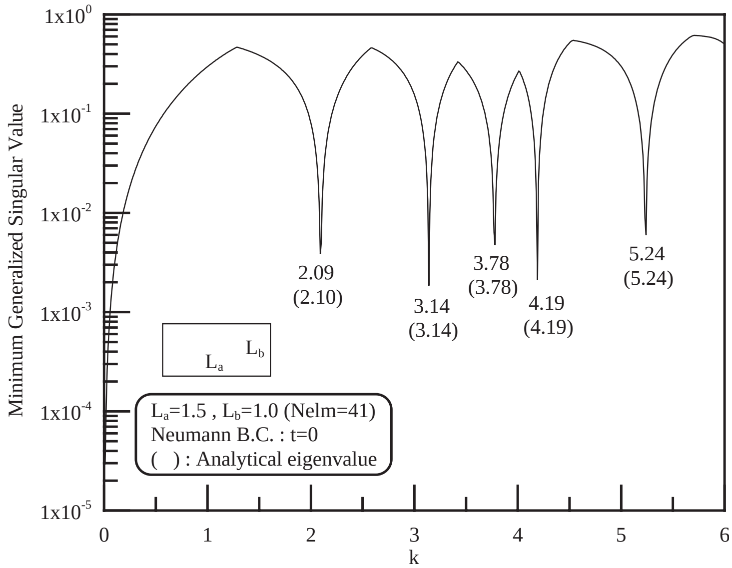

Fig. 5. Eigenvalue searching for a rectangular domain subjected to the Neumann boundary condition using an auxiliary system, $u = 0$.

Example 5. For a multiply connected domain, the outer boundary is a rectangle with the edge lengths $L_a = 2.0$ and $L_b = 1.0$, respectively, and the inner boundary is a circle with a radius $R_i = 0.2$. The origin of the circular hole is the geometric center of the whole domain. The boundary condition is the Dirichlet condition, $u = 0$.

