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Asymptotic modal analysis of dynamical systems: the effect of modal cross-correlation

A. Li, E.H. Dowell*

Department of Mechanical Engineering and Materials Science, Duke University, Box 90300, Durham, NC 27708-0300, USA

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Abstract

Asymptotic modal analysis (AMA) has been shown to be a useful approximation and valid limiting case for classical modal analysis (CMA) when the number of resonant modes in a frequency bandwidth becomes sufficiently large and the cross-correlation between resonant modes is neglected. For CMA the neglect of these cross-correlations is usually justified if the damping is small and the resonant modal frequencies are well separated. In this paper it is shown that the cross-correlations may be neglected in the AMA limit even when the damping is not small. There is a rare exception to this, i.e., when two or more identical resonant frequencies occur. But this exceptional case is easily treated by including the cross-correlations for only such modes that have the same frequencies.

1. Introduction

The standard reference on statistical energy analysis (SEA) is the well-known text by Lyon [1] and is highly recommended to readers. The second author became interested in this subject some years ago because of some questions about the underlying basis for SEA and considered the case of many resonant modes being excited in a frequency bandwidth as an asymptotic limit of classical modal analysis. Since then various authors have considered the basis for SEA and thereby advanced our understanding of it. A more recent edition of Lyon's classic book [1] is that by Lyon and DeJong [2] which contains numerous references to the SEA literature per se.

In Ref. [3], a comparison of results of classical modal analysis (CMA) and asymptotic modal analysis (AMA) was made for the response of a single general linear structure under a random or

*Corresponding author.
E-mail address: dowell@mail.ee.duke.edu (E.H. Dowell).

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sinusoidal excitation. It was shown that the asymptotic behavior of the AMA results depends upon the number of modes in a frequency interval of interest and the location of the point forces, and that all the points on the structure except some special points, including the excited points, asymptotically have the same response. Some numerical examples for a beam were presented. In Ref. [4], Kubota and Dowell also verified these theoretical predictions using an experiment on a rectangular plate. For a recent extension of AMA to multi-component systems see Ref. [5] which contains references to much of the literature.

However, cross-correlations between modal responses were neglected in Refs. [3,4] and the other literature on AMA and SEA to the best of authors' knowledge by assuming they are negligible compared with the auto-correlation of each modal response. In this paper a theoretical and numerical analysis of the cross-correlation effects is carried out to assess whether they have an important effect on the asymptotic response of the structure.

As will be shown, the neglect of cross-correlations among modal responses is usually well justified in the AMA limit even for moderately large damping levels. This is in contrast to the results of CMA where the neglect of such cross-correlations requires the damping to be relatively small and the resonant modal frequencies to be well spaced. For a discussion of CMA, see Lin [6] and To [7]. Also Crandall [8] has provided interesting and valuable insights into modal responses as the number of responding modes becomes large and his work has been an inspiration for much of the work on AMA that has followed.

2. Theoretical modal analysis

2.1. Classical modal analysis under random point forces

According to classical modal analysis, the mean square response of the plate under point forces can be calculated by the following equation (see Ref. [6]):

$$\bar{w}^2(x,y) = \sum_n \sum_r \sum_i \sum_j \psi_n(x,y)\psi_r(x,y)\psi_n(x_i,y_i)\psi_r(x_j,y_j) \times \int_0^\infty H_n(\omega)H_r(-\omega)\Phi_{F_iF_j}\,\mathrm{d}\omega. \tag{1}$$

All the terms are defined in the Nomenclature. If the cross-correlation between forces is neglected, i.e., neglecting cross-correlation terms for which $i \neq j$, then Eq. (1) becomes

$$\bar{w}^2(x,y) = \sum_n \sum_r \sum_i \psi_n(x,y)\psi_r(x,y)\psi_n(x_i,y_i)\psi_r(x_i,y_i) \int_0^\infty H_n(\omega)H_r(-\omega)\Phi_{F_iF_i}\,\mathrm{d}\omega. \tag{2}$$

However, cross-correlation of the modes is included in Eq. (2). When the cross-correlation between modes is neglected (neglecting terms for which $n \neq r$), Eq. (2) further simplifies to

$$\bar{w}^2(x,y) = \sum_n \sum_i \psi_n^2(x,y)\psi_n^2(x_i,y_i) \int_0^\infty H_n(\omega)H_n(-\omega)\Phi_{F_iF_i}\,\mathrm{d}\omega. \tag{3}$$

If the external force is nearly white noise, i.e., $\Phi_{F_iF_i}$ is slowly varying with frequency, the integration over frequency in Eqs. (1) and (2) can be performed analytically, i.e.,

$$\int_0^\infty H_n(\omega)H_r(-\omega)\,\mathrm{d}\omega = \frac{2\pi(\xi_n\omega_n + \xi_r\omega_r)}{M_nM_r[(\omega_n^2-\omega_r^2)^2 + 4(\xi_n^2+\xi_r^2)\omega_n^2\omega_r^2 + 4\xi_n\xi_r\omega_n\omega_r(\omega_n^2+\omega_r^2)]}. \tag{4}$$

In this paper for our damping model, we assume that the modal damping obeys the following relationship (however, similar results are obtained when other assumptions about modal damping are made):

$$\xi_n\omega_n = \xi_1\omega_1 = constant, \tag{5}$$

where $n = 2, 3, \ldots$ Under this assumption, Eq. (2) becomes

$$\bar{w}^2(x,y) = \sum_n \sum_r \sum_i \frac{4\pi\xi_n\omega_n\psi_n(x,y)\psi_r(x,y)\psi_n(x_i,y_i)\psi_r(x_i,y_i)\Phi_{F_iF_i}}{M_nM_r[(\omega_n^2-\omega_r^2)^2 + 8\xi_n^2\omega_n^2(\omega_n^2+\omega_r^2)]}. \tag{6}$$

Further, when a spatial average is computed for the mean square response, Eqs. (2) and (3) become (for a plate of uniform mass and stiffness)

$$\langle \bar{w}^2 \rangle = \frac{\pi}{4}\sum_n \frac{\langle \psi_n^2 \rangle}{M_n^2\omega_n^3\xi_n}\sum_{i=1}^{I}\psi_n^2(x_i,y_i)\Phi_{F_iF_i}(\omega_n), \tag{7}$$

where I is the number of the point forces and the orthogonality of the modes has been used to eliminate the terms for which $n \neq r$. Note that taking the spatial average eliminates any cross-correlation between modes. In the present paper the focus will largely be on the local response of the plate and the difference between Eqs. (2) and (3), i.e., with and without modal cross-correlation.

