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The use of antiresonances for crack detection in beams

M. Dilena[a], A. Morassi[b,*]

[a] DISTART, Dipartimento di Ingegneria delle Strutture, dei Trasporti, delle Acque, del Rilevamento, del Territorio, Università di Bologna, Italy
[b] Dipartimento di Ingegneria Civile, Università degli Studi di Udine, Via delle Scienze 208, Udine 33100, Italy

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Abstract

This paper deals with the identification of a single open crack in a vibrating beam, either under axial or bending vibration, based on measurements of damage-induced shifts in natural frequencies and antiresonant frequencies. It is found that an appropriate use of frequencies and antiresonances may avoid the non-uniqueness of the damage location problem, which occurs in symmetrical beams when only frequency data are employed. The theoretical results are confirmed by a comparison with dynamic measurements on cracked steel beams under free–free boundary conditions.

1. Introduction

This paper is concerned with the identification of a single crack in a vibrating beam, either under axial or bending vibration, from the knowledge of damage-induced shifts in a suitable set of natural frequencies and antiresonant frequencies.

Within the class of diagnostic problems in structural mechanics, the crack detection problem in vibrating beams has received great attention in the scientific community in the last two decades. There are good reasons for this interest: firstly, the mechanical system consisting of a single beam describes the behaviour of many structures, which is important both for the civil and mechanical engineering field. Secondly, the problem of identifying a crack in a beam gives rise to the basic diagnostic problem and, therefore, it represents an important benchmark to test the accuracy of identification techniques.

In most studies of dynamic methods for damage detection, researchers have used changes in natural frequencies as the diagnostic tool, see, for example, Ref. [1]. Frequencies can be measured

*Corresponding author.
E-mail addresses: michele_dilena@virgilio.it (M. Dilena), antonino.morassi@dic.uniud.it (A. Morassi).

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more easily than can mode shapes, and are usually less affected by experimental errors. Most of the diagnostic techniques for vibrating beams are formulated as an optimality criterion, where the stiffness distribution of a chosen reference configuration of the beam is updated so that the first few natural frequencies closely match the measured ones at a certain level of deterioration, see, for example, Ref. [2]. On the one hand, these optimization techniques have the advantage of allowing for investigation of quite general classes of systems. On the other hand, the lack of satisfactory framework of general properties gives rise to several indeterminacy problems, which, in some cases, may obstruct applications to practical problems.

Recently, the crack detection problem in elastic beams has been investigated from a different point of view, namely the attention has been focussed on finding conditions which allow for a rigorous identification of the damage from minimal frequency measurements. Despite very extensive literature on damage detection in beams, few results of this kind are available. In order to recall known results when frequency data are used, the damage detection problem in axially vibrating beams will now be briefly summarized. If the undamaged beam is completely known and the damage is simulated by a linear spring located at the damaged cross-section, then only two parameters need to be determined, namely the stiffness K_A of the spring and the abscissa s of the cracked cross-section. In a relatively recent paper [3], Narkis has shown that if the undamaged rod is a perturbation of the virgin one, namely whenever the crack is small, the only information required for accurate crack localization is the ratio between the variations of the first two natural frequencies caused by the crack. The results were shown in Ref. [3] for uniform free–free rods, and a closed-form solution for the crack location was derived. The above results were subsequently extended in different directions in Refs. [4–6]. The identification procedures presented in these papers are not able to eliminate symmetrical solutions in the damage location problem, namely cracks at any one of a set of symmetrically placed points of a symmetrical rod produce identical changes to natural frequencies. The difficulty to distinguish between real and spurious symmetrical solutions is confirmed also in more recent damage detection studies, see, for example, Refs. [7–10].

Therefore, an important question is left open: *what kind of spectral data might be used in order to avoid the non-uniqueness of the damage location problem due to structural symmetry*?

In this paper it is shown that an appropriate use of resonances and antiresonances may be useful for crack identification in symmetrical beams. In order to illustrate the present results, for the sake of simplicity reference is made again to a free–free uniform beam under axial vibration. It was found that knowledge of the ratio between the variations of the first resonance and the first antiresonance of the point frequency function corresponding to one end of the rod, uniquely determines the position s of the cracked cross-section. Furthermore, the variations of the first resonance and the first antiresonance allow the stiffness K_A of the damage-simulating elastic spring to be uniquely determined. In both cases, simple closed-form expressions are deduced for s and for K_A. It is worth noticing that the required data can be extracted from a single measurement of frequency response function, without further experimental and numerical burden. The proposed identification technique is essentially based on the explicit expression for the damage sensitivity of eigenvalues given in Ref. [11]. Similar results hold true also for initially non-uniform symmetrical rods with a single crack. Part of the results above are also valid for initially uniform cracked beams in bending under various sets of symmetrical boundary conditions.

The predictions of the theory and reliability of the diagnostic technique were checked on the basis of results of several dynamic tests performed on free–free cracked steel beams. Interpretation of experimental results shows that if the frequencies and resonances used as data in identification are affected by (modelling and measurements) relatively small errors with respect to the changes induced by the crack, then damage identification gives satisfactory results. Moreover, in the inverse problem solution, the noise and the modelling errors on antiresonances usually amplified strongly with respect to cases in which frequency data are used.

One of the main motivations on the use of antiresonances in structural identification is that, unlike mode shapes, antiresonances are easily and accurately measurable. However, in spite of this advantage and although experimental modal analysis can nowadays be considered a mature technology, the use of antiresonances for the interpretation of dynamic tests is relatively recent. Wahl et al. [12] discussed the resonance-antiresonance behaviour of frequency response functions for linear lightly damped structures and focussed on the significance of antiresonances in experimental structural analysis. An updating technique that includes antiresonances in the definition of the output residual was presented by D'Ambrogio and Fregolent in Refs. [13,14]. It was observed by the authors that the distribution of antiresonances may be significantly altered by small changes in the structural model and demonstrated that the use of antiresonances extracted from point frequency response functions allows for very robust model updating procedures. Jones and Turcotte [15] considered antiresonances in finite model updating of an experimental full-scale truss and analyzed the physical correctness of the updated model by using it to detect damage. An update using both natural frequencies and antiresonances was shown to produce a better correlation to experimental data than an update that uses only natural frequencies. Bamnios et al. [16] studied the influence of a transverse open crack on the mechanical impedance of cracked beams under various boundary conditions. Monitoring the change of the first antiresonance as a function of the measuring location along the beam, the authors proposed a prediction scheme for crack localization in beams under bending vibrations.

The plan of the present paper is as follows. Theoretical results concerning axial and flexural vibration are presented in Section 2. Numerical and experimental applications of the diagnostic technique are discussed in Section 3.