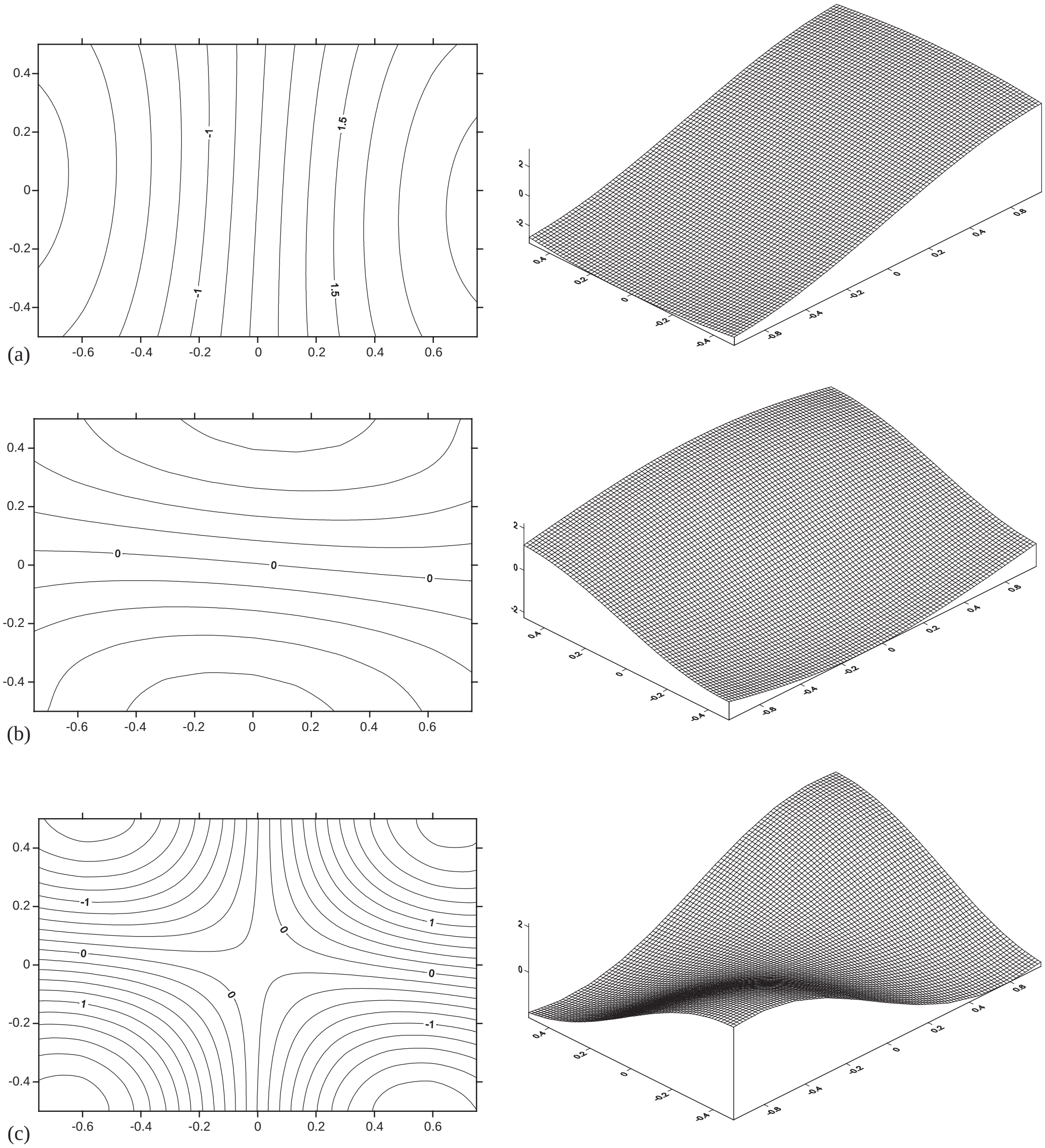

Fig. 6. First three modes of a rectangular domain: (a) the first mode; (b) the second mode; and (c) the third mode.

In this example, no analytical solution is available. We compare our results with those obtained from the complex-valued dual BEM. The auxiliary system is the Neumann problem, $t = 0$. As shown in Fig. 10, numerical results obtained from the proposed indirect asymmetric Trefftz method are close to those obtained from the complex-valued dual BEM. The reason why a complex-dual BEM is required has been explained in Chang's dissertation [33] and Ref. [34].

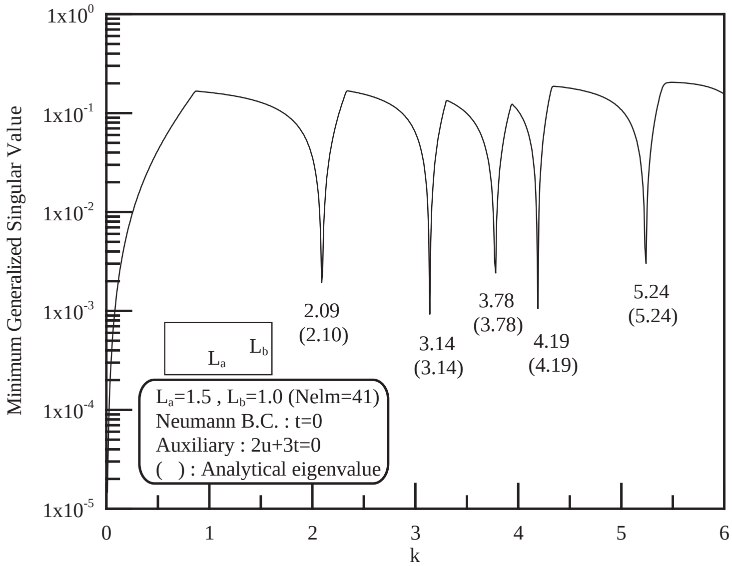

Fig. 7. Eigenvalue searching for a rectangular domain subjected to the Neumann boundary condition using a different auxiliary system, $2u + 3t = 0$.