2.2. *Asymptotic modal analysis under random point forces*

If M_n^2, ω_n^3, ξ_n, and $\langle \psi_n^2 \rangle$ are slowly varying with respect to modal number in the frequency bandwidth of the excited modes, in a certain interval of frequency of $\Delta\omega$, then Eq. (7) becomes [3,4]

$$\langle \bar{w}^2 \rangle = \frac{\pi}{4}\frac{\Delta M}{M_p^2\omega_c^3\xi_c}\sum_{i=1}^{I}\Phi_{F_iF_i}(\omega_c), \tag{8}$$

where ΔM is the number of the modes in the frequency interval. Alternatively Eq. (8) is often written as

$$\langle \bar{w}^2 \rangle = \frac{\pi}{4}\frac{\Delta M}{\Delta\omega}\frac{1}{M_p^2\omega_c^3\xi_c}\bar{F}^2, \tag{9}$$

where

$$\bar{F}^2 = \sum \Phi_{F_iF_i}(\omega_c)\Delta\omega. \tag{10}$$

Eq. (8) or Eq. (9) are the final result of AMA for the response of a plate under point random forces at all but certain special points on the structure. For simplicity of notation, it can be simply called AMA response. Note that Eq. (10) states that the total mean square of all the exciting forces, $\bar{F}^2$, is given by

$$\bar{F}^2 = \bar{F}_1^2 + \bar{F}_2^2 + \bar{F}_3^2 + \cdots. \tag{11}$$

This implies that the point forces are uncorrelated. Of course, this is exactly true when the point forces are uncorrelated in time. However, in the AMA limit there is a diminished spatial correlation because x_i, y_i and x_j, y_j are point forces at different positions which makes Eq. (8) a good approximation when a large number of modes are excited and responding whether or not the individual point forces are uncorrelated in time.

It is important to note that the AMA limit assumes that the frequency bandwidth is so large that the number of excited modes, ΔM, is large but that the frequency bandwidth is not so large that M_n, ω_n or ξ_n vary significantly. Eq. (9) was previously obtained in the context of SEA [1,2] and SEA has the priority in obtaining this result.

As one of the reviewers has noted it is unlikely that multiple modes with identical natural frequencies will occur in a practical structure. However, it is a possibility for simple structures with spatial symmetries and idealized boundary conditions. Hence this rare in practice, but possible in theory, occurence is discussed here so that the reader and practitioner will be aware of this possibility. The more usual (simpler) case of no repeated resonant frequencies is also discussed as well.

3. Comparison of SEA/AMA and CMA

Taking the ratio of Eq. (6) to Eq. (8) and the ratio of Eq. (7) to Eq. (8) one has a measure of the goodness of the asymptotic approximation (AMA or SEA) and one can also see how the modal cross-correlation terms affect the response of the plate. For definiteness consider a uniform, pinned plate under one or more point forces for which the modal function is $\psi_n(x, y) = \sin(n_x \pi x) \sin(n_y \pi y)$, where x, y are non-dimensional co-ordinates. The geometric structure of the system is shown in Fig. 1.

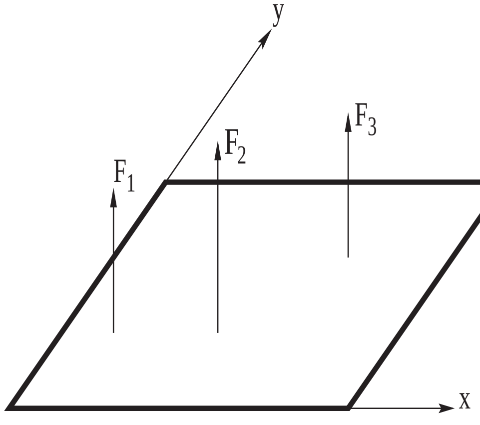

Fig. 1. Geometry of structure.

3.1. Part 1: The effect of the structural geometry

Here the modal damping is chosen as $\xi_1 = 0.01$.

3.1.1. One point force only ($I = 1$)

First, we consider an aluminum plate of dimensions $762 \times 508 \times 0.794\ \mathrm{mm}^3$. Only one point force is applied in the middle of the plate, i.e., $x_f = y_f = 0.5$. The aspect ratio, $762/508 = 1.5$, is a rational number. Thus, this structure has repeated natural frequencies or multiple eigenvalues. As will be seen this is important. The distribution of the natural frequencies of the structure is shown in Fig. 2. Note that although the frequencies are discrete, on the scale shown they appear to be continuous.

3.1.2. Spatial average response

Fig. 3 shows how the ratio of the CMA spatial average response without cross-correlation to the AMA response varies with respect to the minimum frequency of the bandwidth, $f_{\min}$, for various bandwidths, Δf, and $x_f = y_f = 0.5$. Fig. 3 shows the same ratio for frequency intervals of (a) $\Delta f = 300$ Hz, (b) $\Delta f = 100$ Hz, (c) $\Delta f = 30$ Hz and (d) $\Delta f = 10$ Hz. The corresponding number of modes, ΔM, is approximately 45, 15, 5 and 2. It is clear that the ratio of CMA/AMA approaches one as the frequency bandwidth, Δf, becomes large, i.e., as the number of the modes in the frequency interval increases.

3.1.3. Local response

At most points on the plate the response ratio of CMA/AMA still has this kind of property. The ratio approaches one as the frequency bandwidth becomes large (see Fig. 4).

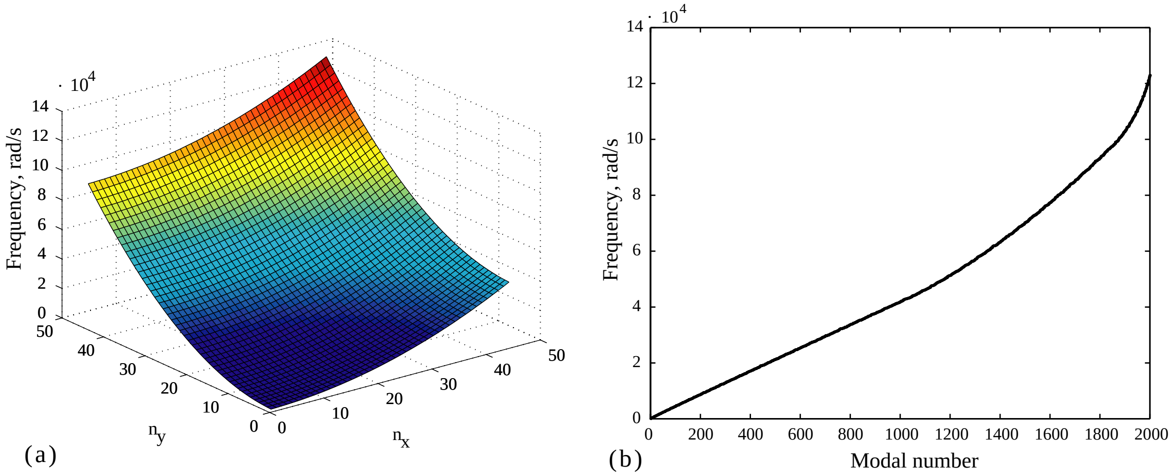

Fig. 2. Natural frequencies of an aluminum plate with dimension of $762 \times 508 \times 0.794\ \mathrm{mm}^3$: (a) natural frequency vs. n_x and n_y, (b) natural frequency vs. modal number.

