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Journal of Sound and Vibration 276 (2004) 440–448

JOURNAL OF SOUND AND VIBRATION

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Letter to the Editor

Preservation of the fundamental natural frequencies of rectangular plates with mass and spring modifications

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Received 25 April 2003; accepted 27 October 2003

1. Introduction

For most cases where plates are used in engineering structures, mass and stiffness modifications become necessary. Ingber et al. [1] investigated experimentally, vibrations of clamped plates with concentrated mass and spring attachments by using a modal analysis technique and mixed boundary-element method. Boay [2] analyzed the natural frequencies of plates with and without a concentrated mass. The Rayleigh-energy method was used in the theoretical formulation. Lin and Lim [3] derived the receptances based on mode superposition and then used it to calculate the receptances of the plate with arbitrary mass and stiffness modification. McMillan and Keane [4] developed the direct modal sum method for shifting resonances from a frequency band by applying concentrated masses to a thin rectangular plate. Cha [5] applied the hybrid approach to analyze the free vibration of a simply supported rectangular plate carrying a concentrated mass. Wu and Luo [6] determined the natural frequencies and mode shapes of a rectangular plate carrying any number of point masses and springs by means of the analytical-and-numerical combined method. Dowell and Tang [7] studied the high-frequency response of a plate carrying a concentrated mass/spring system. Ref. [8] was concerned with satisfying a design aim such that the fundamental frequency of a cantilever beam remains the same in spite of the addition of a mass at some point on the beam. The present study represents to some extent the counterpart of the publication [8] for plate vibrations. Within this framework, the present study aims to investigate the possibility of using springs to preserve the fundamental frequency of a thin rectangular plate carrying any number of point masses. The numerical results obtained in this study are not only related to the fundamental frequency of the plate, but the formulation can also be adopted when any one of the natural frequencies of the plate is desired to be kept constant. The problems on plates carrying concentrated masses are encountered, e.g., in the design of electronic systems. The printed circuit boards and plate-like chassis can be approximated as flat rectangular plates

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.10.044

carrying concentrated masses and subjected to vibration [2]. By calculating the receptance matrix of the unmodified plate and the receptance matrix corresponding to the modification, the receptance matrix of the modified plate are obtained based on substructuring analysis. Then the natural frequencies of the modified plate are calculated by analyzing the receptance data. Finally, the required coefficients of the springs to be placed at certain locations such that the fundamental frequencies will remain the same although there are added point masses that can be calculated.

2. Receptances of a rectangular plate

According to the classical thin-plate theory, the governing equation in terms of the lateral displacement $w(x, y, t)$ is given by

$$D\nabla^4 w + \rho h \partial^2 w/\partial t^2 = P, \tag{1}$$

where ∇^4 is the two-dimensional biharmonic operator, h is the thickness of the plate, D is the flexural rigidity, ρ is the mass density, and P is the lateral load per unit area. The flexural rigidity is given by

$$D = Eh^3/12(1 - v^2), \tag{2}$$

where E is the modulus of elasticity, and v is the Poisson ratio. The receptance of a plate $\alpha_{ij}(\omega)$ is the response at the location $w_i = w(x_i, y_i)$ due to a harmonic force of unit magnitude and frequency ω applied at location $w_j = w(x_j, y_j)$ and can be expressed as [3]

$$\alpha_{ij}(\omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\varphi_{mn}(x_j, y_j)\varphi_{mn}(x_i, y_i)}{(\omega_{mn}^2 - \omega^2)}, \tag{3}$$

where ω_{mn} are the natural frequencies, φ_{mn} are the normalized shape functions of the plate under consideration. For a plate shown in Fig. 1 having edge supports of S–S–S–S and S–C–S–C, the natural frequencies and the normalized shape functions are given as [2]