2. Theoretical results

2.1. Axially vibrating beams

The spatial variation of the infinitesimal free vibrations of an undamaged straight uniform rod is assumed to be governed by the differential equation

$$au'' + \lambda\rho u = 0 \quad \text{in } (0, L), \tag{1}$$

where $u = u(x)$ is the mode shape and $\sqrt{\lambda}$ is the associated natural frequency. Throughout this section, the rod is assumed to have no material damping. The quantities $a \equiv EA$ and ρ denote the axial stiffness and the linear-mass density of the rod. E is Young's modulus of the material and A the cross-section area of the rod. The rod length is denoted by L. The following three sets of boundary conditions will be considered.

$$\text{Free (F):} \quad u'(0) = 0 = u'(L), \tag{2}$$

$$\text{Supported (S):} \quad u(0) = 0 = u(L), \tag{3}$$

$$\text{Cantilever (C):} \quad u(0) = 0 = u'(L). \tag{4}$$

Modes and frequencies are the eigensolutions of the boundary value problem formed by Eq. (1) and one of the three boundary conditions (2)–(4). The mth eigenpair of the undamaged rod, $m \geqslant 0$, is denoted by $\{u_m, \lambda_m\}$, where $0 \leqslant \lambda_0 < \lambda_1 < \cdots$ and $\lim_{m\to\infty} \lambda_m = \infty$.

Suppose that a crack appears at the cross-section of abscissa s, with $s \in (0, L)$. Assuming that the crack remains always open during the longitudinal vibration, by modelling it as a massless translational spring, at $x = s$, see Refs. [17,18], the eigenvalue problem for the damaged rod is the following:

$$a u_{\rm d}'' + \lambda_{\rm d} \rho u_{\rm d} = 0 \quad \text{in } (0, s) \cup (s, L), \tag{5}$$

where, in addition to the boundary conditions at $x = 0$ and L, the jump conditions

$$[u_{\rm d}'(s)] = 0, \quad K_A [u_{\rm d}(s)] = a u_{\rm d}'(s) \tag{6}$$

have to be considered at the cross-section where the crack occurs. In Eqs. (6), the jump $(\phi(s^+) - \phi(s^-))$ of the function $\phi(x)$ at $x = s$ is denoted by $[\phi(s)]$. The expression K_A is the stiffness of the spring simulating the damage, and it can be related to the crack geometry as suggested, for example, in Refs. [17,19]. The undamaged system corresponds to $K_A \to \infty$ or $\varepsilon \equiv 1/K_A \to 0$.

The variational formulation of the eigenvalue problem shows that eigenvalues of the rod are increasing functions of K_A, and thus decreasing functions of ε, so that

$$\lambda_{{\rm d}m} \leqslant \lambda_m, \quad m = 0, 1, 2, \ldots \,. \tag{7}$$

Moreover, as for any constrained system, the following interlacing result holds:

$$\lambda_{m-1} \leqslant \lambda_{{\rm d}m} \leqslant \lambda_m, \quad m = 1, 2, \ldots \,. \tag{8}$$

If the crack is small, namely ε is small enough, the first order variation of the natural frequencies with respect to ε may be found as in Ref. [11]. Putting

$$\lambda_{\rm d} = \lambda + \varepsilon \tilde{\lambda}, \tag{9}$$

it can be shown that

$$\delta\lambda_m \equiv \varepsilon \tilde{\lambda}_m = -\frac{(N_m(s))^2}{K_A}, \quad m = 0, 1, 2, \ldots, \tag{10}$$

where

$$N_m(s) \equiv a u_m'(s) \tag{11}$$

is the axial force in the mth normalized mode shape of the undamaged rod, evaluated at the cracked cross-section.

The effect of the crack on antiresonances of point frequency response functions of the rod will now be investigated. Generally speaking, antiresonances correspond to zeros of frequency response functions (frf) $H(\sqrt{\lambda}, x_i, x_o)$, where x_i, x_o are the abscissas of the excitation point and measurement point, respectively. When $x_i = x_o$, the zeros of the frf $H(\sqrt{\lambda}, x_i, x_i)$ are the frequencies of a rod in which the longitudinal displacement at the cross-section of abscissa x_i is hindered. Therefore, under the assumption of small crack, on proceeding as above and with the same notation, the first order variation of the (square of the) mth antiresonance of the point frf $H(\sqrt{\lambda}, x_i, x_i)$ with respect to $1/K_A$ may be evaluated by Eq. (10).

At this stage it can be shown how a combined use of resonance and antiresonance measurements may be useful for crack identification.

To begin the analysis, a free uniform rod (F) with a small open crack at the cross-section of abscissa s will be considered. The eigenpairs of the (F) rod are given by

$$\lambda_m^{\mathrm{F}} = \frac{a}{\rho}\left(\frac{m\pi}{L}\right)^2, \quad u_m^{\mathrm{F}}(x) = \sqrt{\frac{2}{\rho L}}\cos\frac{m\pi x}{L}, \tag{12}$$

$m = 0, 1, 2, \ldots$. The rigid mode $u_0^{\mathrm{F}}(x)$ obviously is always insensitive to damage. Denote by C_m^{F} the quantity

$$C_m^{\mathrm{F}} \equiv -\frac{\delta\lambda_m^{\mathrm{F}}}{Bm^2}, \tag{13}$$

where m is a positive integer, $m \geqslant 1$, and let B be the constant

$$B = \left(a\sqrt{\frac{2}{\rho L}}\frac{\pi}{L}\right)^2. \tag{14}$$

Putting the expressions of λ_m^{F} and $u_m^{\mathrm{F}}(x)$ for $m \geqslant 1$ into Eq. (10) gives

$$C_m^{\mathrm{F}} = \frac{1}{K_A}\sin^2\frac{m\pi s}{L}. \tag{15}$$

Consider now the point frf $H(\sqrt{\lambda}, 0, 0)$ for $x_i = x_o = 0$. The antiresonances of $H(\sqrt{\lambda}, 0, 0)$ are the (square root of the) eigenvalues of the rod with left end, at $x = 0$, fixed, namely the eigenvalues λ_m^{C} of the cantilever rod (C) in the above notation. It follows that their first order variation with respect to the damage coincides with the first order variation $\delta\lambda_m^{\mathrm{C}}$ of the eigenvalues λ_m^{C} of the cantilever (C) rod. The eigenpairs of the (C) rod are given by

$$\lambda_m^{\mathrm{C}} = \frac{a}{\rho}\left(\frac{(1+2m)\pi}{2L}\right)^2, \quad u_m^{\mathrm{C}}(x) = \sqrt{\frac{2}{\rho L}}\sin\frac{(1+2m)\pi x}{2L}, \tag{16}$$

$m = 0, 1, 2, \ldots$. Denoting $C_m^{\mathrm{C}} \equiv -\delta\lambda_m^{\mathrm{C}}/B((1+2m)/2)^2$, by proceeding as above, it follows that

$$C_m^{\mathrm{C}} = \frac{1}{K_A}\cos^2\frac{(1+2m)\pi s}{2L}, \tag{17}$$

$m = 0, 1, 2, \ldots$. In this case, it turns out that from the knowledge of the $(1+2m)$th frequency under boundary conditions (F) and of the mth antiresonance of the point frf $H(\sqrt{\lambda}, 0, 0)$ it is possible to uniquely determine the stiffness K_A and the position variable $S = \cos(1+2m)\pi s/L$, $m = 0, 1, 2, \ldots$. In fact, using standard trigonometric identities on Eqs. (15) and (17) gives

$$K_A = \frac{1 - C_{1+2m}^{\mathrm{F}}/4C_m^{\mathrm{C}}}{C_m^{\mathrm{C}}}, \quad S = 1 - \frac{C_{1+2m}^{\mathrm{F}}}{2C_m^{\mathrm{C}}}, \tag{18}$$

$m = 0, 1, 2, \ldots$. Considering $m = 0$, it turns out that the damage is uniquely determined by the measurement of the first resonance of the free–free rod and the first antiresonance of the point frf $H(\sqrt{\lambda}, 0, 0)$.