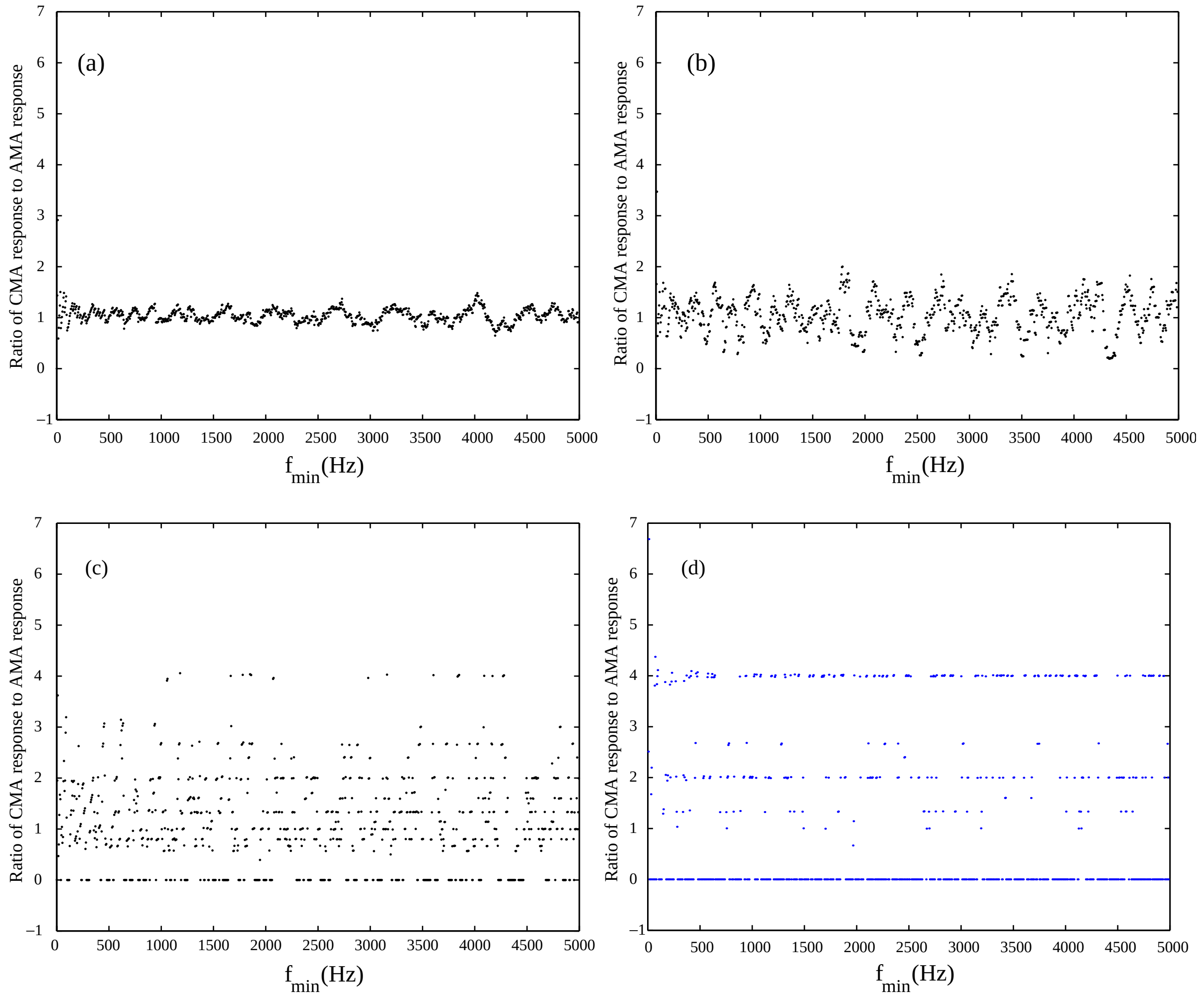

Fig. 3. Ratio of CMA spatial average response to AMA response versus minimum frequency for $x_f = y_f = 0.5$: (a) $\Delta f = 300$ Hz, (b) $\Delta f = 100$ Hz, (c) $\Delta f = 30$ Hz, (d) $\Delta f = 10$ Hz. Cross-correlation between modal responses is neglected.

3.1.4. Spatial distribution of responses

The spatial response of the plate is shown in Fig. 5. Fig. 5(a) shows the ratio of CMA response to AMA response neglecting the cross-correlation between different modes while the result with cross-correlation terms is shown in Fig. 5(b). From these two figures it is clear that the response at the excitation point is much larger than the response at other points. See the interesting discussion of Crandall [8] regarding such special points. Neglecting modal cross-correlation, it is shown in Ref. [4] that the response of the excitation point is four times larger than that of the other points in the AMA limit. The numerical response obtained by neglecting cross-correlation terms is indeed almost four times larger than the general spatial response. It is also noted that the response along

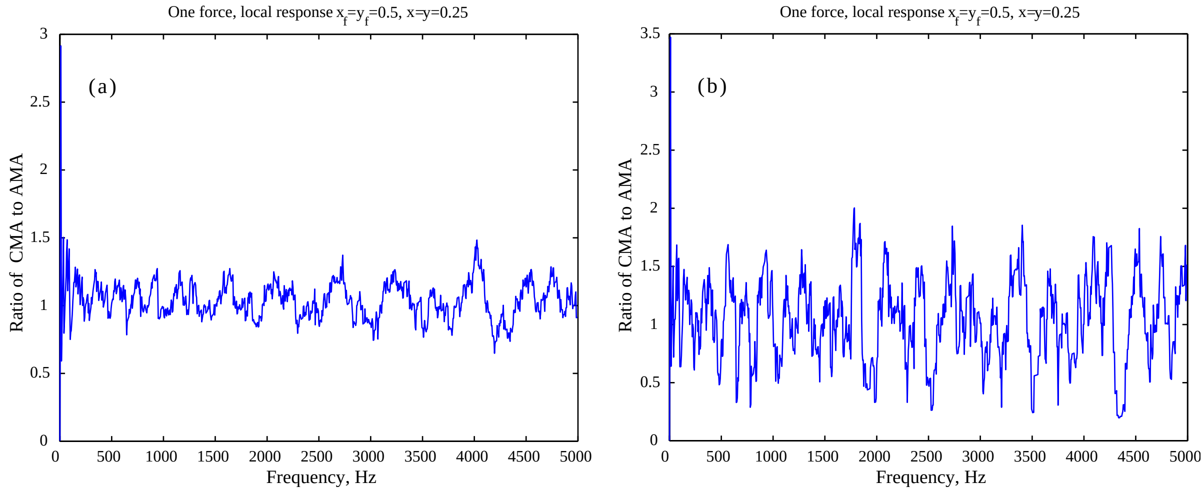

Fig. 4. Ratio of local CMA response to AMA response versus minimum frequency at $x = 0.25, y = 0.25$ for different frequency bandwidth: (a) $\Delta f = 300$ Hz and (b) $\Delta f = 100$ Hz. Cross-correlation between modal responses is neglected.

