



# Sliding bifurcations in the dynamics of mechanical systems with dry friction—remarks for engineers and applied scientists

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Received 20 December 2002; accepted 22 July 2003

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## Abstract

The dynamics of mechanical systems with dry friction is affected by non-smooth bifurcations, which have been recently partially classified as ‘sliding bifurcations’. In applied science a bifurcation is usually seen as the point in which the number of fixed points and/or (quasi-)periodic solutions changes. The paper describes with several detailed examples that ‘sliding bifurcations’ do not always correspond to such definition.

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## 1. Introduction

In recently published papers [1,2] a partial classification of bifurcations occurring in a certain class of non-smooth systems is attempted. Such bifurcations (called ‘sliding bifurcations’ in Refs. [1,2]) are classified according to certain properties of the vector fields and of their derivatives at the bifurcation points. In Refs. [1,2] emphasis is placed especially on the formal mathematical aspects of the classification so that normal forms are derived for maps, which present the same type of dynamic behaviour.

The present paper will deal only with mechanical systems with a dry friction type of non-smoothness and will not attempt any improvement of the classification presented in Refs. [1,2], but will simply try to fill a gap between the formal classification in mathematical terms expressed in them and the physical understanding of the phenomena occurring in the dynamical system at the bifurcation point. For that reason the present paper is intended more for engineers and applied scientists than for mathematicians. Moreover, often in recent papers on bifurcations in

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non-smooth time-continuous systems no explicit definition of bifurcation is given and, therefore, it may not be immediately clear what consequences on the dynamics of the systems can be caused by bifurcations corresponding to different definitions. In the present paper, the definition of bifurcation given in Ref. [3] is preferred because it corresponds to what an engineer or applied scientist would intuitively assume to be a bifurcation.

Consider a continuous-time dynamic system:

$$\dot{x} = f(x, \mu), \quad x \in \mathfrak{R}^n, \quad \mu \in \mathfrak{R}. \quad (1)$$

As the parameter  $\mu$  changes, the limit sets of the system also change and typically a small change in  $\mu$  produces small quantitative changes in a limit set. There is also the possibility that a small change in  $\mu$  can cause a limit set to undergo a *more significant* change. Loosely speaking such a *significant* change is called a bifurcation and the value of  $\mu$  at which a bifurcation occurs is called a bifurcation value.

**Definition 1.** A bifurcation point (with respect to  $\mu$ ) is a solution  $(x^*(t), \mu^*)$  of (1) where the number of fixed points or (quasi-)periodic solutions changes when  $\mu$  passes  $\mu^*$ .

In Refs. [1,2] a different definition is (implicitly) assumed:

**Definition 2.** The appearance of a topologically non-equivalent phase portrait under the variation of a parameter is called a bifurcation [4].

From a practical point of view the most important feature of bifurcations described by the above definitions for smooth systems is that they can easily be identified in a bifurcation diagram as turning or branching points. All usual smooth bifurcations, as fold, flip or Hopf bifurcations [5], are identified by both definitions.

A non-smooth bifurcation is a bifurcation that can only appear in non-smooth systems and, therefore, is different from the bifurcations usually encountered in smooth systems. The examples presented in the paper will make clear that ‘sliding bifurcations’ are bifurcations in the sense of the above Definition 2 but not all of them comply to Definition 1, i.e., at the point  $(x^*, \mu^*)$  corresponding to a ‘sliding bifurcation’ no change in the number of the existing fixed points or (quasi-)periodic solutions may occur. Such inconsistency was already observed in Ref. [6] and no contradiction exists between the present work and Refs. [1,2], but one of the author’s aims is to illustrate many examples of ‘sliding bifurcations’ in stick–slip systems that are not necessarily consistent with Definition 1, and therefore usually neglected by applied scientists.

One of the first works on non-smooth bifurcations is Ref. [7] where grazing bifurcations in impacting systems are defined. The impacting type non-smoothness, characterized by jumps in the velocity state variable, is very different from the dry friction type, where velocity is continuous but non-smooth, and will not be taken into consideration in this paper.

The paper is organized as follows: in Section 2 the equations of motion for the stick–slip systems under investigation are presented and a one-dimensional map, generated by one of the investigated models, is also defined with basic information on non-smooth bifurcations in one-dimensional maps; in Section 3 the ‘sliding bifurcations’ introduced in Ref. [1,2] are recalled. Then

Section 4 will present several examples for each type of ‘sliding bifurcation’. Finally Section 5 will close the paper with the conclusions.

## 2. Mechanical models

### 2.1. Equations of motion and other definitions

The autonomous non-smooth systems to be used to illustrate non-smooth bifurcations are composed either of one or two of blocks on a moving belt, as shown in Fig. 1. Similar models were introduced in Ref. [8] in the seismological field and were later studied in several papers [9–13]. The velocity of the belt is called the driving velocity,  $V_{dr}$ . The surface between the blocks and the belt is rough so that the belt exerts a dry friction force on each block that sticks on it. In the two-block system, Fig. 1(b), each block is connected to a fixed support and to the other block by elastic springs as shown in the figure. It is usually preferable to work with a dimensionless form of the motion equations as shown in Refs. [13,14].

During a stick phase the displacement of any block is given as a function of (dimensionless) time:

$$X(\tau) = X(\tau_0) + V_{dr}(\tau - \tau_0), \tag{2}$$

where  $X$  is the (dimensionless) displacement of the generic block from a reference configuration in which all springs assume their natural length,  $X(\tau_0)$  is the initial displacement and  $\tau$  the dimensionless time. The blocks can stick on the belt only if the following relations are true:

$$\text{One block} \quad -F_s - cV_{dr} < X < F_s - cV_{dr}, \tag{3a}$$

$$\text{Two blocks} \quad -F_{si} < X_i + \alpha(X_i - X_j) < F_{si}, \quad i = 1, 2, \quad j = 2, 1, \tag{3b}$$

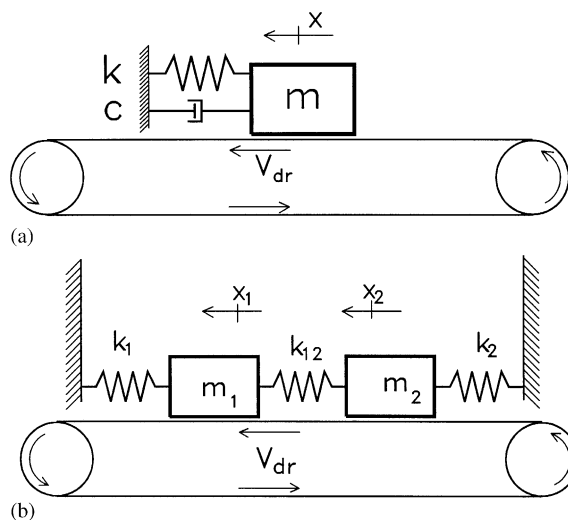


Fig. 1. Stick-slip models.

where  $F_{si}$  is the maximum static friction force acting on any block, the stiffness of the spring connecting any block to the fixed support is unity whereas  $\alpha$  is the stiffness of the coupling spring in the two-block model. A linear viscous damper with constant coefficient  $c$  additionally affects the one-block model because viscous damping will be the varying parameter in a bifurcation example of Section 4.