It is worth noticing that knowledge of the first two natural frequencies of the free–free rod cannot eliminate the (mathematical) symmetrical solution in the damage location problem, see Refs. [3,4]. The analysis above shows how an appropriate use of natural frequencies and antiresonant frequencies can avoid the non-uniqueness of the damage location problem due to structural symmetry. Moreover, the required data for damage identification can be extracted from the frf measurement of $H(\sqrt{\lambda}, 0, 0)$ only, without further experimental or numerical cost.

The present technique can be adapted to analyze also the case of an initially uniform cantilever rod (C) with a small open crack at the cross-section of abscissa s. In this case, the antiresonances of the point frf $H(\sqrt{\lambda}, L, L)$ are the (square of the) eigenvalues $\lambda_m^{\rm S}$ of the supported rod (S) defined by Eqs. (1) and (3). Denoting $C_m^{\rm S} \equiv -\delta\lambda_m^{\rm S}/Bm^2$, with $B = \left(a\pi\sqrt{2/\rho L}/L\right)^2$ one has

$$C_m^{\rm S} = \frac{1}{K_A}\cos^2\frac{(1+m)\pi s}{L}, \tag{19}$$

$m = 0, 1, 2, \ldots$. By using standard trigonometric identities, Eqs. (17) and (19) give a closed-form expressions of the stiffness K_A and of the position variable $S = \cos(1+2m)\pi s/L$

$$K_A = \frac{1}{2C_m^{\rm C} - C_{1+2m}^{\rm S}}, \quad S = \frac{1}{1 - C_{1+2m}^{\rm S}/2C_m^{\rm C}} - 1, \tag{20}$$

$m = 0, 1, 2, \ldots$. In particular, for $m = 0$, the damage is uniquely determined from the first resonance of the cantilever rod and the first antiresonance of the point frf $H(\sqrt{\lambda}, L, L)$.

Up till now only the simple, but very common case of uniform rods has been considered. However, it can be shown that the above results might be extended to also include initially symmetrical non-uniform rods. In order to illustrate the damage identification procedure within this more general setting, the case of the free rod (F) will be investigated in detail in the following. To simplify the analysis it will be assumed that $\rho(x) = \gamma A(x)$, where γ is the (uniform) volume mass density and $A(x)$, $A(x) = A(L - x)$ in $[0, L]$, is strictly positive and continuous differentiable function of x.

From Eq. (10) and from the interpretation of the antiresonances, the ratio of the change in first frequency of the free rod (F) and in first antiresonance of the point frf $H(\sqrt{\lambda}, 0, 0)$ depends only on the damage location, not on its severity. That is

$$\frac{\delta\lambda_1^{\rm F}}{\delta\lambda_1^{\rm C}} = \left(\frac{N_1^{\rm F}(s)}{N_1^{\rm C}(s)}\right)^2 \equiv f(s), \tag{21}$$

where $s \in (0, L)$ and $N_1^{\rm F}$, $N_1^{\rm C}$ are the axial force in the first (normalized) mode shape of the undamaged rod under free–free (F) and cantilever (C) boundary conditions. Note that, if $s \in (0, L)$, then the first mode of the cantilever is always sensitive to damage, e.g., $\delta\lambda_1^{\rm C} < 0$. The inverse problem related to crack location lies in determining the solutions of Eq. (21) in the interval $(0, L)$ for a fixed (measured) value of the ratio $\delta\lambda_1^{\rm F}/\delta\lambda_1^{\rm C}$.

It will be shown that the damage location is uniquely determined by the measurement of the ratio $\delta\lambda_1^{\rm F}/\delta\lambda_1^{\rm C}$.

To prove this property, it suffices to prove that the function $f = f(x)$ defined in Eq. (21) is strictly increasing in the interval $(0, L)$. It is convenient to recall that if $(u_1^F(x), \lambda_1^F)$ is the first eigenpair of the eigenvalue problem (1), (2) for the undamaged rod (F), then $(N_1^F(x) \equiv a(x)u_1^{F'}(x), \Lambda_1^F \equiv \lambda_1^F \gamma/E)$ is the first eigenpair of the eigenvalue problem

$$\begin{aligned} &(\tilde{a}N_1^{F'})' + \Lambda_1^F \tilde{a} N_1^F = 0 \quad \text{in } (0, L), \\ &N_1^F(0) = 0 = N_1^F(L), \end{aligned} \tag{22}$$

where $\tilde{a}(x) \equiv 1/a(x)$. Similarly, if $(u_1^C(x), \lambda_1^C)$ is the first eigenpair of the eigenvalue problem (1), (4) for the undamaged rod (C), then $(N_1^C(x) \equiv a(x)u_1^{C'}(x), \Lambda_1^C \equiv \lambda_1^C \gamma/E)$ is the first eigenpair of the eigenvalue problem

$$\begin{aligned} &(\tilde{a}N_1^{C'})' + \Lambda_1^C \tilde{a} N_1^C = 0 \quad \text{in } (0, L), \\ &N_1^{C'}(0) = 0 = N_1^C(L). \end{aligned} \tag{23}$$

By virtue of well-known properties of the solutions of the Sturm–Liouville problems (22) and (23), $N_1^F(x)$ and $N_1^C(x)$ have no zeros in the interval $(0, L)$, say $N_1^F(x) > 0$ and $N_1^C(x) > 0$ for $x \in (0, L)$, and, moreover, $N_1^C(0) > 0$. It can now be shown that $f(x)$ is well-defined function in the interval $[0, L]$. For this purpose it is sufficient to note that the limit $\lim_{x \to L^-} f(x)$ exists and has finite value. By applying twice the de l'Hôspital rule it is found that

$$\lim_{x \to L^-} f(x) = \left(\frac{N_1^{F'}(L)}{N_1^{C'}(L)} \right)^2. \tag{24}$$

By using the differential equation in Eqs. (22) and (23) to determine $N_1^{F'}(L)$ and $N_1^{C'}(L)$, and considering that $u_1^F(L) \neq 0$, $u_1^C(L) \neq 0$, it follows that

$$\lim_{x \to L^-} f(x) = \left(\frac{\Lambda_1^F}{\Lambda_1^C} \right)^2 \left(\frac{u_1^F(L)}{u_1^C(L)} \right)^2. \tag{25}$$

Computing the first derivative of $f(x)$ gives

$$f'(x) = \frac{2N_1^F(x)N_1^C(x)}{\tilde{a}(x)(N_1^C(x))^4} g(x), \tag{26}$$

where $g(x)$ has the expression

$$g(x) = (N_1^C(\tilde{a}N_1^{F'}) - N_1^F(\tilde{a}N_1^{C'}))(x). \tag{27}$$

Taking into account Eqs. (22) and (23), it follows that

$$g'(x) = (\Lambda_1^C - \Lambda_1^F)\tilde{a}(x)N_1^C(x)N_1^F(x). \tag{28}$$

Since $N_1^C N_1^F > 0$ in $(0, L)$ by assumption, and since $\Lambda_1^C > \Lambda_1^F$ by the variational formulation of the eigenvalue problem, it turns out that $g'(x) > 0$ in $(0, L)$. But $g(0) = 0$ and then $g(x) > 0$ in the whole interval $(0, L)$. In conclusion, $f(x)$ is a positive, strictly increasing function in $(0, L)$, and the assertion is proved.

