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Letter to the Editor

A note on using the collocation method for modelling the dynamics of a flexible continuous beam subject to impacts

D.J. Wagg*

Department of Mechanical Engineering, University of Bristol, Queens Building, University Walk, Bristol BS8 1TR, UK

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1. Introduction

The use of non-smooth modelling techniques to model the dynamics of a flexible impacting beam has recently been reported in Ref. [1]. The method used was based on taking a Galerkin approximation [2] of the partial differential equation (PDE) governing the dynamics of the beam away from impact, and coupling this to a non-smooth coefficient of restitution rule to model the impact [3]. In this letter, the advantages and limitations of using a collocation method instead of the Galerkin method combined with a non-smooth impact law are discussed.

The example of a flexible beam subject to a motion limiting constraint is used, similar to that discussed in Ref. [1].

The general problem of a cantilever beam impacting against an impact stop has been considered by several authors—see for example Refs. [4–7]. The collocation approach has been used for modelling a variety of engineering problems—see for example Refs. [8–11]. In this example, collocation has the advantage that unlike the Galerkin method there is no requirement to integrate the mode shape over the domain of interest in order to decouple the system modal equations. This means that (in general) the collocation method can be applied to a larger range of problems, particularly those with more complex geometry. There is a further advantage in that the Galerkin approach [1] requires the exact solution for the modal equations between impact, whereas with this collocation method a numerical integration routine is used. However, it is noted that in general it is not necessary to use exact solutions for the trial functions when applying the Galerkin method.

For piecewise-linear systems, Wang and Wang [12] describe a collocation method for simulating periodic responses. The use of collocation methods for modelling periodic motions in constrained multi-body systems has also been considered by Franke and Führer [13]. In the approach described here there is no a priori requirement for periodicity.

*Fax: 0117-9294423.
E-mail address: david.wagg@bristol.ac.uk (D.J. Wagg).

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2. Mathematical model

The system considered is a clamped cantilever beam with a motion limiting constraint on one side which is shown schematically in Fig. 1. The stop is positioned at a distance B from the base along the beam, and with an initial transverse distance a from the beam which is harmonically forced at its base. The transverse vibration of the centre line of the beam is denoted by $u(x,t)$, where x is the length along the beam from the base and t is time. Away from the impact constraint, the beam is assumed to be governed by the Euler–Bernoulli equation with damping and external forcing

$$EI\frac{\partial^4 u}{\partial x^4}+\eta\frac{\partial u}{\partial t}+\rho A\frac{\partial^2 u}{\partial t^2}=f(x,t), \quad u<a, \tag{1}$$

where E is the Young's modulus, ρ density, A cross-sectional area, η the damping constant and I the second moment of area for the beam of length L.

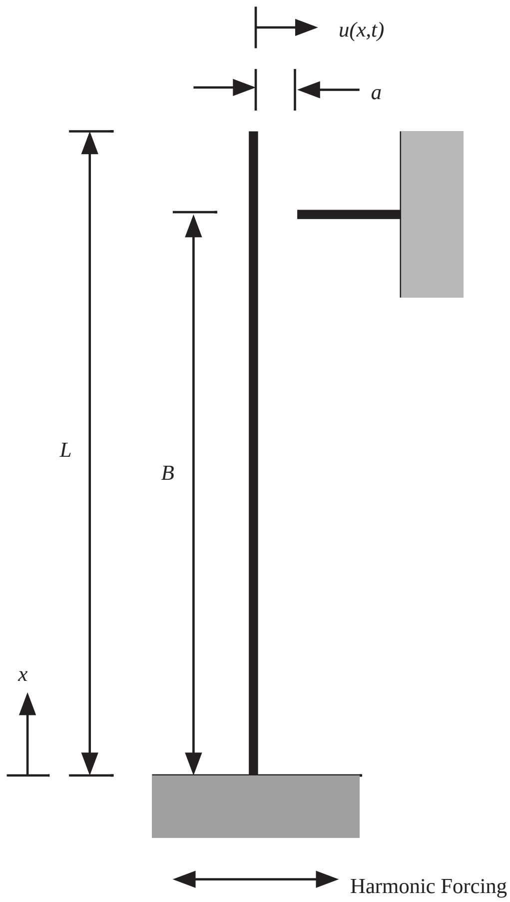

Fig. 1. Schematic representation of the continuous vibro-impact cantilever beam system.