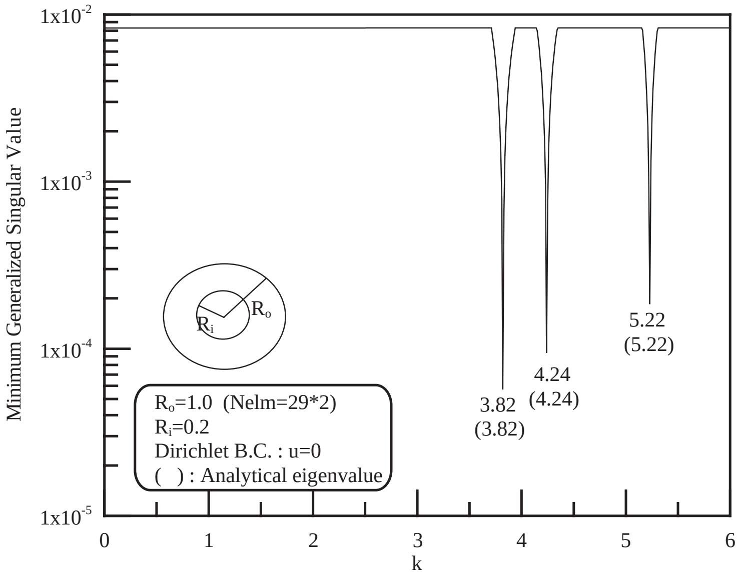

Fig. 8. Eigenvalue searching for an annular domain (a multiply connected domain of genus 1) subjected to the Dirichlet boundary condition.

Chang explained that solving an eigenvalue problem of a multiply connected domain by the complex-valued singular integral equation or the complex-valued hypersingular integral equation will result in an unreasonable numerical resonance. He named this kind of degeneracy of the