The *global sticking region* is the region of the displacement space where all blocks can simultaneously stick. For a one-block model it coincides with the interval defined by relation (3a) whereas for a two-block system it is given by the parallelogram  $\Gamma$  shown in Fig. 2. That is the region of the plane  $X_1$ – $X_2$  where all inequalities (3b) hold. A *global sticking phase* (g.s.p.) is a phase of motion in which all blocks are simultaneously sticking on the belt and is represented in the displacement plane by a straight line at  $45^\circ$ , as shown in Fig. 2. A slip phase is defined by the slipping of at least one block. Since the motion of the belt is in the direction of positive block displacements a global stick phase will end on the right extremes of the intervals (3), i.e., where  $(X = -cV_{dr} + F_s)$ , for one block, or, in the case of two blocks, where  $(X_i + \alpha(X_i - X_j) = F_{si}, i = 1, 2, j = 2, 1; \text{ point } P \text{ in Fig. 2})$ . At one of these points a slip phase will begin according to the following equations of motion:

$$\text{One block } \ddot{X}(\tau) + c\dot{X}(\tau) + X(\tau) = F_k(\dot{X} - V_{dr}), \tag{4a}$$

$$\text{Two blocks } \ddot{X}_i(\tau) + X_i(\tau) + \alpha(X_i(\tau) - X_j(\tau)) = F_{ki}(\dot{X}_i - V_{dr}), \quad i = 1, 2, j = 2, 1, \tag{4b}$$

where  $F_{ki}(\dot{X}_i - V_{dr})$  is the kinetic friction force, usually a function of the velocity of the block relative to the belt.

For the system with two blocks it is possible to define also two *sticking regions*  $\Omega_1$  and  $\Omega_2$ .

$\Omega_1$  is the region of the phase space where block 1 can stick: it is defined by the set of points  $(X_1, X_2, \dot{X}_1 = V_{dr}, \dot{X}_2)$  that satisfy Eq. (3b) with  $i = 1$  and  $j = 2$ ; it is contained between the two

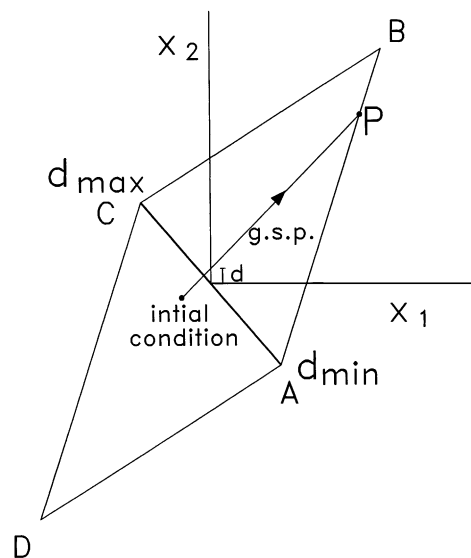


Fig. 2. Global sticking region  $\Gamma$  for the two-block model. AC is the interval of definition of map (10).

manifolds  $\partial_1\Omega_1$  and  $\partial_2\Omega_1$  defined as follows:

$$\partial_1\Omega_1 : \{X_1, X_2, \dot{X}_1 = V_{dr}, \dot{X}_2 : (X_1 + \alpha(X_1 - X_2) = -F_{s1})\}, \tag{5a}$$

$$\partial_2\Omega_1 : \{X_1, X_2, \dot{X}_1 = V_{dr}, \dot{X}_2 : (X_1 + \alpha(X_1 - X_2) = +F_{s1})\}, \tag{5b}$$

$\Omega_2$  is the region of the phase space where block 2 can stick: it is defined by the set of points  $(X_1, X_2, \dot{X}_1, \dot{X}_2 = V_{dr})$  that satisfy Eq. (3b) with  $i = 2$  and  $j = 1$ ; it is contained between the two manifolds  $\partial_1\Omega_2$  and  $\partial_2\Omega_2$  defined as follows:

$$\partial_1\Omega_2 : \{X_1, X_2, \dot{X}_1, \dot{X}_2 = V_{dr} : (X_2 + \alpha(X_2 - X_1) = -F_{s2})\}, \tag{6a}$$

$$\partial_2\Omega_2 : \{X_1, X_2, \dot{X}_1, \dot{X}_2 = V_{dr} : (X_2 + \alpha(X_2 - X_1) = +F_{s2})\}. \tag{6b}$$

Starting from a g.s.p. the first stick–slip transition occurs either along the line  $CB$  or along the line  $AB$  (e.g., point  $P \in AB$  in Fig. 2). After this first transition a block is slipping (block 1) and the trajectory does not belong to the plane  $(X_1, X_2, \dot{X}_1 = V_{dr}, \dot{X}_2 = V_{dr})$  any more because  $\dot{X}_1 \neq V_{dr}$ . The second block (block 2) can slip only if the trajectory intersects the boundaries of the relevant sticking region ( $\partial_1\Omega_2$  or  $\partial_2\Omega_2$ ). The slipping motion will continue to the point in which the velocity of the block will be equal to that of the belt and the elastic forces will be equilibrated by the static friction force (slip–stick transition). This transition can occur in any point of  $\Omega_1$  or  $\Omega_2$ . The projections of  $\Omega_1$  and  $\Omega_2$  on the displacement plane are shown in Fig. 3.

For one-block systems  $\Gamma$  and  $\Omega$  coincide and the boundary  $\partial\Omega$  corresponds to the two points  $X = -cV_{dr} \pm F_s$ .

Sliding bifurcations occur when, under the variation of a parameter, a steady state trajectory evolves in such a way that it intersects one of the boundaries of the sticking regions:  $\partial_i\Omega_j$ ,  $i = 1, 2; j = 1, 2$ .

For a complete characterization of the mechanical model the kinetic friction force must be defined. It is often defined as a function of the relative velocity at the contact surface

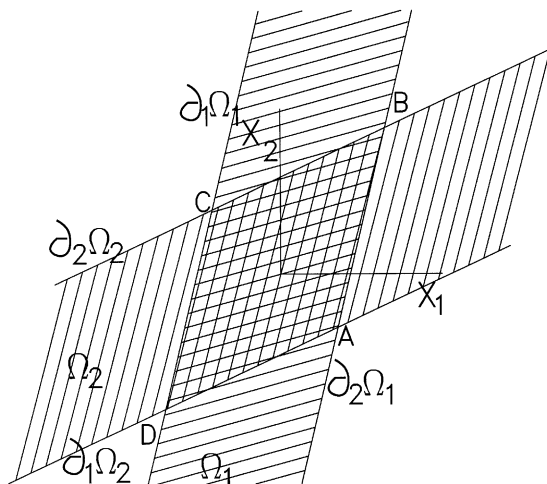


Fig. 3. Projection of the sticking regions  $\Omega_1$  and  $\Omega_2$  of the two-block model on the displacement plane.