When an impact occurs, $u(B,t)=a$ and a coefficient of restitution rule of the form

$$\dot{u}(B,t_+) = -r\dot{u}(B,t_-), \quad u(B,t_-)=a, \tag{2}$$

is applied, where t_- is the time just before impact, t_+ is the time just after impact and $r \in [0,1]$ is the coefficient of restitution. It is assumed that the velocities are normal to the beam centre line, and that the tangential velocity component at impact is negligible. Eq. (2) is applied instantaneously such that $t_- = t_+$, and a non-smooth discontinuity in velocity occurs at impact.

However, for a continuous structural element, such as a beam, the velocity is a continuous function of beam length. Thus, in order to apply the non-smooth impact condition, Eq. (2), at $u=a$, the velocity components for the non-impacting part of the beam $x \neq B$ remain unaffected such that

$$\dot{u}(x\neq B,t_+) = \dot{u}(x\neq B,t_-), \quad u(B,t_-)=a \tag{3}$$

applies. The combination of Eq. (2) and (3) are essentially a non-smooth representation of the physical impact process for the beam. In the physical beam system, the contact time will be finite (though small for materials with high stiffness) and the velocity reversal will propagate outwards from the point of impact, a process which is captured with this type of model.

It is now assumed that there is a series solution to the Euler–Bernoulli equation given by

$$u(x,t) = \sum_{j=1}^{\infty} \phi_j(x) q_j(t), \tag{4}$$

where $\phi_j(s)$ are the normal mode shapes of the beam, and $q_j(t)$ are the modal co-ordinates [14]. Then substituting Eq. (4), into the Euler–Bernoulli equation (1) gives

$$\sum_{j=1}^{N} \left(\phi_j \ddot{q}_j(t) + \beta \phi_j \dot{q}_j(t) + \alpha \phi_j'''' q_j(t) \right) = \gamma f(x,t), \quad j=1,2,3,\ldots,N, \tag{5}$$

where $()'$ represents differentiation with respect to x, an overdot differentiation with respect to t, $\alpha = EI/\rho A$, $\beta = \eta/\rho A$ and $\gamma = 1/\rho A$. As the normal linear beam modes are being used for this example, the standard relationship that $\phi_j'''' = \xi_j^4 \phi_j$, where

$$\xi_j^4 = \omega_{nj}^2 \frac{\rho A L^4}{EI} \tag{6}$$

and ω_{nj} is the jth natural frequency [15] will be used. In the case, when this does not hold, collocation can still be applied providing the fourth derivative of the shape function ϕ_j can be computed for each collocation point. Substituting Eq. (6) into Eq. (5) gives

$$\sum_{j=1}^{N} \left(\phi_j \ddot{q}_j(t) + \beta \phi_j \dot{q}_j(t) + \alpha \xi_j^4 \phi_j q_j(t) \right) = \gamma f(x,t), \quad j=1,2,3,\ldots,N. \tag{7}$$

N collocation points $x_1, x_2, \ldots, x_N$ are now chosen along the length of the beam. Collocation points are usually chosen at evenly spaced intervals, and a key requirement for this method is that the point of contact, $x=B$, is at a collocation point. Now for the N discrete collocation points Eq. (7) can be represented in a matrix form

$$\mathbf{\Phi}\ddot{\mathbf{q}} + \beta \mathbf{\Phi}\dot{\mathbf{q}} + \alpha \mathbf{\Phi}\hat{\boldsymbol{\xi}}\mathbf{q} = \gamma \mathbf{F}, \tag{8}$$

where

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \dots & \phi_N(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \dots & \phi_N(x_2) \\ \vdots & \vdots & \dots & \vdots \\ \phi_1(x_N) & \phi_2(x_N) & \dots & \phi_N(x_N) \end{bmatrix}, \tag{9}$$