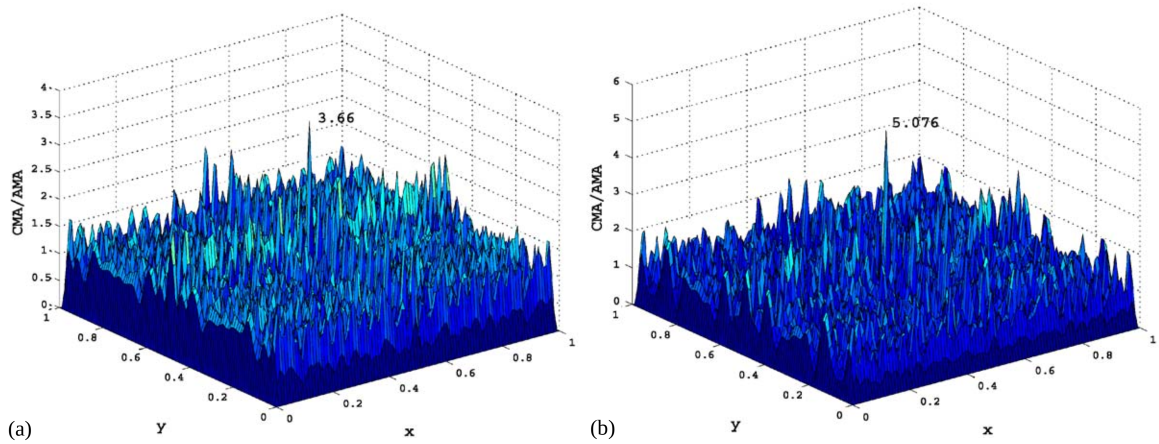

Fig. 5. Ratio of CMA response to AMA response on the plate for $x_f = y_f = 0.5, f_{\min} = 4995$ Hz and $f_{\max} = 5295$ Hz: (a) without cross-correlation and (b) with cross-correlation.

the lines passing through the point of excitation and parallel to an edge is larger than the general response [4].

Comparing the two results with and without the modal cross-correlation included (Figs. 5(a) and (b)) shows there is a measurable difference, suggesting that the cross-correlation terms may have an important effect on the response of the plate. Thus, there is a clear need to reconsider the assumptions of AMA method. For this reason, the ratio of CMA/AMA including the cross-correlation of only those modes having the same natural frequencies is also calculated and shown in Fig. 6. Note that the results of Figs. 5(b) and 6 are virtually the same. This suggests that it is the modes with identical resonant frequencies whose cross-correlation must be included. Fig. 7(a)

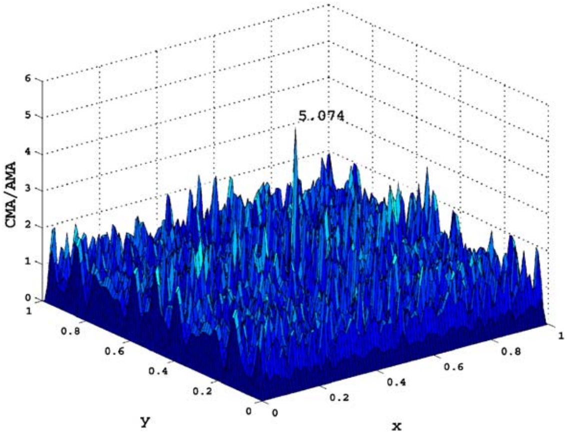

Fig. 6. CMA/AMA including the cross-correlations between the modes of the same frequency for the case: $x_f = y_f = 0.5$, $f_{\min} = 4995$ Hz and $f_{\max} = 5295$ Hz.

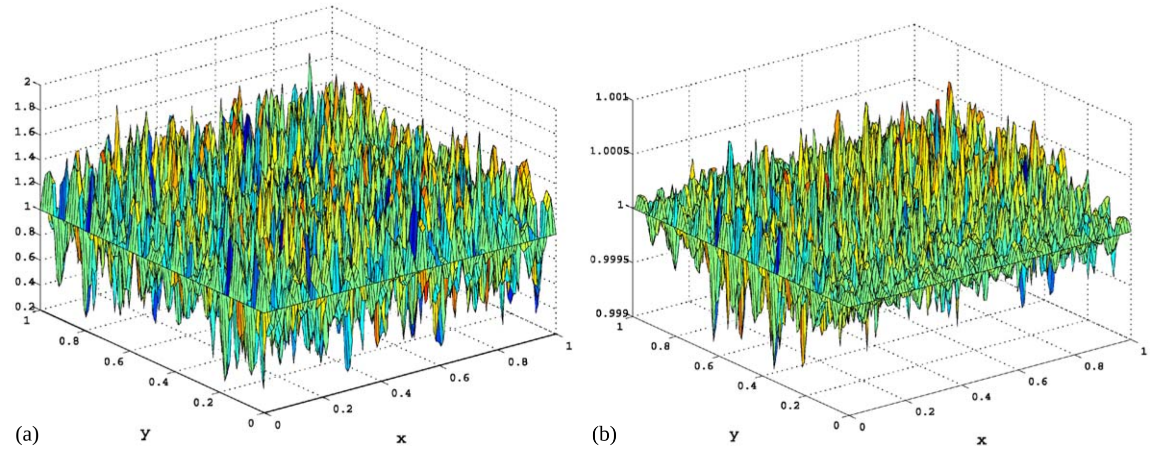

Fig. 7. Comparison of CMA/AMA for $x_f = y_f = 0.5$, $f_{\min} = 4995$ Hz and $f_{\max} = 5295$ Hz: (a) ratio of CMA/AMA with all cross-correlations included to CMA/AMA without cross-correlations and (b) ratio of CMA/AMA with all cross-correlations included to CMA/AMA with cross-correlations of only modes with identical frequencies.

shows a comparison of the ratio of the results of the CMA/AMA with all modal correlations included to those without and Fig. 7(b) shows the ratio of responses with all correlations included to those with only correlations of the modes of the same natural frequencies. Note the difference in vertical scales for Figs. 7(a) and (b). These results indicate the cross-correlations of the modes of the same natural frequency play an important role in the response of the plate. This means it is not quite exact to omit all the terms for which $n \neq r$ in Eq. (1) and one must include the effects of the multiple or repeated frequencies even in the AMA limit.

To gain further understanding of this result, now consider a plate with dimensions of $762 \times 762/\pi \times 0.794\ \text{mm}^3$. The point force is again applied at point $x_f = y_f = 0.5$. The distribution of natural frequencies of this plate is shown in Fig. 8. Note that for this case the aspect ratio of the plate is π which is not a rational number. Thus, no two natural modes have the same frequency.