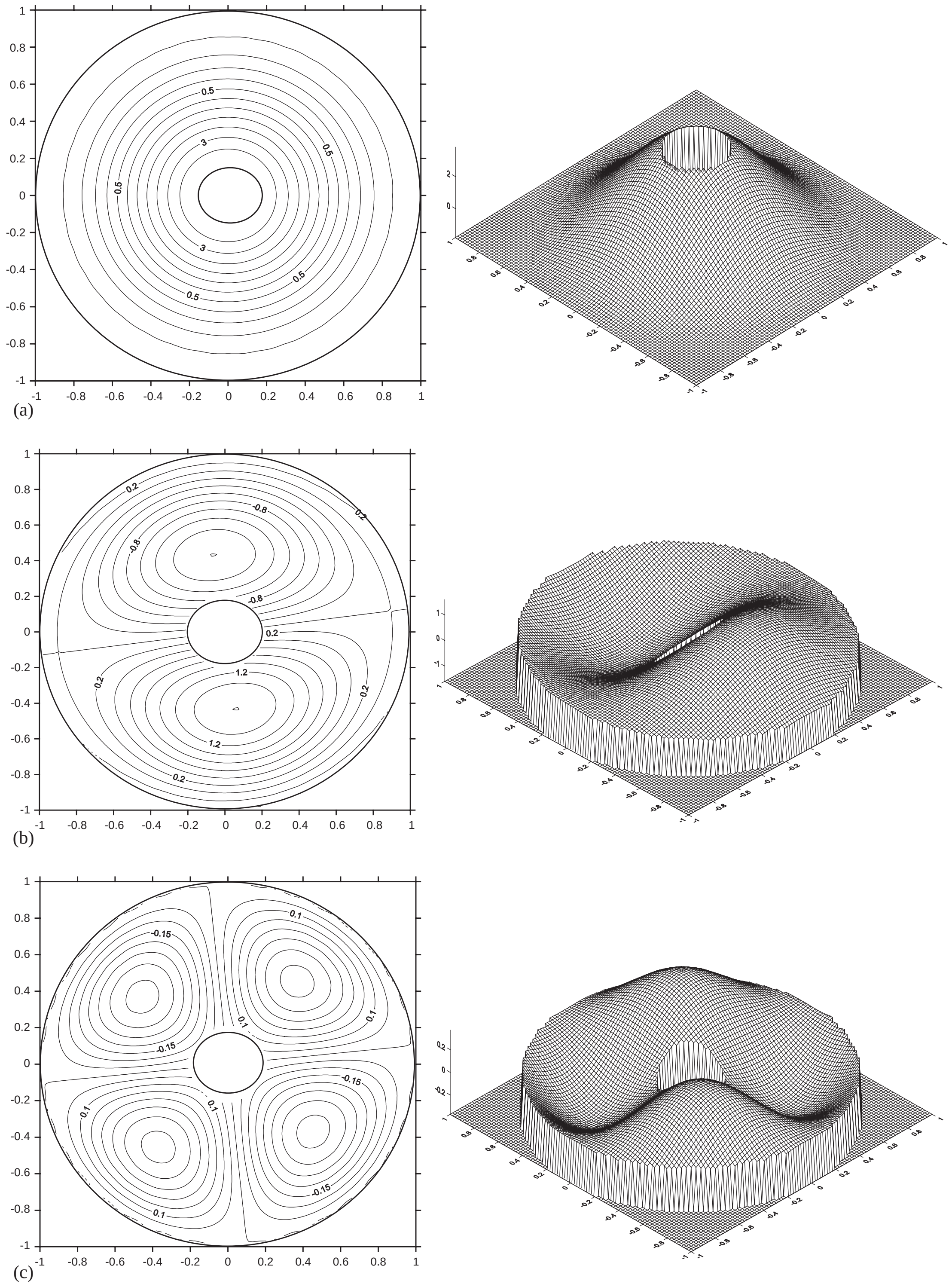

Fig. 9. First three modes of an annular domain: (a) the first mode; (b) the second mode; and (c) the third mode.

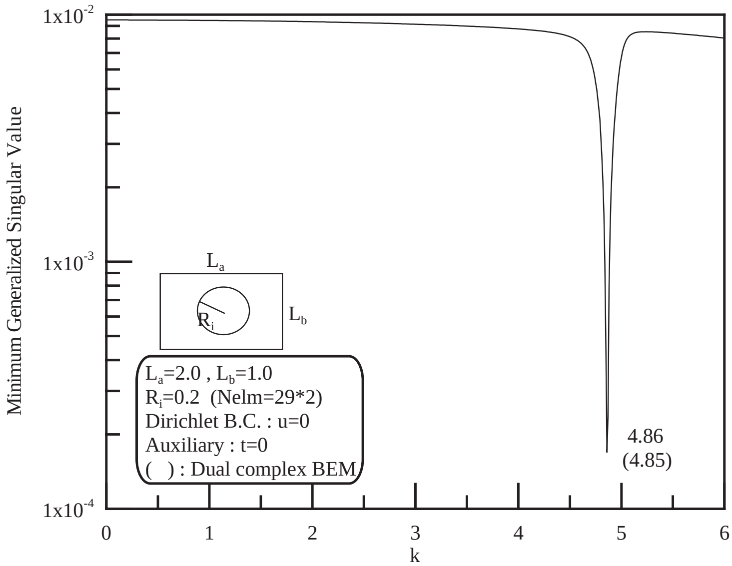

Fig. 10. Eigenvalue searching for an arbitrarily shaped multiply connected domain of genus 1 subjected to the Dirichlet boundary condition.

direct BEM as the pseudo-fictitious eigenvalue. To treat this unreasonable degeneracy, a combined use of singular and hypersingular integral equations was suggested.