$(\dot{X} - V_{dr})$  [11]:

$$F_k(\dot{X} - V_{dr}) = \begin{cases} \frac{1 - \delta}{(1 - \gamma(\dot{X} - V_{dr}))} + \delta & \text{if } V_{dr} > \dot{X}, \\ -\frac{1 - \delta}{(1 + \gamma(\dot{X} - V_{dr}))} - \delta & \text{if } V_{dr} < \dot{X}. \end{cases} \quad (7)$$

In relation (7)  $\gamma$  is taken to be positive because the dynamic friction force is usually decreasing for small values of the relative velocity and  $\delta$  may vary between 0 and 1 to avoid unrealistic changes of sign in the friction force. Note that in the system under investigation the kinetic friction force is defined according to the following relationship:

$$\text{One block} \quad F_k = F_k(\dot{X} - V_{dr}), \quad (8a)$$

$$\text{Two blocks} \quad \begin{aligned} F_{k1} &= F_k(\dot{X}_1 - V_{dr}), \\ F_{k2} &= \beta F_k(\dot{X}_2 - V_{dr}), \end{aligned} \quad (8b)$$

where  $\beta = F_{s2}/F_{s1}$ . The nature of the friction characteristic is important for the type of bifurcations that affect the mechanical system: in particular the standard Coulomb friction, with constant kinetic friction force smaller than the maximum static friction force, may present peculiar bifurcations that are not discussed in the present paper. More details can be found in Refs. [6,15].

**Remark.** The sticking regions  $\Omega_1, \Omega_2$  are called ‘sliding regions’ in Refs. [1,2]. Such terminology finds its origin in the field of electronic dynamics but it is confusing when dealing with dry friction: in fact ‘sliding regions’ are the regions where the blocks are sticking. For that reason the adjective ‘sticking’ has been preferred here.

## 2.2. One-dimensional event map

The two-block stick–slip system can generate a one-dimensional map [9,13]. During a g.s.p. the relative displacement between the two blocks is fixed; therefore the g.s.p.’s of the system may be characterized by the constant value of a variable  $d$ :

$$d = X_2 - X_1. \quad (9)$$

The g.s.p. will finish where one of the two blocks starts slipping. Then the motion of the slipping block may trigger a new slipping phase also for the other block but eventually, if  $V_{dr}$  is sufficiently small [13], both blocks will reach a configuration in which they ride simultaneously on the belt. The new g.s.p., in general, will be characterized by a value of the relative displacement  $d$  different from the one assumed during the previous g.s.p. In this way a motion of the system generates an infinite sequence of values of the variable  $d = d_1, d_2, \dots, d_m, \dots$ , which can be interpreted as a map expressing  $d_{k+1}$  as a function of  $d_k$ :

$$d_{k+1} = f(d_k). \quad (10)$$

Map (10) is defined in the field obtained by projecting the g.s.p. locus  $\Gamma$  on the space of the variable  $d$ , and it is well defined if a g.s.p. is always followed by another g.s.p. In some cases the steady state motion has no g.s.p. and therefore cannot be represented by the 1D map (10), but these cases require driving velocities higher than those used in the examples of Section 4. The boundary of the quadrilateral g.s.p. locus (Fig. 2) contains points  $C$  and  $A$  corresponding to the two limit values  $d_{max}$ ,  $d_{min}$  of the definition field of the map. Map (10) is single valued because a value of the variable  $d$  univocally determines the initial conditions at which one of the blocks will start slipping. The value of  $d$  and one of the two Eqs. (5b) or (6b) determine the displacement values  $X_1, X_2$  at impending slip, while the velocities are equal to  $V_{dr}$ . Therefore, the initial conditions of the slip phase are completely determined and the dynamics forward in time is uniquely defined by the equations of motion and so is the value of  $d$  at the next g.s.p.

The one-dimensional map (10) will be used to describe non-smooth bifurcations of the two-block model, in particular the grazing–sliding bifurcations.

Especially interesting are the discontinuities of the slope of the map and the cases in which the discontinuity points coincide with steady states of the map, in these cases bifurcations are possible and their nature will depend only on the two different slopes of the map at the discontinuity point. If  $a$  indicates the slope of the map to the left of the discontinuity point whereas  $b$  is the slope to the right of the discontinuity point, the possible cases are:

(1)  $0 < a < 1$  and  $-1 < b < 0$  (Fig. 4(a)): according to (Ref. [16, p. 197]) this is not a bifurcation, it is just a transition between two stable period-1 attractors, such a choice is consistent with Definition 1 of Section 1.

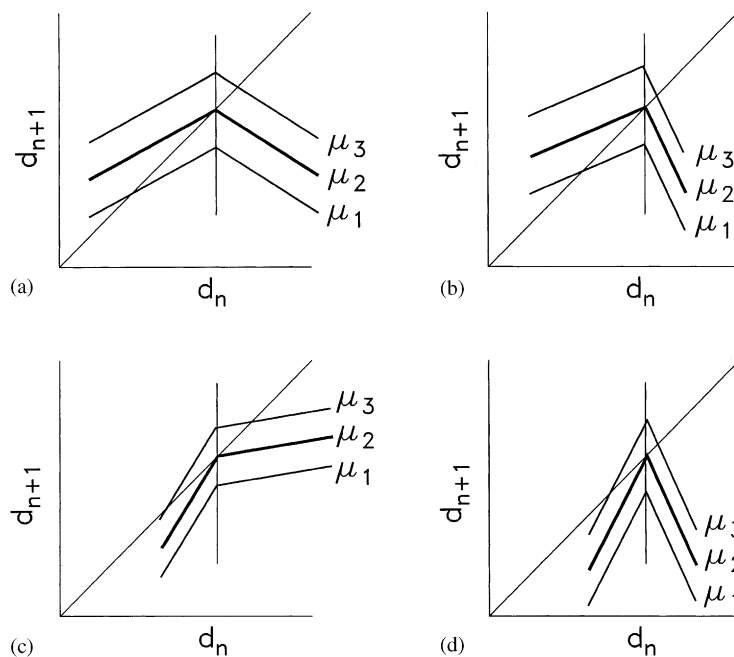


Fig. 4. Possible cases of non-smooth bifurcations for piecewise-smooth maps.  $\mu$  is the varying parameter such that  $\mu_1 < \mu_2 < \mu_3$   $\mu_2$  is the bifurcation value.

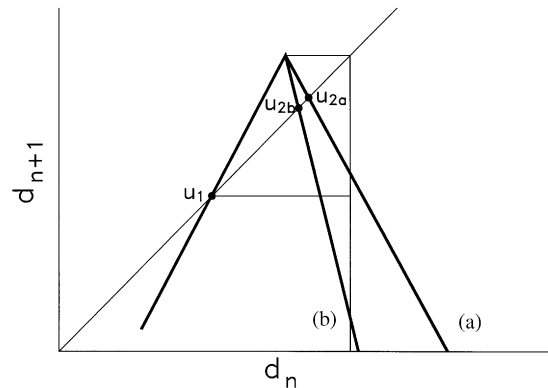


Fig. 5. (a) There exists an attracting region where the magnitude of the slope is everywhere greater than 1 and (b) there is no attracting region.