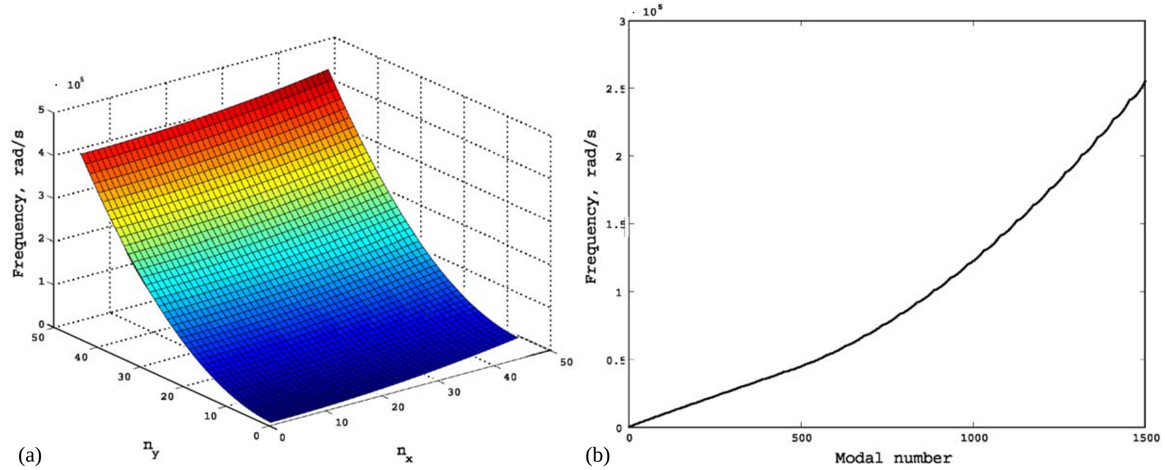

Fig. 8. Natural frequencies of the structure with dimension of $762 \times 762/\pi \times 0.794\ \text{mm}^3$: (a) natural frequency vs. n_x and n_y and (b) natural frequency vs. modal number.

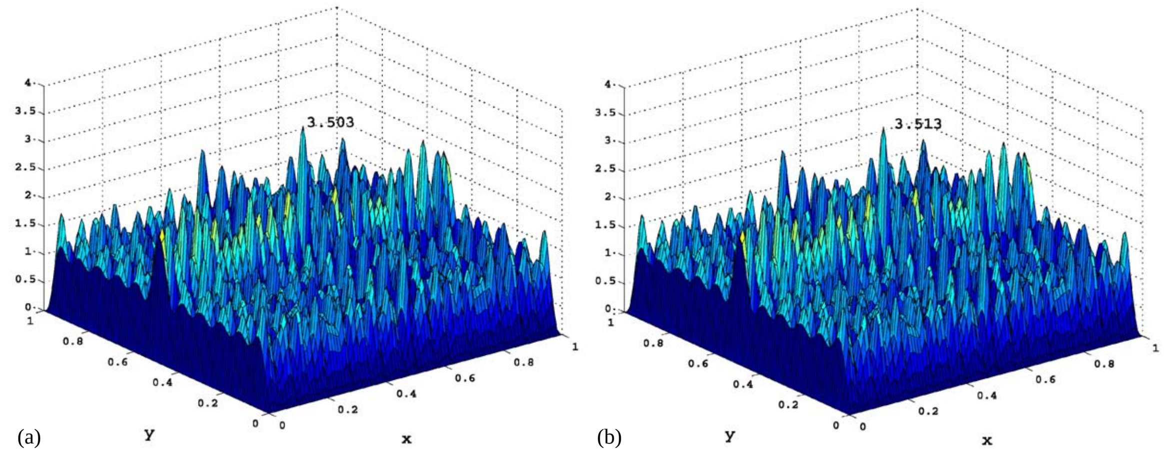

Fig. 9. CMA/AMA for the plate with dimension of $762 \times 762/\pi \times 0.794\ \text{mm}^3, x_f = y_f = 0.5$, $f_{\min} = 4995$ Hz and $f_{\max} = 5495$ Hz: (a) without cross-correlation, and (b) with cross-correlation.

The result for CMA/AMA neglecting cross-correlations is shown in Fig. 9(a) and the result with cross-correlations included is shown in Fig. 9(b). The ratio of these two results is shown in Fig. 10. Clearly these two results are almost same. Therefore, the modal cross-correlation terms are indeed negligible as the AMA limit is approached when the structure does not have multiple natural frequencies.

3.2. Two point forces ($I = 2$)

Consider the aluminum plate with dimension $762 \times 762/1.5 \times 0.794\ \text{mm}^3$. The natural frequencies are same as shown in Fig. 2.

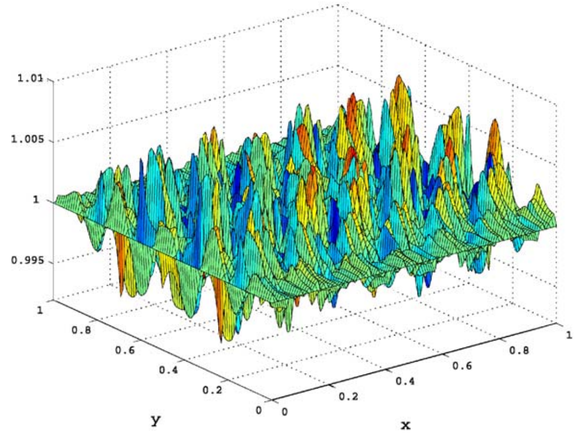

Fig. 10. The ratio of CMA with cross-correlations to CMA without cross-correlations for the case: $x_f = y_f = 0.5$, $f_{\min} = 4995$ Hz and $f_{\max} = 5495$ Hz.

