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Letter to the Editor

Application of the Galerkin-FEM and the improved four-pole parameter method to predict acoustic performance of expansion chambers

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1. Introduction

The linear acoustic wave propagation in a stationary and inviscid medium is described by the well-known Helmholtz equation $\Delta p + \kappa^2 p = 0$, where p is the spatial distribution of a small perturbation of pressure around a steady state and $\kappa = \omega/c$ is a given physical parameter.

The numeric errors associated with the solution of the Helmholtz equation with Galerkin-FEM are related mostly to the parameter κ (κ-singularity) and to the geometry of the problem (λ-singularity).

Considering Ω the domain, h the element size, p the element order, $p \in H^{p+1}(\Omega)$, the exact solution to Helmholtz equation, and $p_h \in S_h^p(\Omega)$ the FEM-solution, the numerical relative error $|e_1|$ measured in the H_1-seminorm is bounded by $|e_1| \leqslant C_1(\kappa h/2p)^p + C_2\kappa(\kappa h/2p)^{2p}$ for oscillating p with frequency κ and a constraint $\kappa h < \pi$. However, numerical results show that $|e_1|$ estimates are sharp [1].

The first term of this error expression is the error due to approximation (interpolation error). This error is under control for constant resolution independent from κ and most acoustic finite element analysis are computed by keeping κh constant. This is called the "rule of the thumb" which determines the minimal mesh refinement of a wavelength and, for example, according to SYSNOISE users manual [2], five or six elements are sufficient for linear elements ($p = 1$).

The other term is the pollution error. The energy norm of the error for FEM-solutions of the Helmholtz equation with $\kappa h \leqslant \beta$ (constant) contains a pollution term, around $(\kappa h)^{2p}$. For reliability of the FEM results, it is hence necessary and sufficient to constrain the pollution error.

Another interesting question is the phase lag. Regular FEM-solutions with κ_h oscillating frequency display a phase difference in relation to the exact solution. The pollution term error

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order is about the same as the phase angle error which can be estimated by $|\kappa - \kappa_h| \leqslant \kappa C(p)(\kappa h/2p)^{2p}$ for $\kappa h < 1$; see Ref. [3].

The λ-singularity only appears on eigenfrequencies of interior acoustic undamped models, for which the system matrix is singular and the error is infinite.

For two-dimensional problems where the phase lag control is much more complex, the h-refinement is an interesting option used with some success; see Refs. [4,5], for example. According to Ihlemburg et al. [1], Galerkin FEM-solutions to two-dimensional Helmholtz equation show the same error behavior as one-dimensional solutions.

The transmission loss (TL) estimate for automotive mufflers are the main goal of this work. First, a one-dimensional problem is studied with two Neumann boundary conditions similar to the improved four-pole parameter method boundary conditions used for TL evaluations. The error analysis is performed for linear, quadratic and cubic elements. The errors for the finite element solutions are evaluated with the nodal values of the sound pressure and with the phase lag angle. Interesting results are observed for the phase lag errors related to even and odd resolutions (elements/wavelength).

The final application is performed with TL calculations for a muffler with inlet and outlet ducts in the frequency range from 0 to 3000 Hz. The numeric results obtained by the FEM show excellent proximity with the experimental results available in the literature and some differences close to λ-singularity are observed when compared to BEM-solutions. Finally, to check the reliability of the results around eigenfrequencies (λ-singularity) the reciprocity relation is used as a FEM-solution quality parameter.

2. One-dimensional problem

The local error analysis is performed for linear ($p = 1$), quadratic ($p = 2$) and cubic ($p = 3$) finite elements using the Galerkin FEM-solutions for the one-dimensional problem

$$\frac{d^2 p(x)}{dx^2} + \kappa^2 p(x) = 0 \quad \forall 0 \leqslant x \leqslant L \tag{1}$$

with boundary conditions (Neumann)

$$\frac{dp(0)}{dx} = 1, \qquad \frac{dp(L)}{dx} = 0, \tag{2}$$

whose exact solution, $p(x) = (1/\kappa)[\sin(\kappa x) + \cos(\kappa L)/\sin(\kappa L) \times \cos(\kappa x)]$, has λ-singularity in $\kappa L = n\pi$, $n = 1, 2, \ldots$. It is important to remember that the boundary conditions are the same used in the improved four-pole parameter method of Kim and Soedel [6–8]; Wu et al. [9], and the eigenfrequencies depend on the L length.