(2)  $0 < a < 1$  and  $b < -1$  (Fig. 4 (b)): according to Ref. [16] this is the case of the border-collision bifurcations, many different scenarios are possible and are investigated in Ref. [16].

(3)  $a > 1$  and  $-1 < b < 0$  (Fig. 4(c)): this case is called a non-smooth fold bifurcation for the clear analogy with the standard fold bifurcation.

(4)  $a > 1$  and  $b < -1$  with  $-b < a/(a - 1)$  (Figs. 4(d) and 5(a)): in this case the non-smooth bifurcation generates a chaotic attractor and a set of unstable periodic orbits, two of which of period one,  $u_1$  and  $u_{2a}$ , are shown in Fig. 5(a).

(5)  $a > 1$  and  $b < -1$  with  $-b > a/(a - 1)$  (Figs. 4(d) and 5(b)): in this case the non-smooth bifurcation generates only unstable orbits ( $u_1$  and  $u_{2b}$  are shown in the figure) and no chaotic attractor.

Negative values of  $a$  generate five other analogous cases.

Numerical simulations suggest that the dynamics of the stick–slip systems generate maps with piecewise-continuous slope; therefore they should exhibit bifurcations similar to those described above for one-dimensional maps. Similar remarks, in a different context, were also presented in Ref. [17].

### 3. ‘Sliding bifurcations’ in stick–slip systems

Refs. [1,2] introduce four possible ‘bifurcations’ due to interactions between a periodic orbit and the boundary of the sticking region  $\partial\Omega$  defined in Section 2.1. The examples of Sections 4 will show that often they do not correspond to bifurcations according to Definition 1 given in Section 1.

*Sliding bifurcation of type II* (Fig. 6(a)): under parameter variation the trajectory, initially completely contained in the sub-space where  $(\dot{X} - V_{dr}) < 0$  and in the sticking region  $\Omega$ , intersects its boundary  $\partial\Omega$  and then passes from the subspace where  $(\dot{X} - V_{dr}) < 0$  to the other, where  $(\dot{X} - V_{dr}) > 0$ , and from this to  $\Omega$ .

*Sliding bifurcation of type I* (Fig. 6(b)): under parameter variation the trajectory, initially composed of three branches, one in the subspace where the relative velocity  $(\dot{X} - V_{dr})$  is positive,



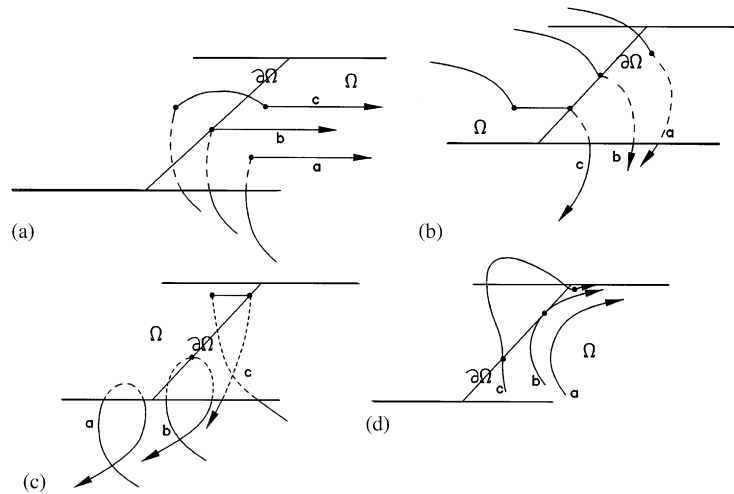


Fig. 6. Sliding bifurcations, trajectories labelled by letter *b* indicate the ‘bifurcating’ orbit. (a) Sliding bifurcation of type II; (b) sliding bifurcation of type I; (c) grazing–sliding bifurcation and (d) multi-sliding bifurcation.

one in the sticking region and one in the subspace where  $(\dot{X} - V_{dr}) < 0$ , intersects the boundary of the sticking region  $\partial\Omega$  and then switches from a subspace to the other with no sticking portion.

Cases of sliding bifurcations of types I and II were given, for example, in Refs. [13,18].

*Grazing–sliding bifurcation* (Fig. 6(c)): in this case a portion of a periodic orbit lying in a subspace grazes the boundary  $\partial\Omega$  of the sticking region, this causes the formation of a portion of sticking motion. Examples of this bifurcation are given in Refs. [14,15,19]. The system described in Ref. [19] was then further examined in Refs. [2,6].

*Multi-sliding bifurcation* (Fig. 6(d)): this last case differs from the scenarios presented above because the segment of the trajectory that undergoes the ‘bifurcation’ lies entirely within the sticking region  $\Omega$ . As a parameter is varied a sticking portion of the trajectory hits the boundary of the sticking region tangentially. Further variation of the parameter causes the formation of an additional segment of trajectory above (or below) the sticking region. A multi-sliding bifurcation was observed in Ref. [18] as Fig. 17(d) in that paper clearly shows.

Apart from Refs. [6,14,15,19], where ‘grazing–sliding bifurcations’ are described in a certain depth, all other references did not pay too much attention to the different types of ‘sliding bifurcations’. The opinion of the author is that such negligence was due to the fact that the above examples do not correspond to the intuitive idea of bifurcation given by Definition 1 of Section 1.

#### 4. Examples of transitions and bifurcations

The present section describes several examples of ‘sliding bifurcations’, which do not necessarily correspond to bifurcations as defined in Definition 1 of Section 1. Whenever possible examples are given for both stick–slip systems with one or two blocks.

#### 4.1. Sliding bifurcation of types II and I

If the friction characteristic (7) is adopted, ‘sliding bifurcations’ of types II and I are easily encountered in stick–slip systems. If the system with only one block is considered, it is possible to show that, when the driving velocity is small, the attractor is composed by a sequence of stick phases and slip phases in which the velocity of the block is always smaller than the velocity of the belt, Fig. 7(a). If the driving velocity is increased the slip–stick transition moves to the left in the phase plane so that for a specific value of driving velocity the slip–stick transition coincides with the left end of the sticking interval  $X = -F_s - cV_{dr}$  (Sliding bifurcation of type II). For higher values of driving velocity a new slipping phase appears in the periodic motion with velocity higher than the driving velocity, Fig. 7(b). At this point the periodic motion is composed of a stick phase that ends up at the right end of the sticking interval where a stick–slip transition generates a slip phase with negative relative velocity ( $\dot{X} - V_{dr}$ ). This slip phase terminates at a slip–slip transition where a new slip phase is started with positive relative velocity. The second slip phase terminates at a slip–stick transition within the sticking interval (3a) where the initial stick phase is resumed. A further increase in the driving velocity move the slip–stick transition to the right in the phase plane so that for a particular value of driving velocity the slip–stick transition coincides with the right end of the sticking interval  $X = F_s - cV_{dr}$  (Sliding bifurcation of type I), Fig. 7(c). Parameter values in Fig. 7 are:  $\delta = 0, \gamma = 3, c = 0$ ;  $V_{dr} \approx 1.241$  corresponds to a ‘sliding bifurcation’ of type II and  $V_{dr} \approx 10.801$  to a ‘sliding bifurcation’ of type I.