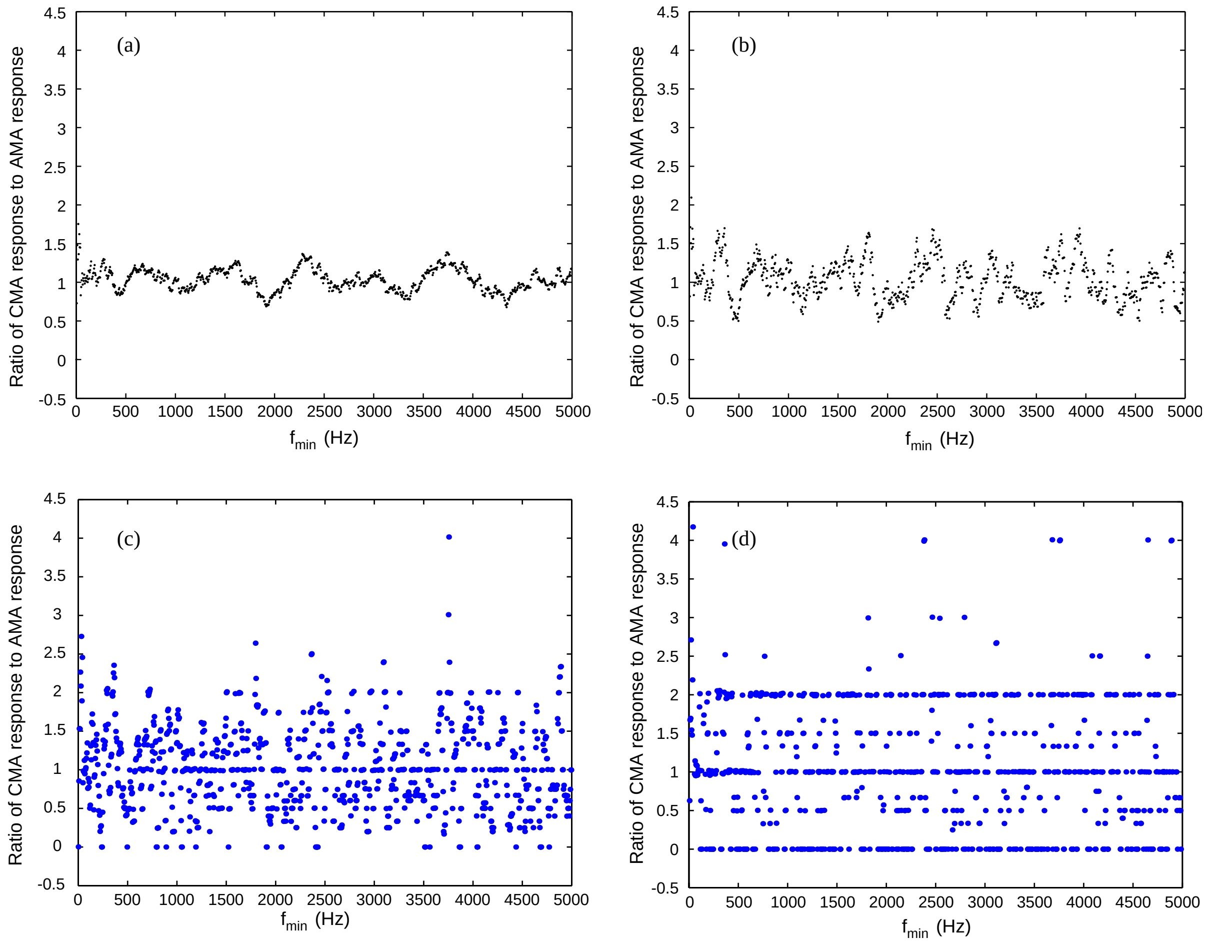

Fig. 11. Ratio of CMA spatial average response to AMA response versus minimum frequency for $x_f(1) = y_f(1) = 0.25$ and $x_f(2) = y_f(2) = 0.75$: (a) $\Delta f = 300$ Hz, (b) $\Delta f = 100$ Hz, (c) $\Delta f = 30$ Hz, and (d) $\Delta f = 10$ Hz.

3.2.1. Spatial average response

Results are shown in Fig. 11 for spatial average response with two point forces whose locations are $x_f(1) = y_f(1) = 0.25$ and $x_f(2) = y_f(2) = 0.75$. Comparing the results of Fig. 11 with those of Fig. 3, we see that the ratio approaches 1 in both cases. This is expected and similar to the theoretical prediction for a beam (see Ref. [3]).

3.2.2. Spatial distribution of response

The spatial response is shown in Fig. 12. Fig. 12(a) shows the ratio of CMA/AMA neglecting the cross-correlation between different modes while the result with cross-correlations included is shown in Fig. 12(b). It is noted that the response along the lines passing through the point of excitation and parallel to an edge is larger than the general response. Comparing these two results also indicates there is a measurable difference, again suggesting that the cross-correlation terms may have an important effect on the response of the plate. The ratio of CMA/AMA including the cross-correlation of only those modes having the same frequencies is shown in Fig. 13. Note the similarity between Figs. 12(b) and 13. The ratio of the results with all modal correlations included to those without correlations is shown in Fig. 14(a) and the ratio of the results with all modal correlations included to those with only correlations of the modes with the same natural frequencies is shown in Fig. 14(b). From Fig. 14(b) it is clear the ratio of two results are almost 1 everywhere. This example again verifies that modal cross-correlation is important in the calculation of the response of the system which has repeated natural frequencies.

Now as before consider another plate with dimension of $762 \times 762/\pi \times 0.794\ \text{mm}^3$. The response of the plate are shown in Fig. 15. Comparing two results in Fig. 15 we see that the ratio of these two results are almost 1 everywhere; see Fig. 16. Thus, the response of the plate with two forces again indicates that the modal cross-correlation terms are indeed negligible as the AMA limit is approached when the structure does not have multiple natural frequencies.

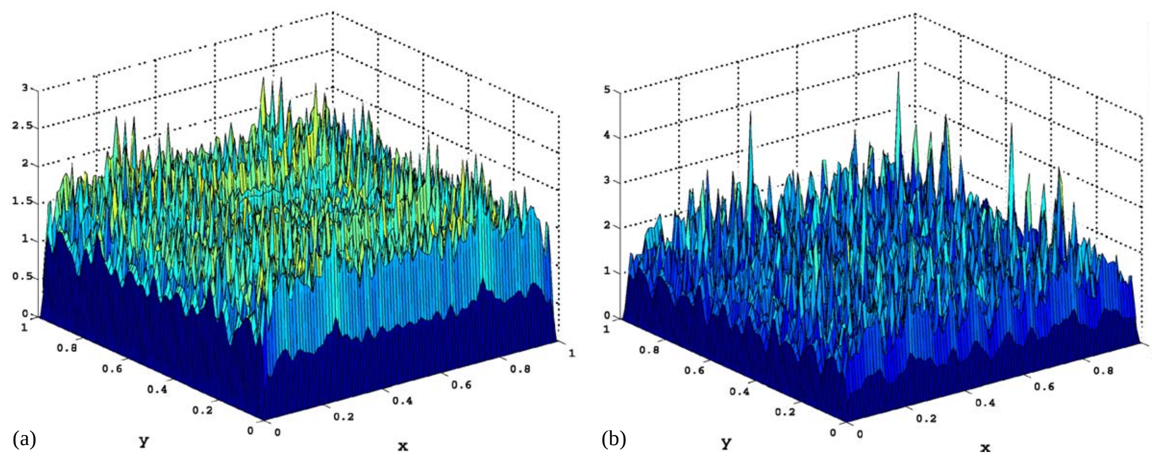

Fig. 12. Ratio of CMA response to AMA response on the plate for $x_f(1) = y_f(1) = 0.25$ and $x_f(2) = y_f(2) = 0.75$, $f_{\min} = 4995$ Hz and $f_{\max} = 5295$ Hz: (a) without cross-correlation, (b) with cross-correlation.

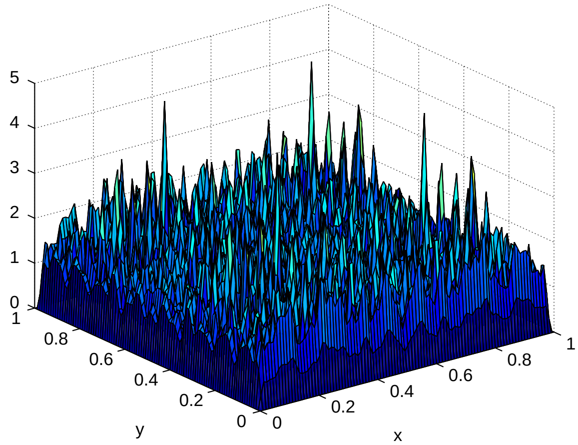

Fig. 13. CMA/AMA including the cross-correlations between the modes of the same frequency for the case: $x_f(1) = y_f(1) = 0.25$ and $x_f(2) = y_f(2) = 0.75$, $f_{\min} = 4995$ Hz and $f_{\max} = 5295$ Hz.