The finite element approximation for $p_h(0)$ is one of the principal variables to determine the constants of the improved four-pole parameters method. Fig. 1 shows the nodal error, $|e| = |1 - p_h(0)/p(0)|$, for $\kappa = 60$ and two different lengths, $L_1 = 9.99999/\kappa$ (near λ-singularity) and $L_2 = 1.12/\kappa$ (without λ-singularity). It can be noticed that for length L_2 the asymptotic behavior of error happens at around rate $2p$ (2.027, 3.937 and 5.948 for $p = 1$, 2 and 3, respectively) and that for length L_1 the same convergence rate is observed (3.912 and 6.012 for $p = 2$ and 3,

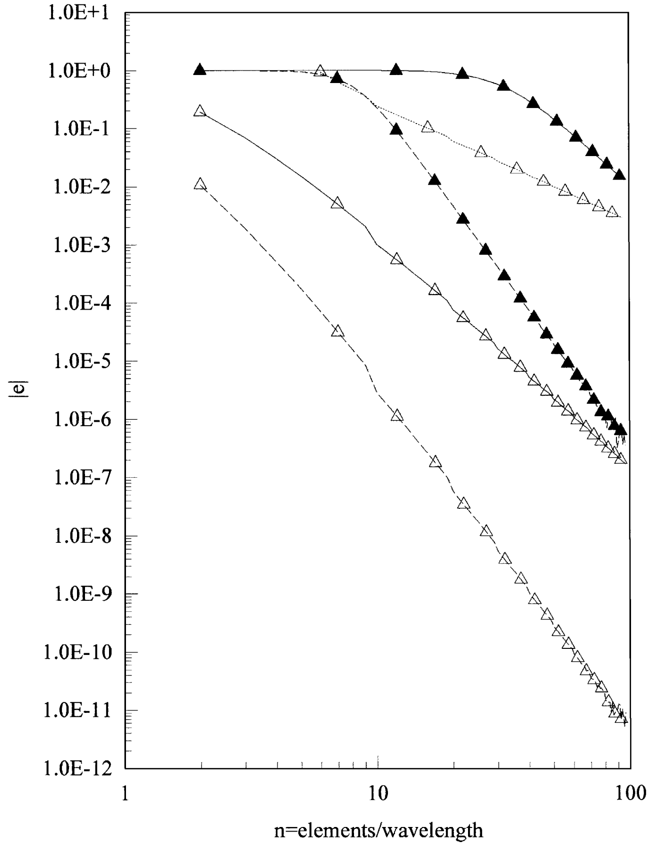

Fig. 1. Convergence for $\kappa = 60$, $|e| = |1 - p_h(0)/p(0)|$: - - -, $p = 1$; —, $p = 2$; – –, $p = 3$; ▲, L_1; △, L_2.

respectively); however, the asymptotic convergence only occurs for elevated n (elements/wavelength) values.

Other interesting result is the phase lag that is associated with the pollution error. The numeric results for the phase angle error, $|e| = 100|\kappa - \kappa_h|/\kappa$, are shown in Fig. 2 for $\kappa = 60$ and $L = L_1$. All the solutions with n odd or even linear elements present the same convergence rate of about $2p$ (1.987 for odd and 1.991 for even). For the quadratic and cubic elements this behavior is different and the FEM-solutions convergence rates are $2p$ for even n (4.001 and 6.012 for $p = 2$ and 3, respectively) and $2p - 1$ for odd n (3.001 and 4.981 for $p = 2$ and 3, respectively).

3. Bi-dimensional application

In this application the FEM is applied to obtain the TL, for the expansion chamber with extended inlet/outlet ducts illustrated in Fig. 3. The improved four-pole parameter method is used to obtain the TL and in this method the pressures and velocities in the muffler inlet, (p_1, u_1), and