It is believed that these examples are the simplest possible examples of sliding ‘bifurcations’ and they clearly do not correspond to bifurcations in the sense of Definition 1. They are transitions between stable attractors, the number of periodic motions of the system does not change and a bifurcation diagram would not reveal any branching or turning point.

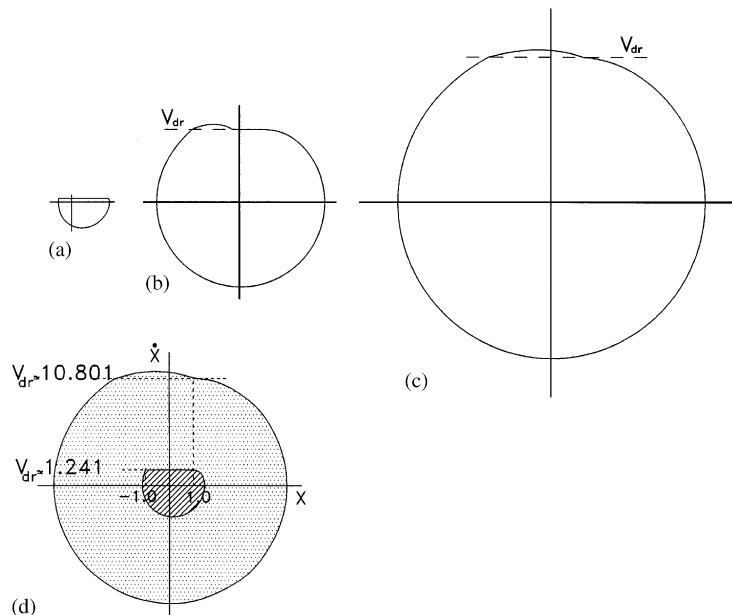


Fig. 7. Sliding bifurcations of types II and I in a one-block system.

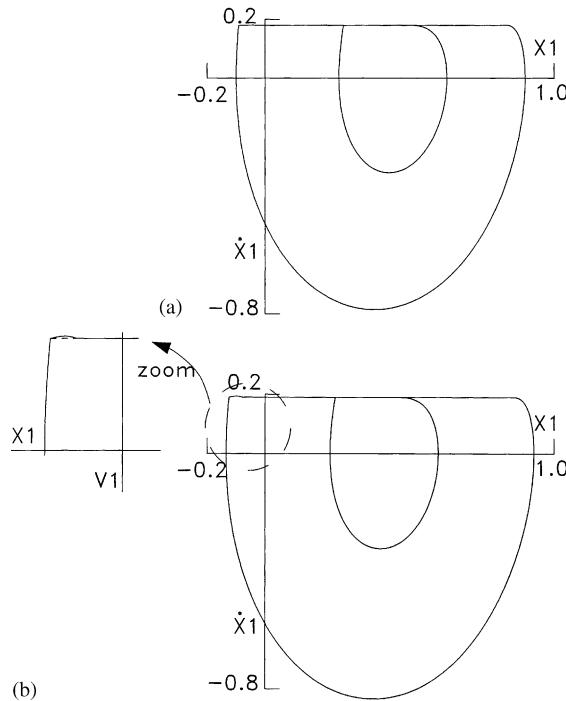


Fig. 8. Sliding bifurcation of type II for a two-block system.  $\alpha = 1.2$ ;  $\beta = 1.3$ ;  $\gamma = 3.0$ ;  $\delta = 0.0$ . (a)  $V_{dr} = 0.18$  and (b)  $V_{dr} = 0.19$ .

A completely analogous behaviour can be found in the two-block system. For the parameter values given in Fig. 8, a ‘sliding bifurcation’ of type II can be found for increasing values of the driving velocity. Also in this case the ‘bifurcation diagram’ would not reveal any apparent bifurcation. For the same system in the velocity range  $0.725 < V_{dr} < 0.740$ , a ‘sliding bifurcation’ of type I can be found for increasing values of the driving velocity, as shown in Fig. 9.

**Remark.** As previously mentioned, the nature of the friction characteristic may have an important influence on the ‘bifurcations’ of the system. In particular, if the usual Coulomb friction is adopted, with kinetic friction constant and smaller than the static friction, the above described transitions cannot take place in the system with one block since, with such a friction force, the slipping phase of the periodic steady states cannot have positive relative velocity, as shown in Ref. [12] for  $c = 0$ .

#### 4.2. Grazing–sliding bifurcation

To the best of the author’s knowledge the first examples of grazing–sliding bifurcation were probably given in Ref. [19], for a non-autonomous system, and in Refs. [14,15] for autonomous systems. The example presented in Ref. [19] was then investigated also in Refs. [6,2]. It is interesting to observe that the bifurcation described in Ref. [19] seems to bear a strong similarity

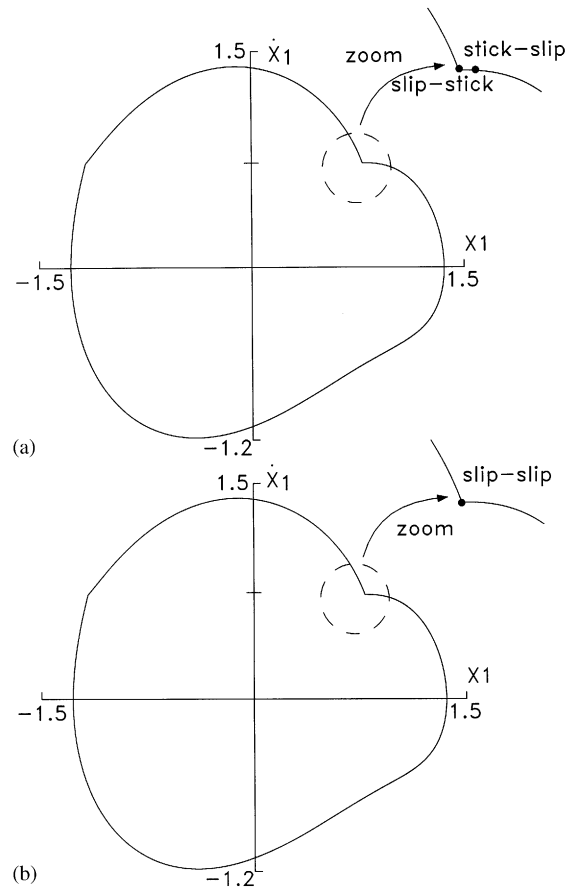


Fig. 9. Sliding bifurcation of type I for a two-block system.  $\alpha = 1.2$ ;  $\beta = 1.3$ ;  $\gamma = 3.0$ ;  $\delta = 0.0$ . (a)  $V_{dr} = 0.725$  and (b)  $V_{dr} = 0.740$ .

with the border-collision bifurcations introduced in Ref. [16] whereas the bifurcation presented in Ref. [14] is a non-smooth fold bifurcation. The two bifurcations presented in Ref. [15] are of the types shown in Figs. 5(a) and (b) since no steady state motion exists before the bifurcation takes place and a chaotic attractor is generated in one case, whereas only unstable periodic motions are generated in the other. In the present section grazing–sliding bifurcations for one block and two block systems will be presented.