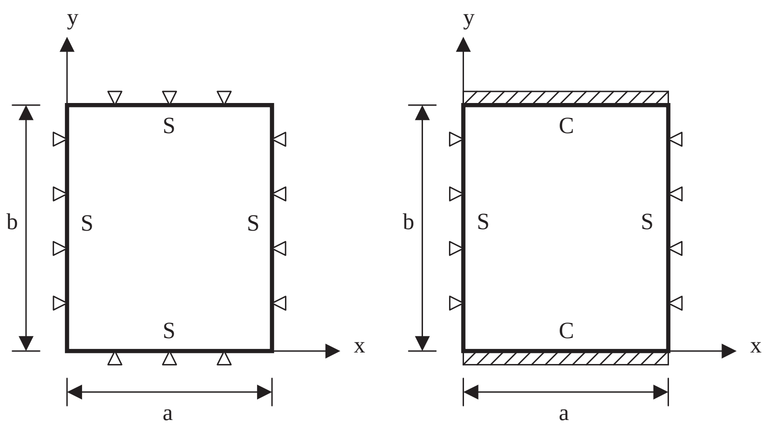

Fig. 1. The two support conditions of the rectangular plate studied: (a) S–S–S–S; (b) S–C–S–C.

S–S–S–S edge supports:
For $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$

$$\omega_{mn} = \sqrt{\frac{D}{\rho h}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right], \tag{4}$$

$$\varphi_{mn} = \frac{2}{\sqrt{\rho abh}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{5}$$

S–C–S–C edge supports:
For $m = 1, 2, 3, \ldots$ and $n = 1$

$$\omega_{mn} = \frac{\pi^2}{a^2b^2}\sqrt{\frac{D(3b^4m^4 + 16a^4 + 8a^2b^2m^2)}{3\rho h}}. \tag{6}$$

For $m = 1, 2, 3, \ldots$ and $n = 2, 3, 4, \ldots$

$$\omega_{mn} = \frac{\pi^2}{a^2b^2}\sqrt{\frac{D[b^4m^4 + a^4(1 + 6n^2 + n^4) + 2a^2b^2m^2(1 + n^2)]}{\rho h}}, \tag{7}$$

$$\varphi_{mn} = \frac{2}{\sqrt{\rho abh}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right). \tag{8}$$

The expressions given in Eq. (8) represent the mode shapes corresponding to the natural frequencies given in Eqs. (6) and (7), where $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$.

3. Frequency response function (FRF) method of coupled structure analysis

This method is often referred to as the 'impedance coupling method' or the 'dynamic stiffness method' [9]. The two components A and B shown in Fig. 2 are to be connected by the coupling

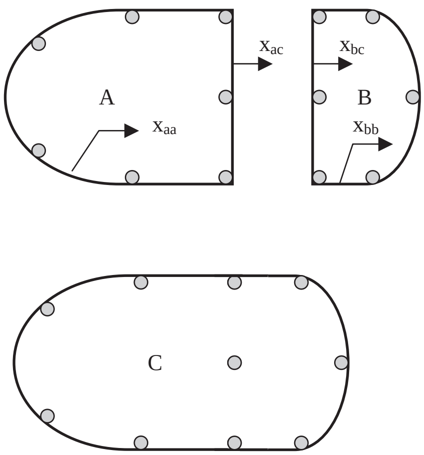

Fig. 2. Basis of modified structure analysis.

co-ordinates to form the connected system C. The substructures, A and B, the 'input' to coupling process will comprise two square FRF matrices, one of $n_A \times n_A$ and the other $n_B \times n_B$ which will then be combined to yield a corresponding FRF matrix for the coupled structure, C, which is of $n_C \times n_C$. Both components A and B have two types of co-ordinates of interest (coupling and slave co-ordinates);

$$\{x_A\}_{n_A\times 1} = \begin{Bmatrix} x_{ac} \\ x_{aa} \end{Bmatrix} = \begin{Bmatrix} \text{coupling} \\ \text{slave} \end{Bmatrix}, \quad \{x_B\}_{n_B\times 1} = \begin{Bmatrix} x_{bc} \\ x_{bb} \end{Bmatrix} = \begin{Bmatrix} \text{coupling} \\ \text{slave} \end{Bmatrix}. \tag{9}$$

The receptance FRF properties for component A are contained in a matrix $[\boldsymbol{\alpha}_A]$ which can be partitioned as shown below, separating those elements which relate to the coupling d.o.f.'s from those which do not:

$$[\boldsymbol{\alpha}_A] = \begin{bmatrix} \alpha^A_{aa} & \alpha^A_{ac} \\ \alpha^A_{ca} & \alpha^A_{cc} \end{bmatrix}_{n_A\times n_A}. \tag{10}$$