$\mathbf{q} = [q_1, q_2, \dots, q_N]^{\mathrm{T}}$, $\hat{\boldsymbol{\xi}} = diag\{\xi_1^4, \xi_2^4, \dots, \xi_N^4\}$ and $\mathbf{F} = [\,f(x_1, t), f(x_2, t), \dots, f(x_N, t)]^{\mathrm{T}}$. Multiplying Eq. (8) by $\mathbf{\Phi}^{-1}$ and putting it into first order form gives

$$\dot{\mathbf{z}} = \mathbf{H}\mathbf{z} + \hat{\mathbf{F}}, \tag{10}$$

where $\mathbf{z} = [q, \dot{q}]^{\mathrm{T}}$, $\hat{\mathbf{F}} = [0_{\mathrm{N}}, \gamma\mathbf{\Phi}^{-1}\mathbf{F}]^{\mathrm{T}}$ and

$$\mathbf{H} = \begin{bmatrix} 0_{\mathrm{N}} & \mathbf{I}_{\mathrm{N}} \\ -\alpha\hat{\boldsymbol{\xi}} & -\beta\mathbf{I}_{\mathrm{N}} \end{bmatrix}. \tag{11}$$

Eq. (10) can now be integrated forward in time from a set of initial conditions using a suitable time-stepping method—in this case a fourth order Runge–Kutta method [16] is used.

To apply the non-smooth impact condition, a coefficient of restitution matrix, $\mathbf{R}$ is defined using Eq. (2) and (3). Eq. (2) applies to the collocation point where impact occurs, $x = B$, and Eq. (3) applies to all other collocation points. For example, for a choice of N collocation points with the impact at point N (the beam tip) the coefficient of restitution matrix is

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -r \end{bmatrix}. \tag{12}$$

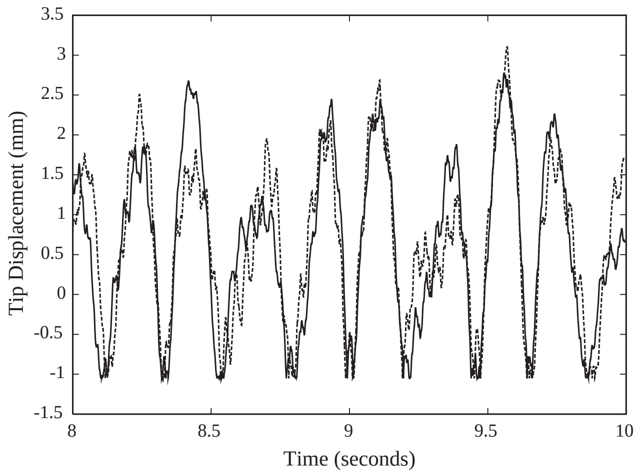

Fig. 2. Impacting beam simulation; parameter values $a = -1.05$, $N = 4$, $P = 0.0006$, $\Omega = 28.3$, $\eta = 0.005$, $r = 0.8$. Solid line, collocation; dashed line, Galerkin.

At each time step the condition for the beam having an impact, $u(B) > a$, is checked. Once an impact is detected a root finding method is used to find the exact time at which $u(B) = a$. Then the modal velocities are updated according to the matrix coefficient of restitution rule [1]

$$\dot{\mathbf{q}}(t_+) = [\mathbf{\Phi}]^{-1}[\mathbf{R}][\mathbf{\Phi}]\dot{\mathbf{q}}(t_-) \tag{13}$$

and time stepping begins again.

3. Example: four mode model of a cantilever beam

As an example a cantilever beam which has dimensions length 300 mm width 25.5 mm and thickness 0.49 mm is considered. The properties of the beam are taken as the following parameter values: Young's modulus $E = 205 \times 10^9\ \mathrm{N/m^2}$, second moment of area $I = 24.4 \times 10^{-14}\ \mathrm{m^4}$,