Finally, it should be observed that the assumption of small changes restricts the range of application of the proposed method to cracked configurations that are a perturbation of the undamaged one. However, this is not a severe limitation, because in most practical situations it is crucial to be able to identify damage as soon as it arises.

2.2. Bending vibrating beams

In the previous section, the problem of locating a crack in an axially vibrating rod from frequency and antiresonance measurements has been discussed. Analogous results concerning cracked beams in bending, as it will be shown below, are less exhaustive.

The physical model, which will be firstly investigated, is a free–free uniform undamped Euler–Bernoulli beam with an open crack at the cross-section of abscissa s. According to Ref. [17], the crack is represented by the insertion of a massless rotational spring at the damaged cross-section. The stiffness K_B of the spring may be related in a precise way to the geometry of the crack as suggested, for example, in Ref. [19]. Denoting Young's modulus of the material by E and the volume mass-density by γ, the mth eigenpair $\{w_{\mathrm{d}m}(x), \lambda^{\mathrm{F}}_{\mathrm{d}m}\}$, $m = 0, 1, 2, \ldots$, of the bending vibrations of the cracked beam satisfies the following boundary eigenvalue problem:

$$EIw''''_{\mathrm{d}m} = \lambda^{\mathrm{S}}_{\mathrm{d}}\gamma A w_{\mathrm{d}m} \quad \text{in } (0, s) \cup (s, L), \tag{29}$$

$$w''_{\mathrm{d}m} = 0, \quad w'''_{\mathrm{d}m} = 0 \quad \text{at } x = 0 \text{ and } L, \tag{30}$$

where the jump conditions

$$[w_{\mathrm{d}m}(s)] = [w''_{\mathrm{d}m}(s)] = [w'''_{\mathrm{d}m}(s)] = 0, \quad EIw''_{\mathrm{d}m}(s) = K_B[w'_{\mathrm{d}m}(s)], \tag{31}$$

hold at the cracked cross-section. In the equations above, I and A represent the moment of inertia and the area of the cross-section, respectively.

The problem of determining the location of a transverse crack in bending free–free beams from frequency measurements has been recently investigated in Ref. [7]. Starting from the idea originally presented in Ref. [20], for each eigenmode considered a curve of the relative crack stiffness (normalized respect to the beam bending stiffness) versus crack position was determined. The possible damage sites are given by the common intersections of the curves for the different modes, see also Refs. [8,10,21] for applications to simply supported beams and cantilevers with not necessarily small cracks. It should be noted that the analytical determination of these curves involves the solution of the direct eigenvalue problem for the damaged beam and the not trivial numerical evaluation of the roots of some determinantal equations usually involving 8×8 matrices. A numerical study of the curves above shows that the curves corresponding to first and second frequency intersect at two points symmetrically placed along the beam axis, see Fig. 3 in Ref. [7]. That is, the first two frequencies determine uniquely the crack location except for a symmetrical position.

This important property can be easily confirmed for the case of a small crack by adopting the arguments presented in the previous section. When the crack is small, namely K_B is large enough, on proceeding as in Ref. [11] and with the above notation, the first order variation $\delta\lambda^{\mathrm{F}}_m$ of the mth eigenvalue with respect to $1/K_B$ is given by

$$\delta\lambda^{\mathrm{F}}_m = -\frac{(M^{\mathrm{F}}_m(s))^2}{K_B}, \tag{32}$$

where $M_m^F(s) \equiv -EIw''_{dm}(s)$ is the bending moment at the cross-section of abscissa s in the mth (normalized) bending mode of the undamaged beam, $m = 0, 1, 2\ldots$.

At this stage, the problem of identifying the position of the crack from the knowledge of the changes in first two natural frequencies can be posed as it was made in Section 2.1. It follows that all, and the only possible, locations of the crack are the abscissas of the points of the $f(x) = (M_2^F(x)/M_1^F(x))^2$ diagram intersecting with the horizontal straight line drawn parallel to the abscissa axis at a distance equal to the measured ratio $\delta\lambda_2^F/\delta\lambda_1^F$. Note that, since $M_1^F(x) \neq 0$ in $(0, L)$, the first eigenvalue of the free–free beam is always sensitive to damage, e.g., $\delta\lambda_1^F < 0$. Moreover, since both two limits $\lim_{x\to 0^+} f(x)$, $\lim_{x\to L^-} f(x)$ exist and have finite value, function $f(x)$ is well defined in the interval $[0, L]$. Because of the structural symmetry of the undamaged beam, it turns out that $|M_m^F(x)| = |M_m^F(L-x)|$ for $x \in [0, L]$, and then $f(x)$ is an even function with respect to the mid-point $x = L/2$. A direct calculation based on the explicit expression of the bending moment shows that $f(x)$ is a strictly decreasing function in $(0, L/2)$, as it is shown in Fig. 1. It turns out that the location of the crack is uniquely determined, except for a symmetrical position, from the first two natural frequencies.

Now it will be shown how a combined use of natural frequencies and antiresonances may exclude the spurious symmetrical damage location. Consider the first (not vanishing) antiresonance of the point frf $H(\sqrt{\lambda}, 0, 0)$ obtained by fixing the transversal displacement of the beam axis at the left end of the beam. Because of the well-known physical interpretation, this antiresonant frequency coincides with the first natural frequency of the supported-free (S–F) beam, say $\sqrt{\lambda_1^{S\text{–}F}}$. Therefore, its first order sensitivity can be evaluated via expression (32) and the following ratio can be defined:

$$\frac{\delta\lambda_1^F}{\delta\lambda_1^{S\text{–}F}} = \left(\frac{M_1^F(s)}{M_1^{S\text{–}F}(s)}\right)^2, \tag{33}$$

where also in this case $\delta\lambda_1^{S\text{–}F} < 0$. A direct calculation based on the exact expressions of the bending moments $M_1^F(x)$ and $M_1^{S\text{–}F}(x)$ shows that the function $f(x) = (M_1^F(x)/M_1^{S\text{–}F}(x))^2$ is well-defined and strictly increasing in the interval $[0, L]$, see Fig. 2. It follows that the damage location is uniquely determined by the measurement of the first resonance of the F–F beam and the first antiresonance of the point frf $H(\sqrt{\lambda}, 0, 0)$. Note that, also in this case, information can be extracted solely from the frf measurement of $H(\sqrt{\lambda}, 0, 0)$.