This receptance FRF matrix can be used to determine the corresponding impedance FRF matrix as follows:

$$[\mathbf{Z}_A] = [\boldsymbol{\alpha}_A]^{-1} = \begin{bmatrix} Z^A_{aa} & Z^A_{ac} \\ Z^A_{ca} & Z^A_{cc} \end{bmatrix}_{n_A\times n_A}. \tag{11}$$

Similarly, one can write a corresponding impedance FRF matrix $[\mathbf{Z}_B]$ for the other component, B, as

$$[\mathbf{Z}_B] = [\boldsymbol{\alpha}_B]^{-1} = \begin{bmatrix} Z^B_{bb} & Z^B_{bc} \\ Z^B_{cb} & Z^B_{cc} \end{bmatrix}_{n_B\times n_B}. \tag{12}$$

By an application of the equilibrium and compatibility conditions which must exist at the connection points, one can derive both a receptance an impedance version of the FRF matrix for the coupled structure of the form as given in Ref. [9]. The results repeated briefly from Ref. [9], will be applied below to a plate modified by a point mass and/or a spring. Then the natural frequencies of the modified plate can be calculated by performing a subsequent modal analysis on its receptance data.

Hence, the original plate (unmodified structure) corresponds to component A, whereas the substructure to be attached to it will correspond to component B. The impedance FRF matrix of the combined system consisting of the plate modified by the mass and spring reads as

$$[\mathbf{Z}_C]_{n_C\times n_C} = \begin{bmatrix} Z^A_{aa} & 0 & Z^A_{ac} \\ 0 & Z^B_{bb} & Z^B_{bc} \\ Z^A_{ca} & Z^B_{cb} & (Z^A_{cc} + Z^B_{cc}) \end{bmatrix}. \tag{13}$$

The FRF of the added substructure B, α_B, which consists of a mass, or a spring, can be expressed as

$$\alpha_B = \begin{cases} \dfrac{1}{-\omega^2 mB} & \text{for a mass,} \\ \dfrac{1}{k_B} & \text{for a spring.} \end{cases} \tag{14}$$

The receptance matrix of the unmodified plate for the co-ordinates of the points of the mass and spring attachment is obtained as

$$[\boldsymbol{\alpha}] = \begin{bmatrix} \alpha_{mm} & \alpha_{mk} \\ \alpha_{km} & \alpha_{kk} \end{bmatrix}, \tag{15}$$

by using Eq. (3). After the connection of mass m_B and spring k_B to the plate, the receptance matrix of the so modified plate is obtained in accordance with Eqs. (13) and (14) as follows:

$$[\boldsymbol{\alpha}]^* = \begin{bmatrix} \alpha^*_{mm} & \alpha^*_{mk} \\ \alpha^*_{km} & \alpha^*_{kk} \end{bmatrix}, \tag{16}$$

$$\alpha^*_{mm} = \frac{(-k_B\alpha_{km}\alpha_{mk} + \alpha_{mm} + k_B\alpha_{kk}\alpha_{mm})}{d_{mk}}, \tag{17}$$

$$\alpha^*_{mk} = \frac{\alpha_{mk}}{d_{mk}}, \tag{18}$$

$$\alpha^*_{km} = \frac{\alpha_{km}}{d_{mk}}, \tag{19}$$

$$\alpha^*_{kk} = \frac{[m_B\alpha_{km}\alpha_{mk}\omega^2 + \alpha_{kk}(1 - m_B\alpha_{mm}\omega^2)]}{d_{mk}}, \tag{20}$$

where

$$d_{mk} = 1 + k_B\alpha_{kk} + m_B[k_B\alpha_{km}\alpha_{mk} - (1 + k_B\alpha_{kk})\alpha_{mm}]\omega^2. \tag{21}$$

The right side of the above expression equated to zero represents the frequency equation of the modified plate.

In case of attaching merely a single mass to the plate, receptances are calculated by placing $k_B = 0$ in equations above and by substituting $m_B = 0$ in case of attaching a sole spring. When more than one masses and springs are attached to the plate at different points, the receptance matrix after each modification is attained by the employment of Eqs. (17)–(20) successively. The process is continued by substituting the new members of matrix $[\boldsymbol{\alpha}]$ into Eqs. (17)–(20), permitting the determination of another $[\boldsymbol{\alpha}]^*$ matrix.