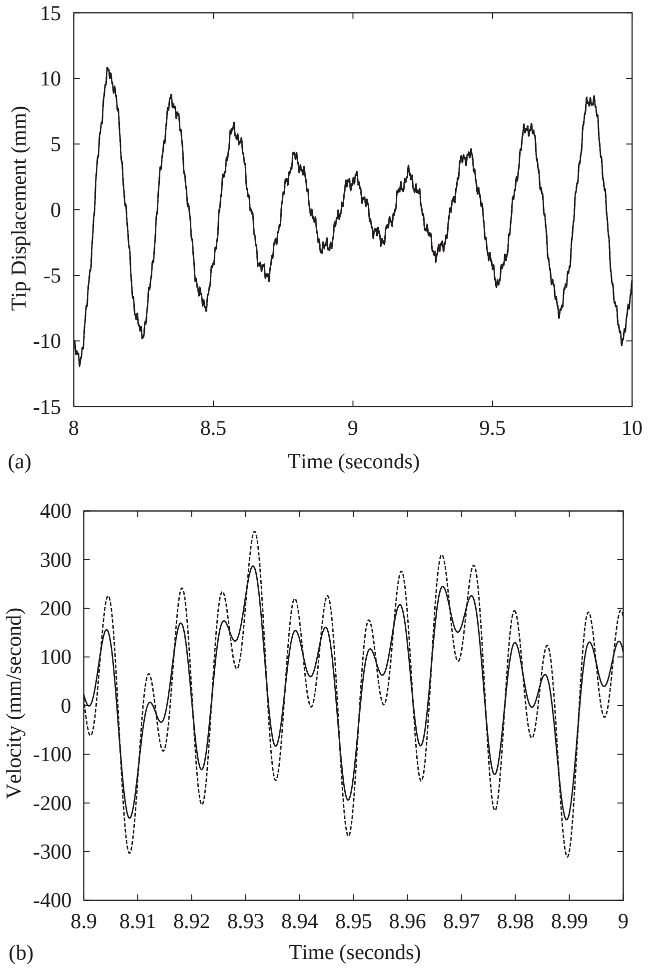

Fig. 3. Non-impacting beam simulation; parameter values $N = 4$, $P = 0.0006$, $\Omega = 28.3$, $\eta = 0.005$. Solid line, collocation; dashed line, Galerkin; (a) displacement, (b) velocity.

density $\rho = 8500\ \mathrm{kg/m^3}$, cross-sectional area $A = 12.4 \times 10^{-6}\ \mathrm{m^2}$, damping constant $\eta = 0.005\ \mathrm{Ns/m}$ and length $L = 0.3$ m. In this example, $N = 4$ is selected and the initial conditions are chosen such that all displacements and velocities of the beam are zero at time $t = 0$. The forcing function is assumed to be separable into space and time-dependant functions such that $f(x,t) = g(x)h(t)$, where for this example $h(t) = P\cos(\Omega t)$, $P = 0.0006$ m and $\Omega = 28.3$ rad/s. Evaluating the forcing functions at the collocation points gives $F = [g(x_1), g(x_2), \ldots, g(x_N)]^{\mathrm{T}} h(t)$ and for this example $g(x_i) = 1$ for $i = 1, \ldots, N$, and it is assumed that the impact occurs at the beam tip $B = L$. Then 10 s of vibro-impact motion is simulated and the last 2 s plotted, which is shown in Fig. 2.

In Fig. 2, the solid line represents the time series simulation computed using the collocation method described in Section 2. For a comparison the non-smooth-Galerkin method used in Ref. [1] is plotted as a dashed line. The two methods give qualitatively similar responses in that the maximum amplitudes and times of impacts are similar. The periodicity of the response is clearly shown by both simulations.

However, there are considerable differences between the two simulation results. This is demonstrated more clearly when exactly the same simulation without impacts is plotted—Fig. 3. In Fig. 3(a), showing displacement, the solid line and dashed line are indistinguishable, but in Fig. 3(b), showing a short section of the velocity signal, there are significant differences between the simulations. As a result when an impact occurs the non-smooth jump in velocity causes the post impact behaviour of the two simulations to differ slightly. As more impacts occur this initially small difference is compounded and the higher frequency behaviour of the two approaches diverge—demonstrated in Fig. 4. Defining the parameter regimes where reasonable quantitative agreement between the two methods occurs is an area of future study.