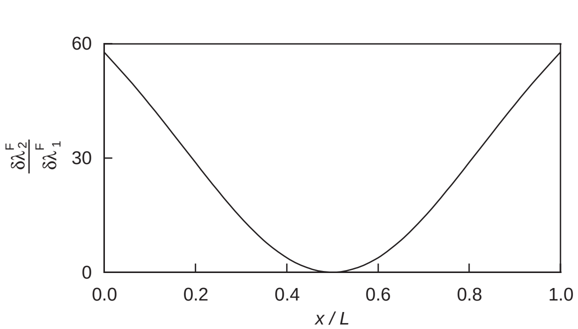

Fig. 1. Plot of the function $f(x) = (M_2^F(x)/M_1^F(x))^2$ for uniform free–free beam in bending vibration.

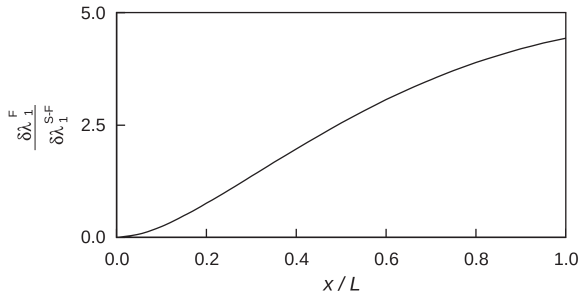

Fig. 2. Plot of the function $f(x) = (M_1^{\mathrm{F}}(x)/M_1^{\mathrm{S-F}}(x))^2$ for uniform free–free beam in bending vibration.

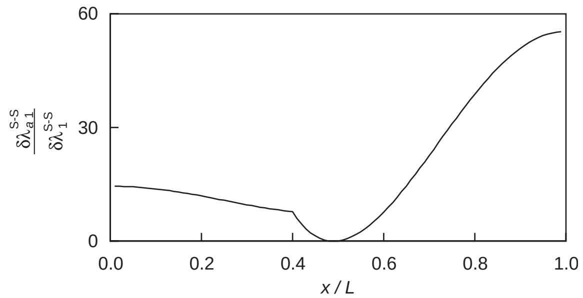

Fig. 3. Plot of the function $f(x) = (M_{a1}^{\mathrm{S-S}}(x)/M_1^{\mathrm{S-S}}(x))^2$ for uniform supported-supported beam in bending vibration, with $x_0 = 0.4L$.

Damage identification in a simply supported beam leads to a slightly different situation. It is well-known that, under this set of boundary conditions, measurement of the first two natural frequencies determines uniquely the severity of the damage and the damage location, except for a symmetrical position, see Refs. [3,4]. This situation seems to be, in a certain sense, an optimal one since, differently from the free–free case, knowledge of the first resonance and the first antiresonance of a point frf $H(\sqrt{\lambda}, x_o, x_o)$, where $0 < x_o < L$, it is not enough to uniquely determine the crack location. As an example, Fig. 3 shows the behaviour of the ratio $(M_{a1}^{\mathrm{S-S}}(x)/M_1^{\mathrm{S-S}}(x))^2$ between the change of first antiresonance and the change of first frequency for $x_o = 0.4L$. Here, $M_{a1}^{\mathrm{S-S}}(x)$ denotes the bending moment when the transversal displacement of the beam axis at the point x_o is hindered. Note that $M_1^{\mathrm{S-S}}(x)$ does not vanish in $(0, L)$. It can be observed from Fig. 3 that only for sufficiently high values of the ratio $\delta\lambda_{a1}^{\mathrm{S-S}}/\delta\lambda_1^{\mathrm{S-S}}$, i.e., approximately greater than 15, the damage location problem admits a unique solution. However, measurement of shifts in the first antiresonance may be equally useful to exclude the spurious symmetrical locations which occur when only frequency measurements data is used in identification.

The case of clamped–clamped (C–C) boundary conditions can be similarly discussed and it leads to a more involved situation. The behaviour of the function $f(x) = (M_{a1}^{\mathrm{C-C}}(x)/M_1^{\mathrm{C-C}}(x))^2$ for $x_o = 0.4L$ is drawn in Fig. 4. Since $M_1^{\mathrm{C-C}}(x)$ vanishes at two distinct points of the interval $(0, L)$, the function $f(x)$ has two vertical asymptotes at these points. Therefore, there are several points of the beam axis which correspond to the same ratio $\delta\lambda_{a1}^{\mathrm{C-C}}/\delta\lambda_1^{\mathrm{C-C}}$ and the damage location problem does not have a unique solution.

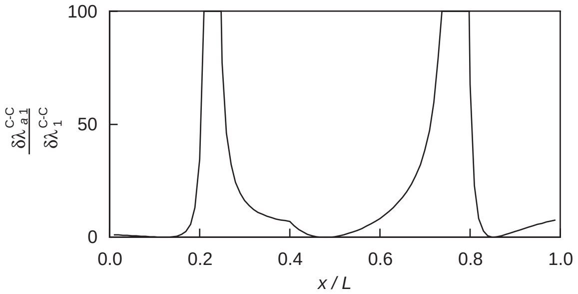

Fig. 4. Plot of the function $f(x) = (M_{a1}^{C-C}(x)/M_1^{C-C}(x))^2$ for uniform clamped-clamped beam in bending vibration, with $x_0 = 0.4L$.

3. Experimental results

In previous sections it was shown how measurements of natural frequencies and antiresonant frequencies may be employed to assess the location as well as the severity of a crack in a beam. The present section is devoted to illustrating some applications of experimental character. Several cracked steel beams were studied during the experiments. In particular, the following analysis concerns with a double T beam (beam 1) and a beam with rectangular solid cross-section (beam 2) under axial and flexural vibration, respectively. The results obtained for these two specimens are representative of the experimental and theoretical questions arisen in the course of the damage identification. A complete account of the experiments is presented in Ref. [22].

3.1. Axially vibrating beams

In the first experiment a steel beam of series HE100B (beam 1) was considered. The beam was suspended by two steel wire ropes to simulate free–free boundary conditions, see Fig. 5. By using an impulsive dynamic technique, the first lower natural frequencies and the antiresonant frequencies related to the point frf $H\sqrt{\lambda}, 0, 0)$ were determined for the undamaged beam and the beam under two damage configurations D1 and D2. The damage was obtained by saw cutting the beam at the cross-section at $s = 0.55$ m far from the left end. Levels D1 and D2 correspond, respectively, to a symmetrical cut of depth 6 and 15 mm. The width of each cut was equal to about 1.5 mm and, because of the small level of the excitation, during the dynamic tests each crack can be considered always open. Throughout experiments, the excitation was introduced at the left end by means of a PCB 086B03 impulse force hammer with a metallic tip. The axial response of the beam was measured with a PCB 303A03 piezoelectric accelerometer (with mass equal to 4×10^{-3} kg) fixed in the centre of the left end cross-section. Vibration signals were acquired by a HP35670A dynamic analyzer and then processed in the frequency domain to determine the related frf term. Output signals were weighted by an exponential window, while a force window was applied to the input signal. The well-separated vibrating modes and the very small damping allowed identification of the frequencies by means of the single-mode technique. Antiresonances were determined as dips in the frf magnitude associate with a phase variation of +180, see Ref. [14].