The denominators, which are common to all elements of the receptance matrix (16), represent the frequency equation of the modified plate. If ω_i denotes one of the natural frequencies of the modified plate, the value of the spring coefficient k_B to be attached for compensating the effect of the attachment of the mass m_B at this frequency is obtained simply by solving this equation with respect to k_B which yields

$$k_B = \frac{m_B\alpha_{mm}\omega_i^2 - 1}{(\alpha_{kk} + m_B\alpha_{km}\alpha_{mk}\omega_i^2 - m_B\alpha_{kk}\alpha_{mm}\omega_i^2)}. \tag{22}$$

Eq. (22) can be used not only for the fundamental frequency of the plate, but also for any one of the natural frequencies which is desired to be kept constant. In the special case of the attachment of the mass and spring to the plate at the same point $(a\xi_m, b\eta_m)$, i.e., when the attachment points are collocated, one has

$$\alpha_{mm} = \alpha_{kk} = \alpha_{km} = \alpha_{mk}. \tag{23}$$

The receptance expression of the modified plate for the same point of the mass and spring attachment is obtained as

$$\alpha^*_{mm} = \left[\alpha^{-1}_{mm} + \left[\frac{1}{(k_B - m_B\omega^2)}\right]^{-1}\right]^{-1}. \tag{24}$$

In this case, Eq. (21) reduces to

$$d_{mm} = d_{mk} = 1 + \alpha_{mm}(k_B - m_B\omega^2). \tag{25}$$

The elements of the receptance matrix are now as follows:

$$\alpha^*_{mm} = \alpha^*_{kk} = \alpha^*_{km} = \alpha^*_{mk} = \frac{\alpha_{mm}}{d_{mm}}. \tag{26}$$

As seen from Eqs. (25) and (26), to keep the receptance unchanged at a prescribed frequency ω, the natural frequency of the attached mass–spring system with one degree-of-freedom, should be equal to this frequency ω $(\omega = \sqrt{k_B/m_B})$. This enables one to calculate the required coefficient of the spring to be placed such that the fundamental frequency remains unchanged, although a point mass is added.

4. Numerical results

This section is devoted to the testing of the expressions obtained. The rectangular plates shown in Fig. 1 are taken as examples. One of them is simply supported on its four edges denoted as (S–S–S–S) and two edges of the second plate are simply supported whereas the remaining two are clamped, denoted as (S–C–S–C). Before following the design aim that the fundamental frequency remains unchanged despite mass attachment, the two natural frequencies of the simply supported (S–S–S–S) rectangular plate carrying three concentrated masses: $m_1 = 70$ kg, $m_2 = 50$ kg, $m_3 = 60$ kg located at $\xi_{m_1} = x_1/a = 0.375$, $\eta_{m_1} = y_1/b = 0.25$, $\xi_{m_2} = 0.5$, $\eta_{m_2} = 0.625$, $\xi_{m_3} = 0.75$, $\eta_{m_3} = 0.5$ and three springs $k_1 = 10^6$ N/m, $k_2 = 10^4$ N/m, $k_3 = 10^5$ N/m located at $\xi_{k_1} = 0.125$, $\eta_{k_1} = 0.25$, $\xi_{k_2} = 0.5$, $\eta_{k_2} = 0.5$, $\xi_{k_3} = 0.625$, $\eta_{k_3} = 0.625$ are determined by using the FRF method for $m = n = 3$, $m = n = 6$, $m = n = 12$, $m = n = 20$, $m = n = 30$, and the results are compared with those of the analytical-and-numerical combined method (ANCM) [6] given in Table 1. The effect of m and n values upon the first and second natural frequencies for the example is illustrated in Table 1, which shows that five-digit accuracy has been achieved assuming $m = 20$ and $n = 20$. Table 1 makes it clear that when the number of the considered modes increase, calculated natural frequency value decreases indicating that the results get more precise. The physical properties of the plate are chosen as: $a = 2$ m, $b = 3$ m, $h = 0.005$ m, $\rho = 7850$ kg/m^3, $E = 2.051 \times 10^{11}$ N/m^2, and $\nu = 0.3$.