The bifurcation affecting the one-block system is a clear example of discontinuous fold. It is possible to show [14] that in the one-block system with friction law given by Eq. (7) with parameters  $\delta = 0$ ,  $\gamma = 3$  and viscous damping coefficient  $c \neq 0$ , there exist two critical values of the viscous damping coefficient:  $c_1$  and  $c_2$ . The dynamics is as follows:

- In the range  $c < c_1$  a stable stick–slip periodic attractor exists (see Fig. 10(a)). Motions starting from points external to the periodic cycle are attracted to it. Motions generated from the internal points are repelled by an unstable fixed point and they attain the attractor when the trajectories intersect the line  $\dot{X} = V_{dr}$ .

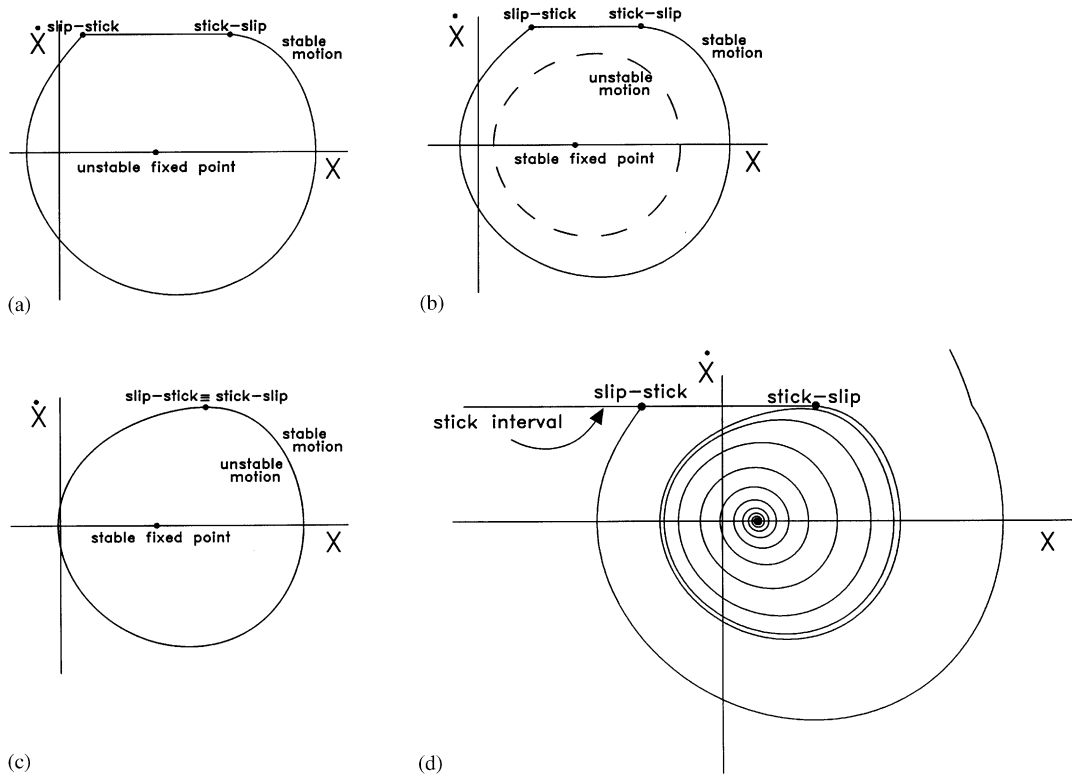


Fig. 10. Grazing bifurcation for a system with one block, it is a non-smooth fold bifurcation: (a)  $c < c_1$ , (b)  $c_1 < c < c_2$ , (c)  $c = c_2$  and (d)  $c > c_2$ .

- For  $c = c_1$  a subcritical Hopf bifurcation stabilises the fixed point. In the range  $c_1 < c < c_2$  the stick–slip periodic attractor is still stable but a stable point attractor also exists separated by an unstable motion (Fig. 10(b)). The increase of viscous damping from  $c_1$  to  $c_2$  has a two-fold effect on the motions appearing in Fig. 10. The amplitude of the unstable motion becomes larger and the length of the stick phase decreases since the slip–stick transition moves towards right, because of the higher damping, and the stick–slip transition moves towards left, according to relation (3a). There exists a critical value  $c = c_2$  for which these two transition points coincide. In this case periodic attractor and unstable motion also coincide at a non-smooth fold bifurcation. For  $c = c_2$  the unstable motion of the continuous system is tangent to the line  $\dot{X}_{\max} = V_{dr}$ , it is a repelling motion for the internal points, which are initial conditions of motions attracted by the point attractor. The tangential point is also the point where slip–stick transition and stick–slip transition of the attracting solution have converged. This motion is, therefore, an attracting motion for external initial conditions (Fig. 10(c)).
- Finally, in the range  $c_2 < c$  only the stable point attractor remains, motions originated far from the fixed point are attracted towards the stick locus, but in this case the right extreme of the stick interval falls in the basin of attraction of the fixed point and therefore, when the trajectory

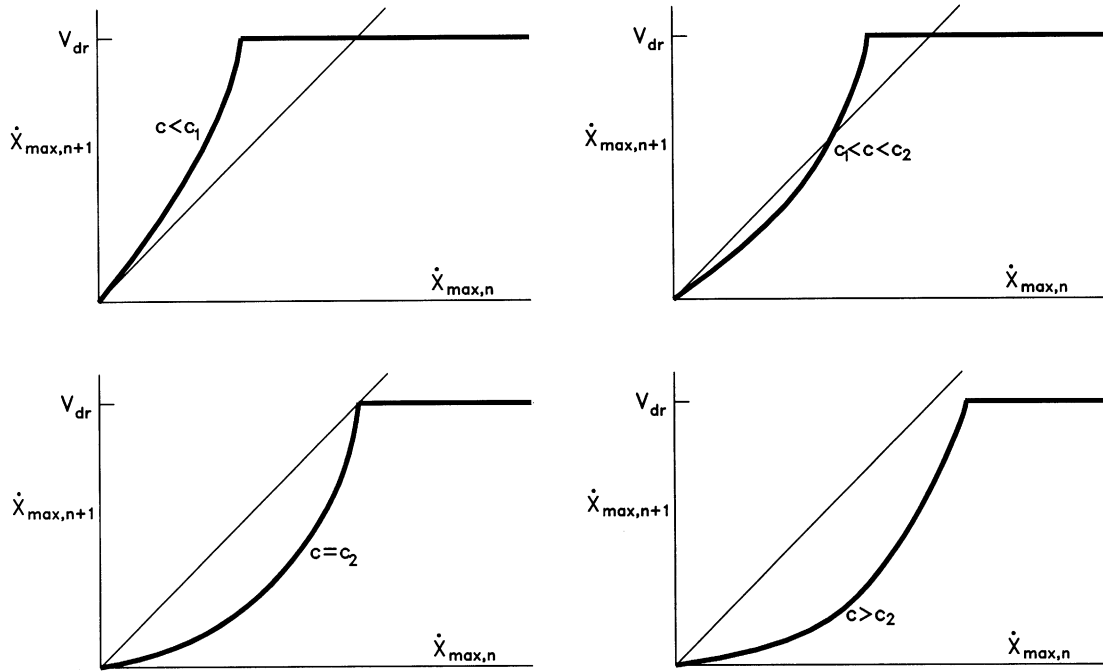


Fig. 11. One-dimensional map generated by the one-block model at the non-smooth fold bifurcation shown in Fig. 10.