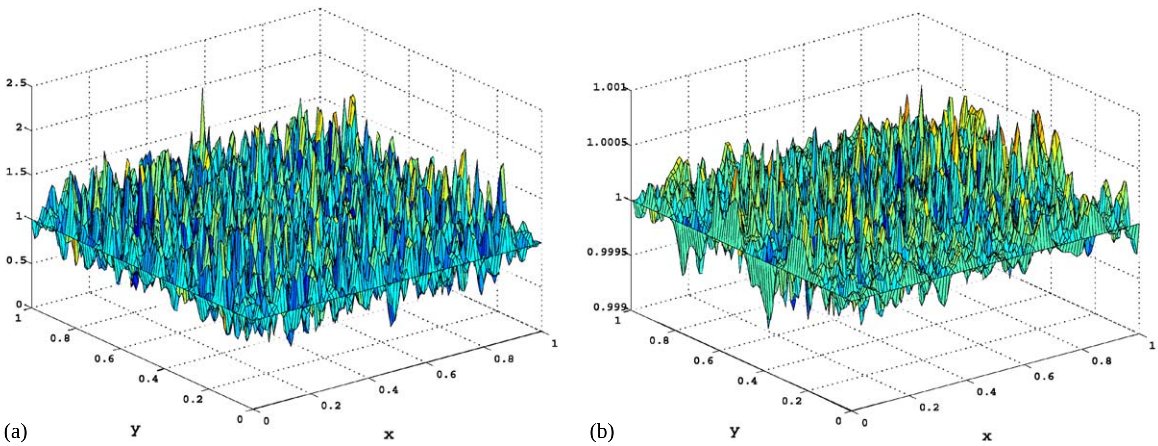

Fig. 14. Comparison of CMA/AMA for $x_f(1) = y_f(1) = 0.25$, $x_f(2) = y_f(2) = 0.75$, $f_{\min} = 4995$ Hz and $f_{\max} = 5295$ Hz: (a) ratio of CMA/AMA with all cross-correlations included to CMA/AMA without cross-correlations, (b) ratio of CMA/AMA with all cross-correlations included to CMA/AMA with cross-correlations of only modes with identical frequencies.

3.3. Part 2: The effect of the modal damping

In Part 1 the numerical results are obtained by choosing $\xi_1 = 0.01$. In this part of the paper we will increase the value of ξ_1 to see whether the modal damping will affect the response of the plate. The dimension of the plate is again selected as $762 \times 762/1.5 \times 0.794\ \mathrm{mm}^3$ and the excitation point is at $x_f = y_f = 0.5$.

Comparing Eqs. (7) and (8), it can be seen that the ratio of CMA/AMA without cross-correlation is independent of modal damping. This is also seen to be true when only cross-

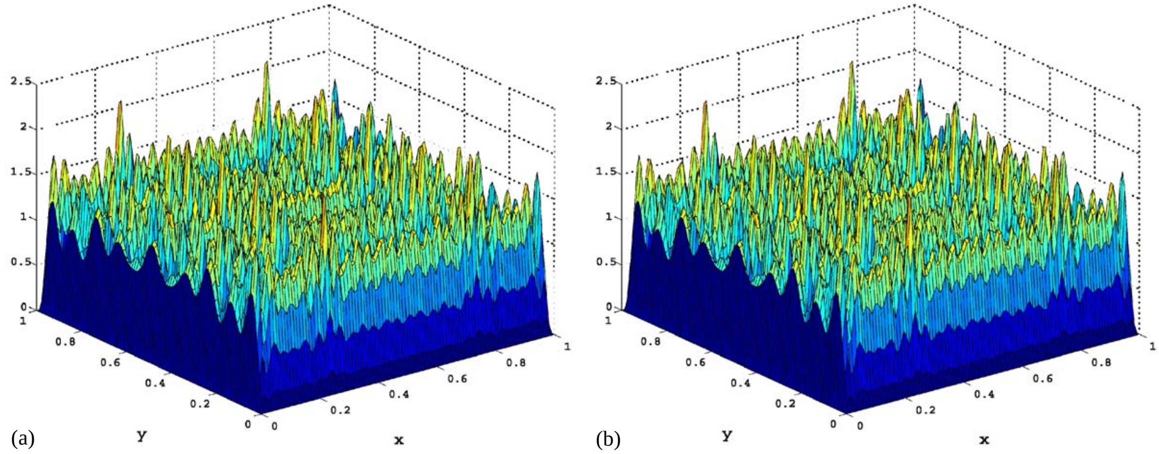

Fig. 15. CMA/AMA for the plate with dimension of $762 \times 762/\pi \times 0.794\ \mathrm{mm}^3, x_f(1) = y_f(1) = 0.25, x_f(2) = y_f(2) = 0.75$, $f_{\min} = 5500$ Hz and $f_{\max} = 6000$ Hz: (a) without cross-correlation, (b) with cross-correlation.

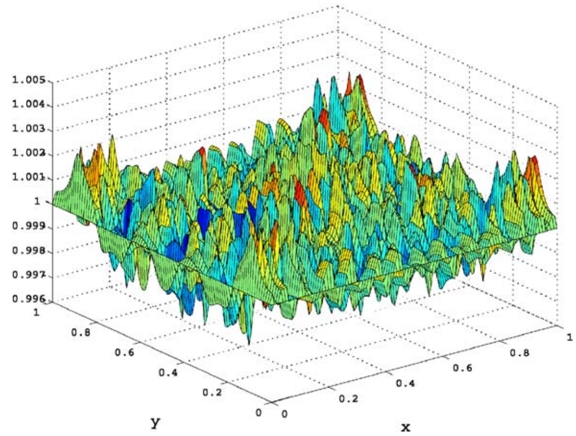

Fig. 16. The ratio of CMA with cross-correlations included to CMA without cross-correlation for the case: $x_f(1) = y_f(1) = 0.25, x_f(2) = y_f(2) = 0.75$, $f_{\min} = 5500$ Hz and $f_{\max} = 6000$ Hz.

correlations between modes of the same natural frequencies are included by considering Eqs. (6) and (8). Therefore, the results shown in Figs. 3, 5(a) and 6 are also the results for any value of ξ_1. The only difference for different ξ_1 is the result for the ratio of CMA/AMA with all cross-correlations included. The ratios of CMA/AMA with all correlations included for $\xi_1 = 0.1$ and 1.0 are shown in Fig. 17. However, note that for $\xi_1 = 0.1$, the result of Fig. 17(a) is similar to that of Fig. 5(b) and Fig. 6 for $\xi_1 = 0.01$, suggesting that it is only the cross-correlation of modes with identical frequencies that is important at this damping level. For $\xi_1 = 1.0$, however, cross-correlations of other modes appear to be important. Therefore, Figs. 17 and 5(b) indicate that the cross-correlation becomes more and more important as ξ_1 increases. However, the damping must be quite large, $\xi_1 \sim O(1)$, for this effect to be detectable.