It is worth reiterating at this point that the non-smooth Galerkin method described in Ref. [1] uses the exact solutions of the decomposed normal mode equations between impacts, and requires integration of the normal mode shapes across the length of the beam. In principle, the collocation approach can be applied with neither of these requirements, and can therefore be applied to a

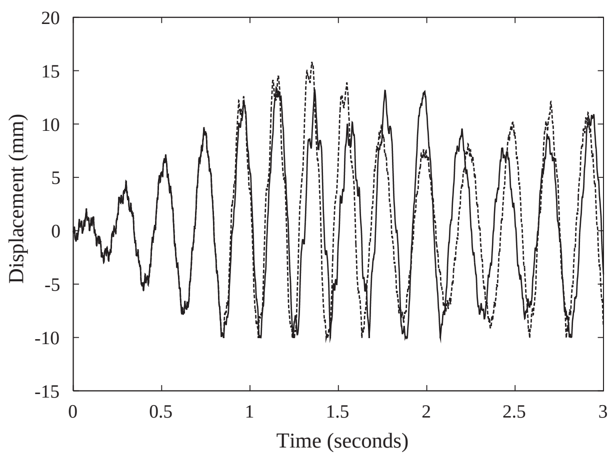

Fig. 4. Impacting beam simulation; parameter values $a = -10.05$, $N = 4$, $P = 0.0006$, $\Omega = 28.3$, $\eta = 0.005$, $r = 0.8$. Solid line, collocation; dashed line, Galerkin.

wider range of problems. The trade off is that there is a cumulative reduction in accuracy for the high frequency part of the simulation. However, for the examples considered here the qualitative behaviour of the system is still captured by the collocation method.

References

[1] D.J. Wagg, S.R. Bishop, Application of non-smooth modelling techniques to the dynamics of a flexible impacting beam, *Journal of Sound and Vibration* 256 (5) (2002) 803–820.
[2] C.A.J. Fletcher, *Computational Galerkin Methods*, Springer, New York, 1984.
[3] S.R. Bishop, M.G. Thompson, S. Foale, Prediction of period-1 impacts in a driven beam, *Proceedings of the Royal Society of London A* 452 (1996) 2579–2592.
[4] J.L. Escalona, J. Valverde, J. Mayo, J. Domingez, Reference motion in deformable bodies under rigid body motion and vibration. Part ii: evaluation of the coefficient of restitution for impacts, *Journal of Sound and Vibration* 264 (5) (2003) 1057–1072.
[5] I. Svensson, Dynamic response of a constrained axially loaded beam, *Journal of Sound and Vibration* 252 (4) (2002) 739–749.
[6] A. Fathi, N. Popplewell, Improved approximations for a beam impacting a stop, *Journal of Sound and Vibration* 170 (3) (1994) 365–375.
[7] J. Wang, J. Kim, New analysis method for a thin beam impacting against a stop based on the full continuous model, *Journal of Sound and Vibration* 191 (5) (1996) 809–823.
[8] J. Kouatchou, Comparison of time and spatial collocation methods for the heat equation, *Journal of Computational and Applied Mathematics* 150 (2003) 129–141.
[9] K.-Y. Lee, A.A. Renshaw, Stability analysis of parametrically excited systems using spectral collocation, *Journal of Sound and Vibration* 258 (4) (2002) 725–739.
[10] R. Baltensperger, J.-P. Berrut, The linear rational collocation method, *Journal of Computational and Applied Mathematics* 134 (2001) 243–258.
[11] B. Li, A Crank–Nicolson orthogonal spline collocation method for vibration problems, *Applied Numerical Mathematics* 33 (2000) 299–306.
[12] Y. Wang, Z. Wang, Periodic response of piecewise-linear oscillators using trigonometric collocation, *Journal of Sound and Vibration* 177 (4) (1994) 573–576.
[13] C. Franke, C. Führer, Collocation methods for the investigation of periodic motions of constrained multibody systems, *Multibody System Dynamics* 5 (2001) 133–158.
[14] R. Vichnevetsky, *Computer Methods for Partial Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1981.
[15] R.D. Blevins, *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, New York, 1979.
[16] W.H. Press, S.A. Teukolsky, W.T. Vettering, B.P. Flannery, *Numerical Recipes in C*, 2nd Edition, Cambridge University Press, Cambridge, 1994.