Table 2 shows the values of the calculated spring coefficients for 16 positions of the spring which compensates for the decreasing effect of the attached point mass ($m_B = 0.4$ kg at the location $\xi_m = 0.625$, $\eta_m = 0.375$) on the fundamental frequency. The mass is attached

Table 1
The effect of number of modes on the first two natural frequencies of the modified plate with S–S–S–S supporting condition

	$m=3,\ n=3$	$m=6,\ n=6$	$m=12,\ n=12$	$m=20,\ n=20$	$m=30,\ n=30$	ANCM
ω_1	30.894	29.012	28.536	28.429	28.429	28.632
ω_2	42.038	39.811	39.073	38.914	38.914	39.392

Table 2
The spring coefficients necessary for preservation of the fundamental natural frequency of the rectangular plate with S–S–S–S supported condition ($\xi_m = 0.625,\ \eta_m = 0.375,\ m_B = 0.4$ kg)

	$\xi_k = 0.125$	$\xi_k = 0.375$	$\xi_k = 0.625$	$\xi_k = 0.875$
$\eta_k = 0.875$	9.23328×10^7	7.92495×10^6	7.83593×10^6	8.76105×10^7
$\eta_k = 0.625$	7.27805×10^6	1.11025×10^6	1.08940×10^6	6.95162×10^6
$\eta_k = 0.375$	7.00683×10^6	1.05948×10^6	1.00920×10^6	6.45712×10^6
$\eta_k = 0.125$	7.77206×10^7	7.09827×10^6	6.83444×10^6	6.73664×10^7

to the plate with the S–S–S–S supporting condition shown in Fig 3. The material properties used for generating the results are $E = 2.1 \times 10^{11}\ \mathrm{N/m^2}$, $v = 0.3$ and $\rho = 7800\ \mathrm{kg/m^3}$. The dimensions of the plate are: $a = 0.4$ m, $b = 0.5$ m and $h = 0.01$ m. For calculation of the receptances of the modified plate, $m = 50$ and $n = 50$ were chosen. A look at Table 2 makes clear that spring coefficients get higher values at the points closer to the edges of the plate, as expected.

The spring coefficients of the modified plate with S–C–S–C supporting conditions are shown in Table 3. When corresponding data of Tables 2 and 3 for the same spring locations are compared, it is seen that higher spring coefficients are required at S–C–S–C supported conditions as can be expected. If mass and spring attachments are collocated ($\xi_k = \xi_m = 0.625,\ \eta_k = \eta_m = 0.375$), the natural frequency of the mass–spring system is equal to the natural frequency of the unmodified plate. In this case, in accordance with $k_B = \omega_1^2 m_B$ equation for both supporting conditions, the desired spring coefficient can be calculated via $\omega_1 = 1588.45$ and 2142.09 rad/s which are the corresponding fundamental frequency values for S–S–S–S and S–C–S–C supporting conditions, respectively with $m_B = 0.4$ kg.

Table 4 aims to show on how m and n, i.e., the numbers of the modes considered, affect the results of the spring coefficients k_B, calculated, which should be attached to the points ($\xi_k = 0.125,\ \eta_k = 0.625$), ($\xi_k = 0.375,\ \eta_k = 0.625$), ($\xi_k = 0.625,\ \eta_k = 0.625$), ($\xi_k = 0.875,\ \eta_k = 0.625$) for both of the plates with S–S–S–S and S–C–S–C supporting conditions. A scrutiny of this table shows the result that as the coefficients m and n considered increase, values of the calculated spring coefficients also rise. The reason for this is the lowered natural frequency value, which is due to the increasing mode numbers. In order to keep the natural frequency value at its original level, decreasing natural frequency value should be compensated by raising the spring coefficient.