4. Conclusions

In this paper, the AITM has been developed to deal with free-vibration problems. It is proved that the proposed method is equivalent to the direct Trefftz method mathematically. Consequently, no special effort on programming is needed and some known results from the direct Trefftz method can be adopted directly. It then can be said that the proposed method does not have spurious eigensolution but encounters numerical instability when the number of elements increases. In order to deal with the numerical instability, a combined use of Tikhonov's regularization and the GSVD is suggested. The proposed method can easily treat a multiply connected domain of genus 1 without introducing the artificial boundary, and it can also represent mode shapes within its own formulations. Several numerical examples have shown the validity of the proposed approach.

Acknowledgements

The first author would like to express his thank to the National Science Council, Taiwan for their financial support under Contract Number: NSC-90-2611-E-019-018.

References

[1] D.E. Beskos (Ed.), *Boundary Element Methods in Mechanics*, North-Holland, Amsterdam, 1987.

[2] G. De Mey, A simplified integral equation method for calculation of eigenvalue of Helmholtz equation, *International Journal for Numerical Methods in Engineering* 11 (1977) 1340–1342.

[3] J.R. Chang, W. Yeih, J.T. Chen, Determination of the natural frequencies and natural modes of a rod using the dual BEM in conjunction with the domain partition technique, *Computational Mechanics* 24 (1999) 29–30.

[4] S.R. Kuo, J.T. Chen, C.X. Huang, Analytical study and numerical experiments for true and spurious eigensolutions of a circular cavity using the real part BEM, *International Journal for Numerical Methods in Engineering* 48 (2000) 1401–1422.

[5] J.R. Hutchinson, An alternative BEM formulation applied to membrane vibration, in: C.A. Brebbia, G. Maier (Eds.), *Boundary Elements VII*, Springer, Berlin, 1985, pp. 6.13–6.25.

[6] A.J. Nowak, C.A. Brebbia, The multiple reciprocity method—a new approach for transforming BEM domain integral to the boundary, *Engineering Analysis with Boundary Elements* 6 (1989) 164–167.

[7] N. Kamiya, E. Andoh, A note on multiple reciprocity integral formulation for Helmholtz equation, *Communications in Numerical Methods in Engineering* 9 (1993) 9–13.

[8] N. Kamiya, E. Andoh, Helmholtz eigenvalue analysis by boundary element method, *Journal of Sound and Vibration* 160 (1993) 279–287.

[9] J.T. Chen, F.C. Wong, Analytical derivations for one-dimensional eigenproblems using dual BEM and MRM, *Engineering Analysis with Boundary Elements* 20 (1997) 25–33.

[10] W. Yeih, J.T. Chen, K.H. Chen, F.C. Wong, A study on the multiple reciprocity method and complex-valued formulation for Helmholtz equation, *Advances in Engineering Software* 29 (1998) 1–6.

[11] W. Yeih, J.T. Chen, C.M. Chang, Applications of dual MRM for determining the natural frequencies and natural modes of an Euler–Bernoulli beam using the singular value decomposition method, *Engineering Analysis with Boundary Elements* 23 (1999) 339–360.

[12] D.Y. Liou, J.T. Chen, K.H. Chen, A new method for determining the acoustic modes of a two-dimensional sound field, *Journal of the Chinese Institute of Civil and Hydraulic Engineering* 11 (1999) 89–100.

[13] S.R. Kuo, W. Yeih, Y.C. Wu, Applications of the generalized singular-value decomposition method on the eigenproblem using the incomplete boundary element formulation, *Journal of Sound and Vibration* 235 (2000) 813–845.

[14] I.L. Chen, J.T. Chen, S.R. Kuo, M.T. Liang, A new method for true and spurious eigensolutions of arbitrary cavities using the combined Helmholtz exterior integral equation formulation method, *Journal of Acoustical Society of America* 109 (2001) 982–998.