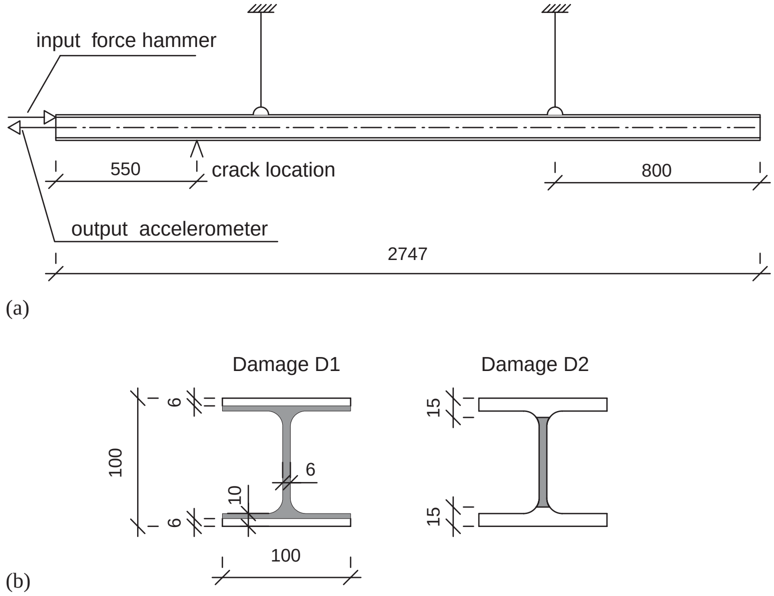

Fig. 5. Beam 1 in axial vibration: experimental model (a) and damage configurations (b). Lengths in millimeters.

Preliminary tests showed that antiresonances are rather sensitive to the position of the impact location and to the reproducibility of the impulse. This aspect was emphasized by the simultaneous presence in the axial response of the beam of transversal vibration with frequency content close to the antiresonance values. To reduce this indeterminacy, a significant number of averages for the estimation of frf were required. After a comprehensive series of tests, it was considered sufficient to estimate each frf term as an average of 20 measurements. Moreover, special care was necessary to ensure that the different impacts, which are required in the averaging process, always fall on the same location. Finally, from an extensive session of preliminary experiments, the intersection point between the upper flange and the vertical web of the cross-section was selected as an excitation point. In this way, a good reproducibility of the frf measurements was guaranteed and slight changes of the antiresonances were observed in different impulsive tests.

Table 1 compares the experimental natural frequencies and antiresonances and their corresponding analytical estimates for the undamaged and damaged beam. Since the accelerometer mass is about the 0.007 per cent of the total beam mass, its presence was disregarded in studying the dynamic behaviour of the system. The analytical model (5), (6) of the cracked beam was defined by assuming the position s of the damage as known and by determining the theoretical value of the stiffness K_A such that, for each damage configuration, the measured and the analytical fundamental elastic frequency coincide. Damage generally causes a fall in natural frequencies and antiresonances. A slight increase in third antiresonance was observed from the undamaged configuration to D1, but its source has not been understood. Frequency changes induced by damage are around 0.2–0.6 and 2–10 per cent for D1 and D2 configurations, respectively. Antiresonance decreasing is about 3–7 per cent for configuration D2.

Table 1
Experimental and analytical frequencies and antiresonances (of the point frf $H(\sqrt{\lambda}, 0, 0)$) of beam 1 in axial vibration

Mode	Frequencies			Antiresonances		
	Exp.	Model	$\Delta f_n\%$	Exp.	Model	$\Delta f_n\%$
Undamaged (1)						
1	941.1	941.1	0.00	468.6	470.6	0.41
2	1879.1	1882.2	0.16	1411.7	1411.7	0.00
3	2814.9	2823.3	0.30	2328.4	2352.8	1.05
4	3725.4	3764.4	1.05	3265.8	3293.9	0.86
Damage D1 (2)						
1	939.3	939.3	0.00	439.5	468.2	6.52
2	1868.3	1872.5	0.23	1409.3	1408.9	−0.03
3	2798.5	2809.1	0.38	2337.0	2352.7	0.67
4	3718.3	3757.3	1.05	n.a.	3287.3	—
Damage D2 (3)						
1	901.8	901.8	0.00	432.9	427.7	−1.19
2	1693.3	1697.6	0.25	1365.6	1365.4	−0.02
3	2608.9	2616.8	0.30	2324.4	2352.7	1.22
4	3637.0	3667.1	0.83	3102.5	3129.5	0.87

Abscissa of the cracked cross-section: $s = 0.550$ m. (1) $EA = 5.454 \times 10^8$ N, $\rho = 20.4$ kg/m, $L = 2.747$ m ($K_A = \infty$). (2) $K_{A,\text{anal}} = 3.507 \times 10^{10}$ N/m. (3) $K_{A,\text{anal}} = 1.736 \times 10^9$ N/m. Frequency values in Hz. $\Delta f_n\% = 100(f_n(\text{model}) - f_n(\text{exp.}))/f_n(\text{exp.})$.

As natural frequencies are concerned, the analytical model turns out to be extremely accurate for all the configurations under investigation, with maximum and average differences between experimental and theoretical values equal to 1 and 0.3 per cent, respectively, up to the 6th vibrating mode, see Ref. [22]. Apart from the first antiresonance for damage configuration D1, average errors on antiresonances are around 1 per cent. The significative reduction measured for first antiresonant frequency from the undamaged to damage configuration D1 implies a modelling error around 6 per cent. The source of this disagreement has not been explained. Summing up, the comparison between measurements and theoretical estimates is quite satisfactory since average modelling errors, both for natural frequencies and antiresonances, are less then 1 per cent in most of the vibrating modes considered.

Although the analytical model can be considered very accurate in the frequency range considered, percentage crack-induced changes in frequency are small and comparable with the accuracy of the beam model for damage level D1. In fact, both average modelling errors and average shifts caused by the crack are around 0.4 per cent for the first level of damage. Concerning antiresonances, it is expected that the important error on first antiresonance will produce wrong estimates of the damage parameters for damage configuration D1.

The results of the identification are summed up in Tables 2 and 3. For the sake of completeness, the results of identification using the changes $\{C^{\mathrm{F}}_m, C^{\mathrm{F}}_{2m}\}$ in mth and $2m$th natural frequencies, $m = 1, 2$, have been included in Tables 2 and 3. Corresponding formulae for damage location and

Table 2
Determination of the crack location in beam 1 by using frequencies (pair $C_m^F, C_{2m}^F, m = 1, 2$) and frequencies–antiresonances (pair $C_m^C, C_{2m+1}^F, m = 0, 1$)

m	Damage D1		Damage D2	
	s_{exp}	s_{anal}	s_{exp}	s_{anal}
Frequencies				
1	0.477	0.552	0.623	0.625
	2.270	2.195	2.124	2.122
2	0.561	0.552	0.577	0.571
	0.813	0.822	0.796	0.802
	1.934	1.925	1.951	1.945
	2.186	2.195	2.170	2.176
Frequencies–antiresonances				
0	0.159	0.553	0.671	0.612
1	0.690	0.547	0.484	0.485
	1.141	1.284	1.347	1.347
	2.521	2.379	2.316	2.316

Experimental (s_{exp}) and analytical (s_{anal}) estimates of the crack location. Actual crack location $s = 0.550$ m. Lengths in metres.