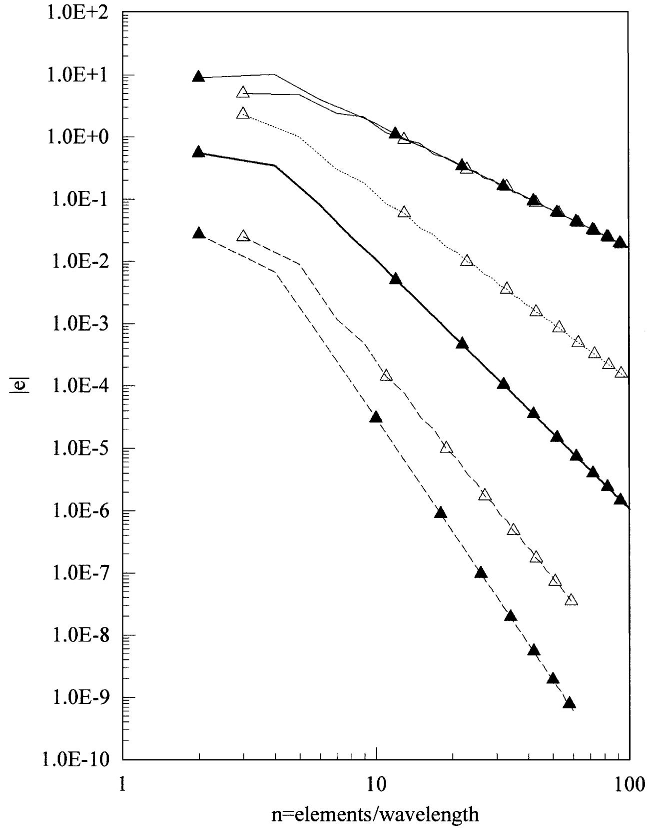

Fig. 2. Phase angle error near to λ-singularity, $|e| = 100|\kappa - \kappa_h|/\kappa$: —, $p = 1$; - - -, $p = 2$; — —, $p = 3$; ▲, n even; △, n odd.

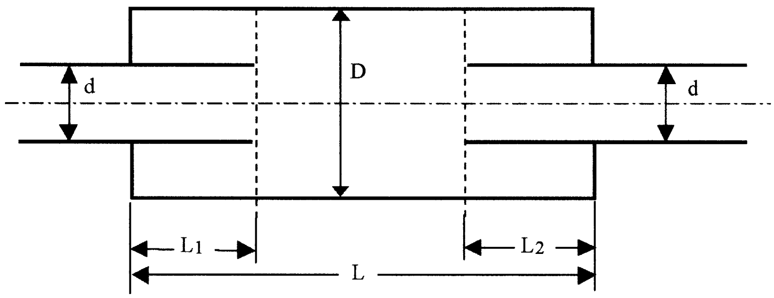

Fig. 3. Expansion chamber with extended inlet/outlet ducts.

muffler outlet, (p_2, u_2), can be related by the equation

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix} \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}, \tag{3}$$

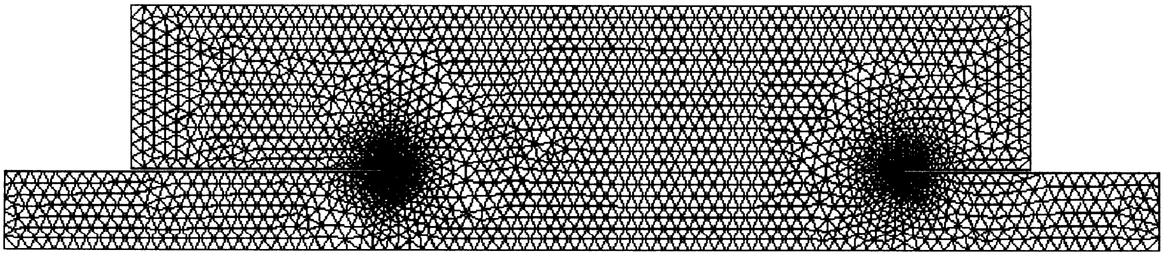

Fig. 4. Finite element mesh for expansion chamber with extended inlet/outlet ducts.

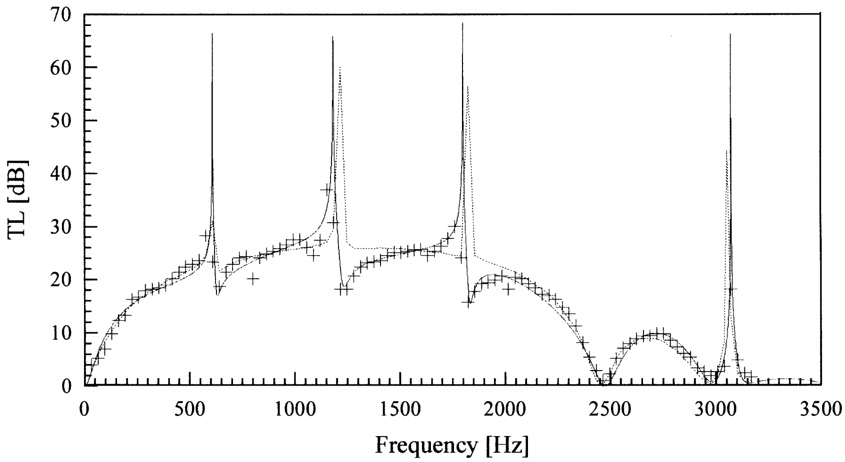

Fig. 5. Comparative results for the TL of a concentric expansion chamber with extended inlet/outlet: —, GFEM; - - -, BEM [10]; +, experimental [10].

where A^*, B^*, C^* and D^* are the improved four-pole parameters associated with the well-known four-pole parameters (four-pole parameters method) by $A = A^*/C^*$, $B = B^* - A^*D^*/C^*$, $C = 1/C^*$ and $D = -D^*/C^*$.