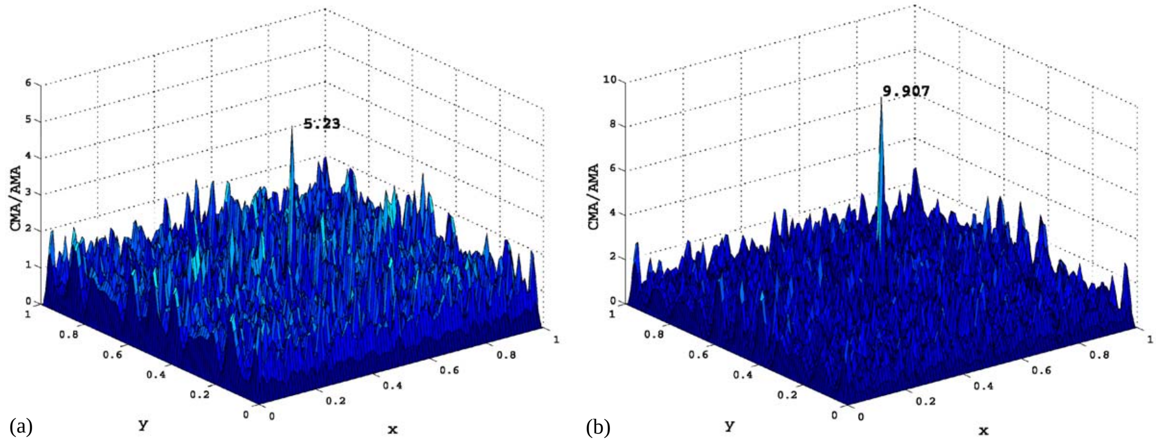

Fig. 17. CMA/AMA for the plate with dimension of $762 \times 762/1.5 \times 0.794\ \mathrm{mm}^3, x_f = y_f = 0.5$, $f_{\min} = 4995$ Hz and $f_{\max} = 5295$ Hz: (a) $\xi_1 = 0.1$ and (b) $\xi_1 = 1.0$.

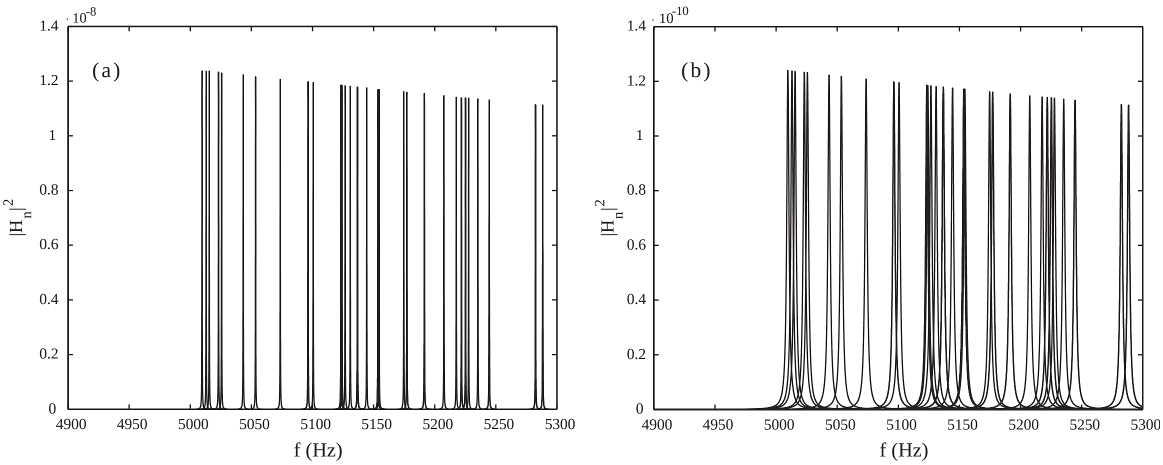

Fig. 18. Mean square of transfer functions versus frequency for $f_{\min} = 4995$ Hz and $f_{\max} = 5295$ Hz: (a) $\xi_1 = 0.01$ and (b) $\xi_1 = 0.1$.

To find the reason of this tendency, consider the transfer functions for all modes. The transfer functions are expressed by

$$H_n(\omega) = \frac{1}{M_n[\omega_n^2 - \omega^2 + 2i\xi_n\omega_n\omega]}. \tag{12}$$

The transfer functions vs. frequencies are shown in Figs. 18 and 19 in which only transfer functions with the natural frequency in the frequency interval are plotted. From these three figures it is obvious that the overlapping between different modes gets larger and larger as ξ_1 becomes larger. This explains the results shown in Fig. 17. That is to say, the cross-correlation between

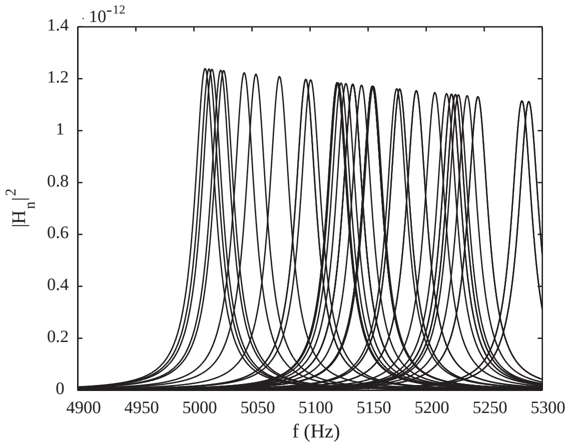

Fig. 19. Mean square of transfer functions versus frequency for $f_{\min} = 4995$ Hz, $f_{\max} = 5295$ Hz and $\xi_1 = 1.0$.

modes is important for very large modal damping but not for the small to moderate levels of damping typical of most structures.

4. Concluding remarks

Cross-correlation of modal responses is shown to be important for AMA only under special circumstances. It should be noted that the modal cross-correlation has no effect on the result for the spatial average response of uniform structures because of the orthogonality of the modes. In this paper AMA with modal cross-correlations neglected is shown to be a reasonable approximation when modal damping is small to moderate (and this includes the case of most structural elements) and the structure does not have multiple modes with the same natural frequencies. For systems which have multiple modes with the same natural frequencies, the cross-correlation of only those modes with identical natural frequencies must be included to get a better approximation. For very large damping the cross-correlations between other modes also appear to be important but such levels of damping are not usually found in most structures.

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Appendix A. Nomenclature

w	vertical displacement of the plate
ψ_n	the nth modal function of the plate
$\Phi_{F_iF_j}$	cross-correlation of the point forces, F_i and F_j
M_n	modal mass of the nth mode of plate: $M_n \equiv \int\int m(x,y)\psi_n^2\,\mathrm{d}x\,\mathrm{d}y$

M_p	total mass of the plate
ω_n	the nth natural frequency of plate
$f_{\min}$	the minimum frequency in a certain frequency interval
$f_{\max}$	the maximum frequency in a certain frequency interval
f_c, ω_c	center frequency: $f_c = \sqrt{f_{\min} * f_{\max}}$
ξ_n	the nth modal critical damping ratio of plate
ξ_c	critical modal damping ratio corresponding to the center frequency
n_x, n_y	modal numbers of plate in x-, y-directions, respectively
$H_n(\omega)$	transfer function of the nth mode
$\bar{w}^2$	mean square response (averaged over time)
$\langle \bar{w}^2 \rangle$	mean square average over (time and) space
x_f, y_f	location of point forces
x, y	location of structural response

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