[15] Y.Y. Kim, J.H. Kang, Free vibration analysis of membranes using wave-type base functions, *Journal of Acoustical Society of America* 99 (1996) 2938–2946.

[16] S.W. Kang, J.M. Lee, Y.J. Kang, Vibration analysis of arbitrarily shaped membranes using non-dimensional dynamic influence function, *Journal of Sound and Vibration* 221 (1999) 117–132.

[17] W. Chen, Meshfree boundary particle method applied to Helmholtz problems, *Engineering Analysis with Boundary Elements* 26 (2002) 577–581.

[18] W. Chen, M. Tanaka, A meshless, integration-free, and boundary-only RBF technique, *Computers and Mathematics with Applications* 43 (2002) 379–391.

[19] W. Chen, Symmetric boundary knot method, *Engineering Analysis with Boundary Elements* 26 (2002) 489–494.

[20] W.G. Jin, Y.K. Cheung, O.C. Zienkiewicz, Applications of the Trefftz method in plane elasticity problems, *International Journal for Numerical Methods in Engineering* 30 (1990) 1147–1161.

[21] W.G. Jin, Y.K. Cheung, O.C. Zienkiewicz, Trefftz method for Kirchhoff plate bending problems, *International Journal for Numerical Methods in Engineering* 36 (1993) 765–781.

[22] I. Harari, P. Barai, P.E. Barbone, M. Slavutin, Three-dimensional infinite elements based on a Trefftz formulation, *Journal of Computational Acoustics* 9 (2001) 381–394.

[23] J. Jirousek, A. Wroblewski, T-elements: state of the art and future trends, *Archives of Computational Methods in Engineering* 3 (1996) 323–434.

[24] E. Kita, N. Kamiya, Trefftz method: an overview, *Advances in Engineering Software* 24 (1995) 3–12.

[25] Y.K. Cheung, W.G. Jin, O.C. Zienkiewicz, Solution of Helmholtz equation by Trefftz method, *International Journal for Numerical Methods in Engineering* 32 (1991) 63–78.
[26] S.C. Huang, R.P. Shaw, The Trefftz method as an integral equation, *Advances in Engineering Software* 24 (1995) 57–63.
[27] N. Kamiya, S.T. Wu, Generalized eigenvalue formulation of the Helmholtz equation by Trefftz method, *Engineering Computations* 11 (1994) 177–186.
[28] J.R. Chang, R.F. Liu, W. Yeih, S.R. Kuo, Applications of the direct Trefftz boundary element method to the free-vibration problem of a membrane, *Journal of Acoustical Society of America* 112 (2002) 518–527.
[29] J.R. Chang, R.F. Liu, S.R. Kuo, W. Yeih, Application of symmetric indirect Trefftz method to free vibration problems in 2D, *International Journal for Numerical Methods in Engineering* 56 (2003) 1175–1192.
[30] S.G. Mikhlin, K.L. Smoltskiy, *Approximate Method for Solution of Differential and Integral Equations*, Elsevier, New York, 1967.
[31] C.X. Huang, A Study on True and Spurious Eigensolutions of Two-Dimensional Acoustic Cavities, Master Thesis, National Taiwan Ocean University, Keelung, Taiwan, ROC, 1999.
[32] S.W. Kang, J.M. Lee, Eigenmode analysis of arbitrarily shaped two-dimensional cavities by the method of point-matching, *Journal of Acoustical Society of America* 107 (2000) 1153–1160.
[33] J.R. Chang, Applications of Domain Partitioning on Vibration Problems using the Dual BEM, PhD Dissertation, National Taiwan Ocean University, Keelung, Taiwan, ROC, 1999.
[34] W. Yeih, J.R. Chang, S.R. Kuo, R.F. Liu, On the false degeneracy of the Helmholtz boundary integral equations, *Journal of Sound and Vibration* 258 (2002) 95–122.