The main advantages of using this method to evaluate TL with FEM are based on the fact that all the calculations are performed with real variables, the final system of equation is just solved once and the constants A^*, B^*, C^* and D^* are obtained without velocity calculations (FEM-solutions post-processing).

The geometrical data for this muffler are the same as the ones used by Selamet and Ji [10]: $L = 28.23$ cm, $d = 4.86$ cm, $D = 15.32$ cm, $L_1 = 13.1$ cm, $L_2 = 6.1$ cm and wall thickness, $t = 0.2$ cm. The physical parameter, sound speed, used in numerical evaluations is 346.1 m/s.

The expansion chamber is modelled using 5352 (11,073 nodes) quadratic axisymmetric triangular finite elements and the mesh refinements are shown in Fig. 4. The coarse size h for the finite element mesh is around 0.4 cm (almost 29 elements per wavelength at approximately 3000 Hz) and some calculated errors are expected to be around the λ-singularity, see Fig. 1.

In Fig. 5 the finite elements results are compared to the data obtained experimentally and numerically (BEM) by Selamet and Ji [10]. It is easy to visualize the differences obtained by the

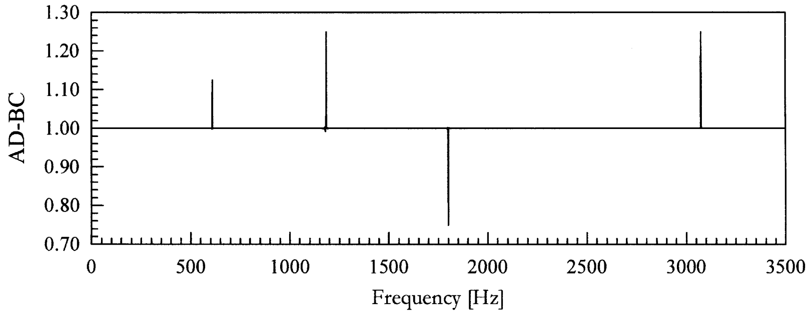

Fig. 6. Reciprocity relation *AD-BC*.

numerical solutions using different methods (BEM and FEM). Around the λ-singularity the TL values present variations near to 10 dB and some phase differences.

According to Selamet and Ji [10] the minor discrepancies between BEM-solution and experimental results are associated with the neglected viscous effects and the wall thickness of extended ducts in the numerical model and minor geometrical imperfections in the experimental setup. However, the numerical solution obtained by using only the Helmholtz equation and the Galerkin-FEM are very close to the experimental results presented by Selamet and Ji [10].

Comparison with more details of these methods (FEM and BEM) for muffler TL calculations based on accuracy and computational processing time was performed recently by Bilawchuk and Fyfe [11]. Using numerical solutions obtained by SYSNOYSE [2] and two other methods for TL evaluations (the four-pole method and the three-point method) the conclusion is that the FEM is better suited for this kind of application.

Finally, as seen previously for one-dimensional problems, using only 30 quadratic elements per wavelength around the λ-singularity, some perturbations are expected in the FEM-solutions due to the singularity in the equation system. In these situations the reciprocity relation, *AD-BC*, is used as a parameter to evaluate the FEM-solutions quality. For all frequencies, the *AD-BC* value must remain constant and equal to 1; see Ref. [12]. This is verified by the results of Fig. 6, except near λ-singularities.

4. Conclusion

To evaluate the transmission loss, TL, numerically in acoustic mufflers by using the improved four-pole parameter method, it is necessary and sufficient to use the correct determination of sound pressure values at the muffler's inlet and outlet.

It was shown in convergence analyses for one-dimensional problems solved by Galerkin-FEM, that the determination of the element size has a fundamental importance for the study of problems governed by Helmholtz's equation. These results showed that the control of pollution errors lead to reliable finite elements models. Near the λ-singularity the convergence rate of the phase angle remains similar to $2p$ for linear element. However, for quadratic and cubic elements the numeric results show a different convergence rate. For those elements the convergence rate found is about $2p$ for even resolution and $2p-1$ for odd resolution.

Finally, the two-dimensional numerical results show excellent proximity to available experimental results found in literature. Adequate geometric models with correct thickness wall modelling and mesh refinements near the changes in geometry are fundamental for those muffler simulations. The reciprocity relation can be used as an FEM-solutions quality parameter, even when next to the λ-singularities.

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