Table 3
Determination of the spring stiffness in beam 1 by using frequencies (pair $C_m^F, C_{2m}^F, m = 1, 2$) and frequencies–antiresonances (pair $C_m^C, C_{2m+1}^F, m = 0, 1$)

m	Damage D1		Damage D2	
	$10^{10} K_{A,exp}$	$10^{10} K_{A,anal}$	$10^9 K_{A,exp}$	$10^9 K_{A,anal}$
Frequencies				
1	2.717	3.524	2.073	2.086
2	3.178	3.520	1.989	1.983
Frequencies–antiresonances				
0	0.330	3.535	2.343	2.016
1	1.680	3.590	2.810	2.794
Actual value	3.507		1.736	

Experimental ($K_{A,exp}$) and analytical ($K_{A,anal}$) estimates of the spring stiffness. Stiffness values in N/m.

damage severity were deduced in Ref. [4] (Eqs. (29) and (30)), namely

$$K_A = \frac{1 - C_{2m}^F / C_m^F}{C_m^F}, \quad S \equiv \cos\left(\frac{2m\pi s}{L}\right) = \frac{C_{2m}^F}{2C_m^F} - 1, \tag{34}$$

where $C_m^F = (1/K_A)\sin^2(m\pi s/L)$.

With reference to the localization of the cracked cross-section, analytical results are in good agreement with the theory presented in Section 2. The set of solutions predicted by the theory for the mathematical problem contains a satisfactory estimate of the real solution of the damage problem. Deviations from the correct values are negligible for configuration D1 and, due to the assumption of small amount of damage, which is of the order 4–14 per cent for D2 configuration. Estimates of the crack location are consistent even when experimental data are employed. However, for configuration D1, the inaccuracy of the antiresonance estimates prejudices the reliability of the identification based on experimental data.

Identified values for K_A are generally less accurate with respect to the corresponding damage location estimates, and they become worse when the identification is based on experimental data, as was already observed in previous studies, see Refs. [4,5]. In particular, estimates for K_A are rather rough for damage configuration D2 even when low frequencies are used in identification. This is because, in the problem under investigation, the damage is rather severe from the beginning.

In conclusion, experiments show that if antiresonant frequencies used as data in identification are affected by relatively small errors with respect to the shifts induced by the crack, then antiresonance measurements are useful to exclude symmetrical solutions which occur when damage identification is based on natural frequencies only.

3.2. Bending vibrating beams

The second experimental model (beam 2), shown in Fig. 6, is a steel beam of rectangular solid cross-section. By adopting an experimental technique similar to that used for beam 1, the undamaged beam and four damaged configurations D1–D4 were studied, see Ref. [22] for more details on the experiments. Damage was obtained by introducing a symmetrical saw-cut of depth

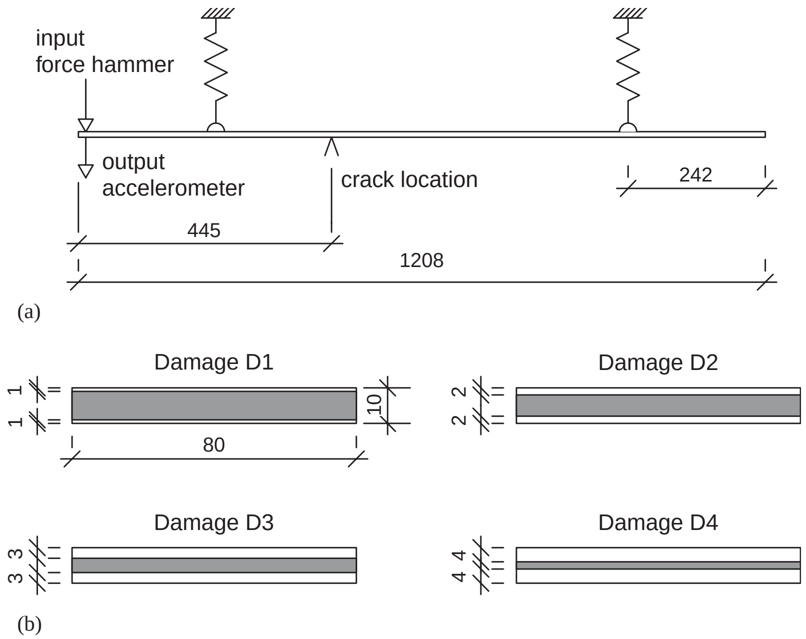

Fig. 6. Beam 2 in bending vibration: experimental model (a) and damage configurations (b). Lengths in millimeters.

1, 2, 3, 4 mm for damaged configurations D1–D4, respectively, at the cross-section 0.445 m far from the left end. The width of each cut was equal to about 1.0 mm and, because of the small level of the excitation, during the dynamic tests each crack can be considered always open. The beam was excited transversally at the left end by means of a PCB 086B03 impulse force hammer, with a soft tip and a tip of intermediate stiffness for range 0–200 and 200–800 Hz, respectively. The transversal response at the same end was acquired by a PCB 303A3 piezoelectric accelerometer (with mass equal to 4×10^{-3} kg). Dynamic tests show a good reproducibility of the frf measurements and slight variations of the antiresonance values were observed in different impulsive tests. Fig. 7 shows a typical inertance frf obtained as the average of 20 measurements.

Table 4 shows the measured and analytical values for the first four lower modes. Also in this case, the presence of the accelerometer mass, which is about 0.053 per cent of the total beam mass, was disregarded in studying the dynamic behaviour of the system. The analytical model of the cracked beam was obtained in the same way as before assuming that the damage location is known and determining the stiffness K_B of the rotational spring by taking the measured value for the analytical fundamental frequency. The analytical model generally fits very well with the real behaviour of the cracked beam. Absolute percentage deviations are negligible for frequencies up to the 6th vibrating mode. As in experiments on axially vibrating beams shown in previous section, an analytical model is less accurate for antiresonances. In fact, percentage errors are of the order of 1 per cent for first six antiresonances. Maximum deviation was measured for the lower antiresonance and it ranges from 0.7 to 2.0 per cent for configurations D1, D2 to 2.9–3.2 per cent for configurations D3, D4.