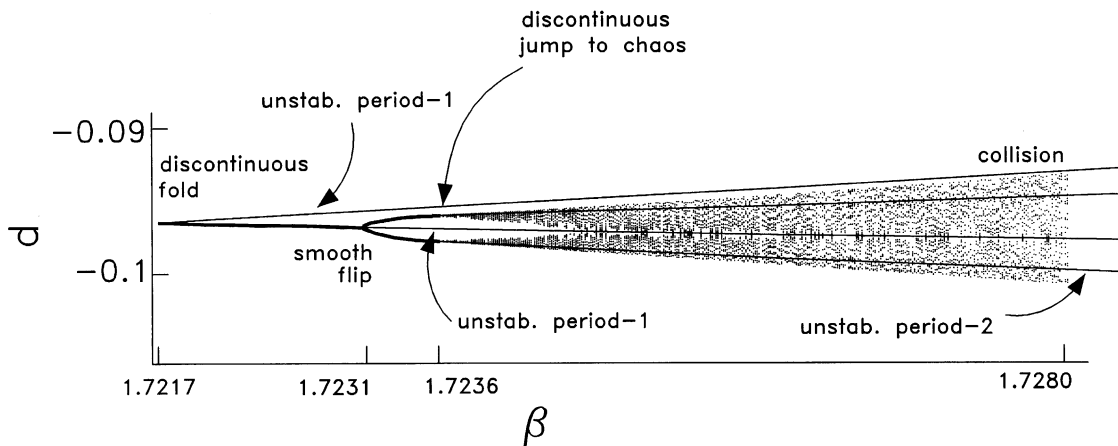


Fig. 12. Bifurcation diagram for the two-block model,  $\alpha = 1.2$ ,  $\beta$  is the varying parameter,  $\gamma = 3.0$ ,  $\delta = 0.0$ ,  $V_{dr} = 0.25$ . Thick lines indicate periodic attractors; thin lines indicate unstable steady state motions.

leaves the point  $X = F_s - cV_{dr}$ , a slip trajectory begins which is attracted by the stable fixed point (Fig. 10(d)).

A one-dimensional map, as shown in Fig. 11, can represent the evolution of the dynamics of the system for varying  $c$ . More details on this bifurcation can be found in Ref. [14].

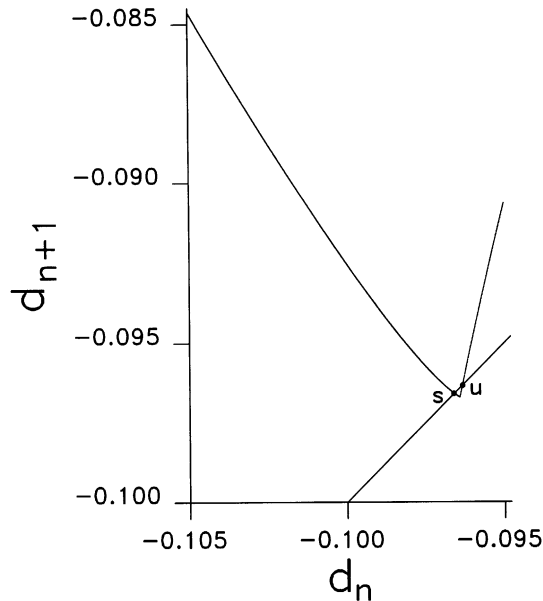


Fig. 13. One-dimensional map for the system of Fig. 12 with a value of  $\beta$  slightly larger than the value at the non-smooth fold bifurcation:  $s$  and  $u$  indicate, respectively, stable and unstable fixed points of the map.

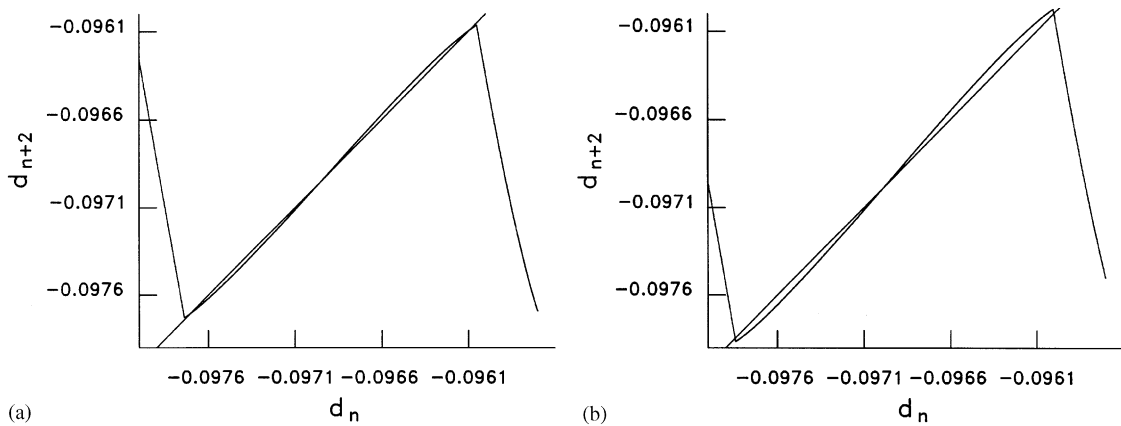


Fig. 14. One-dimensional maps for the system of Fig. 12 with (a)  $\beta = 1.7234$  and (b)  $\beta = 1.7238$ .

An interesting sequence of grazing–sliding bifurcations affecting the two-block system is described in the bifurcation diagram of Fig. 12:

(1) A period-1 discontinuous fold bifurcation, of the type shown in Fig. 4(c), occurs for  $\beta \approx 1.7217$ , a period-1-stable and period-1-unstable motion are generated as shown in Fig. 13. ‘Period-1’ motion means that the steady state motion of the blocks generates a period-1 fixed point of the one-dimensional map.

(2) The period-1-stable motion loses stability at an apparently smooth period doubling bifurcation for  $\beta \approx 1.7231$ .

(3) The period-2 stable motion disappears at a further non-smooth bifurcation where a period-2-unstable motion and chaos are born for  $\beta \approx 1.7236$ . This bifurcation seems to be of border-collision type with a transition from periodic to chaotic motion, Fig. 14. The chaotic attractor will disappear at a collision with the unstable period-1 motion generated at the non-smooth fold bifurcation, see Fig. 12.