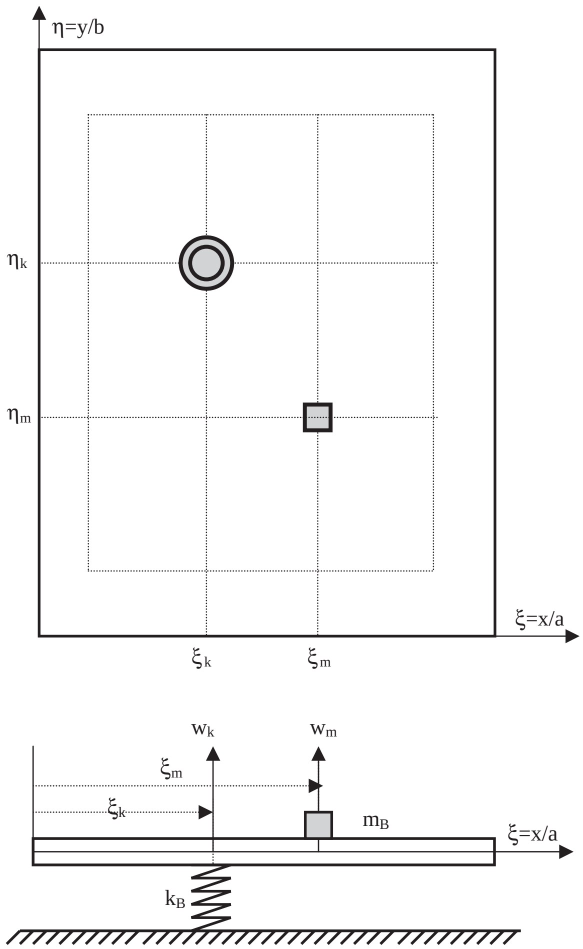

Fig. 3. A rectangular plate carrying a concentrated mass m_B at the position (ξ_m, η_m) and a spring k_B at point (ξ_k, η_k) to compensate the decrease in the fundamental frequency.

Table 3
The spring coefficients necessary for preservation of the fundamental natural frequency of the rectangular plate with S–C–S–C supported condition ($\xi_m = 0.625$, $\eta_m = 0.375$, $m_B = 0.4$ kg)

	$\xi_k = 0.125$	$\xi_k = 0.375$	$\xi_k = 0.625$	$\xi_k = 0.875$
$\eta_k = 0.875$	1.60731×10^9	8.46949×10^7	8.27329×10^7	1.36421×10^9
$\eta_k = 0.625$	1.41627×10^7	2.03812×10^6	1.97713×10^6	1.31233×10^7
$\eta_k = 0.375$	1.38169×10^7	1.96630×10^6	1.83532×10^6	1.22758×10^7
$\eta_k = 0.125$	1.35943×10^9	7.83263×10^7	7.41753×10^7	1.01011×10^9

5. Conclusions

The present study is concerned essentially with the derivation of the receptance matrix of a rectangular thin plate to which several point-masses and springs are attached, by the so-called

Table 4
The effect of number of modes used on the spring coefficients ($\xi_m = 0.625,\ \eta_m = 0.375,\ m_B = 0.4$ kg)

			$\xi_k = 0.125$	$\xi_k = 0.375$	$\xi_k = 0.625$	$\xi_k = 0.875$
$\eta_k = 0.625$	S–S–S–S	$m = 6,\ n = 6$	7.18330×10^6	1.10591×10^6	1.08644×10^6	6.86644×10^6
		$m = 12,\ n = 12$	7.25344×10^6	1.10929×10^6	1.08855×10^6	6.92911×10^6
		$m = 20,\ n = 20$	7.26976×10^6	1.10992×10^6	1.08911×10^6	6.94405×10^6
		$m = 40,\ n = 40$	7.27705×10^6	1.11021×10^6	1.08936×10^6	6.95070×10^6
		$m = 50,\ n = 50$	7.27805×10^6	1.11025×10^6	1.08940×10^6	6.95162×10^6
	S–C–S–C	$m = 6,\ n = 6$	1.38701×10^7	2.02610×10^6	1.96900×10^6	1.28748×10^7
		$m = 12,\ n = 12$	1.40841×10^7	2.03538×10^6	1.97475×10^6	1.30557×10^7
		$m = 20,\ n = 20$	1.41360×10^7	2.03718×10^6	1.97630×10^6	1.31004×10^7
		$m = 40,\ n = 40$	1.41595×10^7	2.03800×10^6	1.97702×10^6	1.31205×10^7
		$m = 50,\ n = 50$	1.41627×10^7	2.03812×10^6	1.97713×10^6	1.31233×10^7

'impedance coupling method'. The study enables one to obtain the eigenfrequencies of the combined system described above. Further, an examination was carried out of the problem of determining the stiffness coefficient of the spring to be placed at a specified position so that the fundamental frequency of the plate subject to two different (S–S–S–S and S–C–S–C) boundary conditions does not change, despite the attachment of a point mass at a predefined position.

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