The results of identification are reported in Tables 5 and 6. For the sake of completeness, crack location is determined by also using the second and fourth frequencies and the second frequency

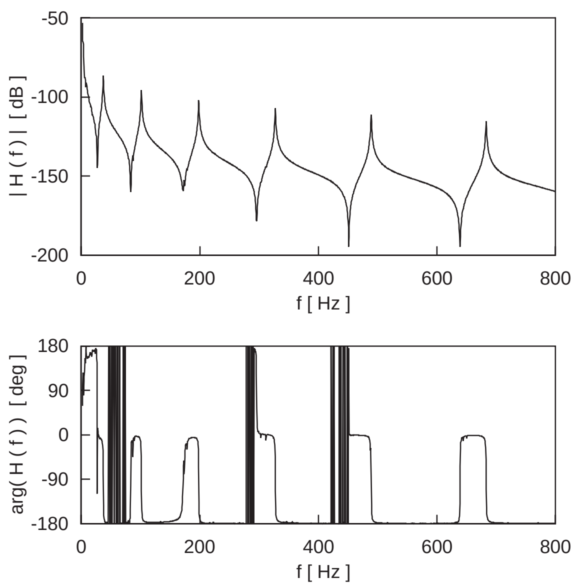

Fig. 7. Typical behaviour of the inertance for beam 2 in bending vibration.

Table 4
Experimental and analytical frequencies and antiresonances of beam 2 in bending vibration

Mode	Frequencies			Antiresonances		
	Exp.	Model	$\Delta f_n\%$	Exp.	Model	$\Delta f_n\%$
Undamaged (1)						
1	36.6	36.6	0.00	25.1	25.2	0.38
2	100.8	100.9	0.04	83.1	81.7	−1.63
3	197.7	197.8	0.05	171.5	170.5	−0.55
4	326.8	326.9	0.03	295.2	291.6	−1.23
Damage D1 (2)						
1	36.5	36.5	0.00	25.0	25.2	0.78
2	100.7	100.7	0.00	82.6	81.7	−1.05
3	197.7	197.8	0.06	171.7	170.4	−0.77
4	326.4	326.3	−0.01	295.1	291.1	−1.38
Damage D2 (3)						
1	36.2	36.2	0.00	24.4	24.9	2.05
2	100.1	100.1	−0.04	82.3	81.5	−0.92
3	197.6	197.7	0.07	171.6	169.9	−0.96
4	324.0	323.9	−0.04	293.5	288.8	−1.60
Damage D3 (4)						
1	35.4	35.4	0.00	23.6	24.3	2.94
2	98.4	98.3	−0.08	81.6	81.1	−0.71
3	197.5	197.7	0.08	170.9	168.6	−1.30
4	317.9	317.7	−0.06	287.1	283.1	−1.39
Damage D4 (5)						
1	32.2	32.2	0.00	22.5	21.8	−3.19
2	92.9	92.6	−0.25	80.5	79.4	−1.28
3	197.3	197.5	0.13	165.5	164.1	−0.81
4	298.8	299.0	0.07	269.7	266.8	−1.07

Abscissa of the cracked cross-section: $s = 0.445$ m. (1) $EI = 1477$ N m^2, $\rho = 6.30$ kg/m, $L = 1.208$ m ($K_B = \infty$). (2) $K_{B,\text{anal}} = 602.5 \times 10^3$ N m/rad. (3) $K_{B,\text{anal}} = 111.9 \times 10^3$ N m/rad. (4) $K_{B,\text{anal}} = 33.27 \times 10^3$ N m/rad. (5) $K_{B,\text{anal}} = 7.766 \times 10^3$ N m/rad. Frequency values in Hz. $\Delta f_n\% = 100(f_n(\text{model}) - f_n(\text{exp.}))/f_n(\text{exp.})$.

and the second antiresonance. Concerning the determination of crack location, the results are in good agreement with theoretical expectations when exact analytical data are used. It can be noted that the employment of the antiresonances, which are affected by larger modelling errors, prejudices the capability of obtaining accurate estimates of the crack location. In such cases, however, it is advisable, firstly, to resort to frequency data and, subsequently, to intersect the results with those from antiresonance measurements in order to eliminate symmetrical locations of the cracked cross-section. Identification based on frequency measurements gives accurate estimates for the stiffness K_B. On the contrary, large errors are observed when antiresonances are used as data in identification.

Table 5
Determination of the crack location in beam 2 by using frequencies (pair $\delta\lambda_m^F, \delta\lambda_{2m}^F, m = 1,2$) and frequencies–antiresonances (pair $\delta\lambda_m^F, \delta\lambda_m^{S-F}, m = 1,2$)

m	Damage D1		Damage D2		Damage D3		Damage D4	
	s_{exp}	s_{anal}	s_{exp}	s_{anal}	s_{exp}	s_{anal}	s_{exp}	s_{anal}
Frequencies								
1	0.467	0.445	0.455	0.447	0.454	0.450	0.464	0.461
	0.741	0.763	0.753	0.761	0.754	0.758	0.744	0.747
2	0.242	0.244	0.242	0.244	0.244	0.245	0.246	0.248
	0.448	0.445	0.447	0.444	0.445	0.443	0.441	0.439
	0.760	0.763	0.761	0.764	0.763	0.765	0.767	0.769
	0.966	0.786	0.966	0.964	0.964	0.963	0.962	0.960
Frequencies–antiresonances								
1	0.214	0.445	0.240	0.447	0.315	0.451	0.582	0.466
2	0.100	—	0.238	—	0.336	—	—	—
	0.585	0.445	0.574	0.444	0.569	0.442	0.418	0.436
	0.643	0.638	0.754	0.561	1.017	0.562	0.564	0.562

Experimental (s_{exp}) and analytical (s_{anal}) estimates of the crack location. Actual crack location $s = 0.445$ m. Lengths in metres.

Table 6
Determination of the spring stiffness in beam 2 by using frequencies (pair $\delta\lambda_m^F, \delta\lambda_{2m}^F, m = 1$) and frequencies–antiresonances (pair $\delta\lambda_m^F, \delta\lambda_m^{S-F}, m = 1$)

Damage D1		Damage D2		Damage D3		Damage D4	
$K_{B,exp}$	$K_{B,anal}$	$K_{B,exp}$	$K_{B,anal}$	$K_{B,exp}$	$K_{B,anal}$	$K_{B,exp}$	$K_{B,anal}$
Frequencies							
655321	604357	118224	114666	36389	35842	10409	10300
Frequencies–antiresonances							
84790	604357	23030	114666	15849	35979	13104	10481
602491[a]		111903[a]		33270[a]		7766[a]	

Experimental ($K_{B,exp}$) and analytical ($K_{B,anal}$) estimates of the spring stiffness. Stiffness values in N m/rad.
[a] Actual value.

4. Conclusions

This paper was concerned with the identification of a single open crack in a vibrating beam, either under axial or bending vibration, from the knowledge of the damage-induced changes in natural frequencies and antiresonant frequencies.

It was shown how an appropriate use of frequencies and antiresonances may be useful to avoid the non-uniqueness of the damage location problem, which occurs in symmetrical beams when only frequency data are used.

Numerical results are in good agreement with the theory when exact analytical data are employed in identification. A series of dynamic tests on cracked steel beams showed that, in the inverse problem solution, the noise and the modelling errors on antiresonances are usually amplified strongly with respect to cases in which only frequency data are used. This peculiar behaviour suggests that damage identification techniques based on antiresonance data should be carried out with some caution when mechanical systems of greater complexity are considered.

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