Numerical computations show that all points of the maps with discontinuous slope in Figs. 13 and 14 correspond to trajectories of the mechanical system that have a branch grazing the border of the sticking region  $\partial_i \Omega_j$  ( $i = 1, 2; j = 1, 2$ ) and therefore could be classified as grazing–sliding bifurcations.

Finally Fig. 15 shows another portion of bifurcation diagram for the system with the same parameters as the one investigated in Fig. 12, but with  $\beta$  in the range 1.7652–1.7656. The diagram seems to have a point with discontinuous slope for  $\beta = \beta^*$ . Numerical calculations show that the steady state motion for values of  $\beta < \beta^*$  are characterized by a short stick phase of block 2 that disappears at a ‘grazing–sliding bifurcation’ for  $\beta = \beta^*$ . Therefore, the figure presents the numerical evidence of a ‘grazing–sliding bifurcation’ corresponding to the transition between two periodic attractors with the same period.

The possibility of having different scenarios corresponding to ‘grazing–sliding bifurcations’ was expressed also in Ref. [2, Section 5.2]. In the present work, several different scenarios are presented in a simpler theoretical framework in order to make the concept more clear to application-oriented researchers.

**Remark 1.** Also in this case the friction characteristic can change the nature of the non-smooth bifurcations. If the above-mentioned Coulomb friction is adopted in the single block model, the discontinuous fold bifurcation will be characterized by different properties. More details on that are given in Ref. [6, Section 6.6] for one-block systems and in Ref. [15, Section 3] for a two-block model.

**Remark 2.** Grazing sliding bifurcations, as the one shown in Fig. 14, mark the sudden transformation of a periodic attractor in a chaotic motion. In practical applications such a

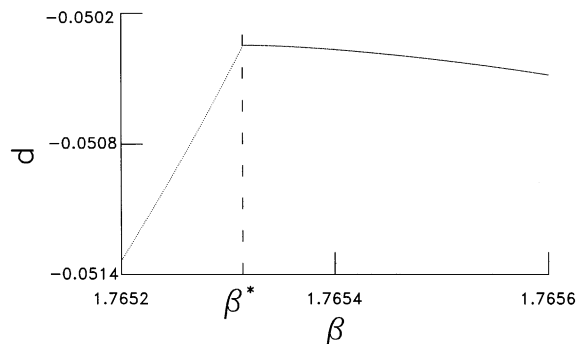


Fig. 15. Bifurcation diagram for the two-block model, parameters as in Fig. 12. The point with discontinuous slope corresponds to a grazing–sliding bifurcation of the stick–slip system.



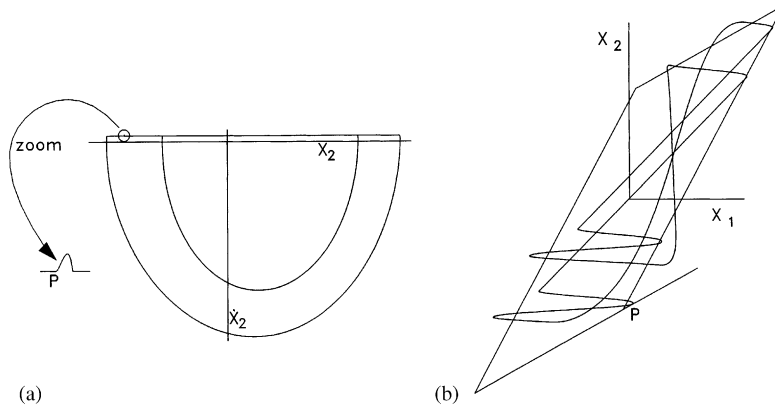


Fig. 16. Multi-sliding bifurcation for a two-block system.  $\alpha = 1.2, \beta = 2, \delta = 0.0, V_{dr} = 0.05, \gamma = 10.756$ . Initial condition  $d \approx 0.1969494$ . (a) Phase plane of block 2 and (b) displacement plane.

bifurcations would not cause sudden changes due to the unavoidable presence of noise and to the initial infinitesimal amplitude of the chaotic motion. An experiment would just show the increase in amplitude of a noisy motion and its chaotic nature would become apparent only when the chaotic fluctuations reach appreciable amplitude.

#### 4.3. Multi-sliding bifurcation

To the best of the author's knowledge the example of the present section is the first case of multi-sliding 'bifurcation' explicitly recognized in stick–slip systems. The phase plane of block 2 (Fig. 16(a)) shows that a new slipping phase with positive relative velocity is generated from a stick phase for the value  $\gamma \approx 10.75265$ . For smaller values of the parameter  $\gamma$  such a slip phase does not exist, whereas it becomes more evident for larger values of  $\gamma$ . Also in this case no change in the number of the attractors can be observed at the 'bifurcation' point. Fig. 16 shows the stable periodic motion for a value of the parameter slightly bigger than the value at the 'bifurcation' point. Point  $P$  in Fig. (b) is the point where the trajectory of block 2 assumes a positive relative velocity starting from a stick phase.

### 5. Conclusions

The paper deals with 'sliding bifurcations' in mechanical systems with dry friction. When a portion of the trajectory of a steady state motion intersects the frontier of the sticking region a 'sliding bifurcation' may occur. The paper presents several examples to stress that such an occurrence does not always correspond to a change in the number of fixed points or (quasi-) periodic cycles and therefore 'sliding bifurcations' do not necessarily correspond to bifurcations as they are usually intended by engineers or applied scientists. In particular 'sliding bifurcations' of types I and II, described in Section 4.1, and 'multi-sliding bifurcations', presented in Section 4.3, correspond just to a transition between attractors with the same period.

Moreover it is interesting to notice that many and deep analogies may be found between ‘grazing–sliding bifurcations’ and the behaviour of piecewise-smooth one-dimensional maps. In particular:

(1) An example of ‘grazing–sliding bifurcation’, presented in Section 4.2, corresponds to the transition between two stable motions with the same period as for the 1D maps of Fig. 4(a).

(2) Two of the grazing–sliding bifurcations presented in Section 4.2 correspond to the non-smooth fold bifurcation of 1D maps as shown in Fig. 4(c).

(3) The grazing–sliding bifurcation from periodic to chaotic attractors presented in Section 4.2 corresponds to a border-collision bifurcation of 1D maps as shown in Fig. 4(b).

(4) Finally the grazing–sliding bifurcations presented in Ref. [15] correspond to the non-smooth bifurcations of 1D maps as shown in Figs. 4(d) and 5(a) and (b).

The problem of classifying non-smooth bifurcations is still open and extremely intricate: the same scenario can correspond to different ‘sliding bifurcations’ (as in the case of transitions between periodic motions) and, vice versa, the same ‘sliding bifurcation’ can correspond to completely different scenarios (as in the case of grazing–sliding bifurcations). The present paper does not attempt any form of classification but just highlights significant problems from an engineering oriented point of view.

## Acknowledgements

The financial support of the European project SICONOS is gratefully acknowledged.

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