Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 276 (2004) 1145–1149
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the editor

A modified method of equivalent linearization that works even when the non-linearity is not small

H. Hu

Department of Mathematics and Physics, Xiangtan Polytechnic University, Xiangtan 411201, Hunan, People's Republic of China

Received 23 December 2002; accepted 14 November 2003

1. Introduction

There are various perturbation techniques for constructing analytical approximations to the oscillatory solutions of second order, non-linear differential equations. But many of them apply to weekly non-linear cases only. To overcome the limitations, many novel techniques have been proposed in recent years. For example, He [1] proposed a perturbation technique which is valid for large parameters, and Lim et al. [2] presented a modified Mickens procedures for certain non-linear oscillators.

Consider a non-linear oscillator modelled by the equation

$$x'' + f(x) = 0, \quad x(0) = A, \quad x'(0) = 0, \tag{1}$$

where A is a given positive constant and $f(x)$ satisfies the condition

$$f(-x) = -f(x) \tag{2}$$

and its derivative near $x = 0$ is non-negative. The system will oscillate between symmetric limits $[-A, A]$. The starting point in Refs. [1,2] is to rewrite Eq. (1) in the form

$$x'' + \omega^2 x = \omega^2 x - f(x) := g(x), \tag{3}$$

where ω is a priori unknown frequency of the periodic solution $x(t)$ being sought. A similar technique was used in Refs. [3,4].

This research was motivated by the method used in Refs. [1–4]. The purpose of this paper is to propose a modified method of equivalent linearization by using Eq. (3), which works even when the non-linearity is not small.

E-mail address: huihuxt@yahoo.com.cn (H. Hu).

doi:10.1016/j.jsv.2003.11.006

2. Classical method of equivalent linearization

Consider the non-linear differential equation

$$x'' + \omega_0^2 x = \varepsilon f(x) = g(x), \quad x(0) = A, \quad x'(0) = 0, \tag{4}$$

where ε is a small positive parameter. In the first approximation, the solution takes the form

$$x = A\cos\omega t, \tag{5}$$

where according to the method of equivalent linearization [5]

$$\omega(A) = \omega_0 - \frac{1}{2\pi A}\int_0^{2\pi} g(A\cos\varphi)\,\cos\varphi\,\mathrm{d}\varphi. \tag{6}$$

3. Basic ideas of the modified method

We assume that the equivalent linear equation corresponding to Eq. (1) is

$$x'' + \omega^2 x = 0. \tag{7}$$

Then, $g(x) = \omega^2 x - f(x)$ in Eq. (3) is probably "small". Comparing Eq. (3) with Eq. (4), we obtain from Eq. (6)

$$\omega = \omega - \frac{1}{2\pi A}\int_0^{2\pi} [\omega^2 A\cos\varphi - f(A\cos\varphi)]\cos\varphi\,\mathrm{d}\varphi$$

or

$$\omega = \sqrt{\frac{1}{\pi A}\int_0^{2\pi} f(A\cos\varphi)\cos\varphi\,\mathrm{d}\varphi}. \tag{8}$$

This formula is the main result of this paper.

4. Examples

Example 1. Consider the well-known Duffing equation

$$x'' + x + \varepsilon x^3 = 0, \quad x(0) = A, \quad x'(0) = 0. \tag{9}$$

Substituting $f(x) = x + \varepsilon x^3$ into Eq. (8) yields

$$\omega = \sqrt{1 + \frac{3\varepsilon A^2}{4}}. \tag{10}$$

Therefore, the first approximation is

$$x = A\cos\left(\sqrt{1 + \frac{3\varepsilon A^2}{4}}t\right). \tag{11}$$

Example 2. The second example is the non-linear differential equation [5]

$$x'' + |x|x = 0, \quad x(0) = A, \quad x'(0) = 0. \tag{12}$$

In this case,

$$\begin{aligned}\int_0^{2\pi} f(A\cos\varphi)\cos\varphi\,\mathrm{d}\varphi &= A^2\int_0^{2\pi}\cos^2\varphi|\cos\varphi|\,\mathrm{d}\varphi\\ &= A^2\left[\int_0^{\pi/2}\cos^3\varphi\,\mathrm{d}\varphi - \int_{\pi/2}^{3\pi/2}\cos^3\varphi\,\mathrm{d}\varphi + \int_{3\pi/2}^{2\pi}\cos^3\varphi\,\mathrm{d}\varphi\right]\\ &= \frac{8A^2}{3}.\end{aligned} \tag{13}$$

Then, we have

$$\omega = \sqrt{\frac{8A}{3\pi}} \tag{14}$$

and

$$x = \cos\left(\sqrt{\frac{8A}{3\pi}}t\right). \tag{15}$$

Eqs. (10) and (14) are in agreement with the results obtained by using the method of harmonic balance [5].

Example 3. We now consider the non-linear differential equation [6]

$$x'' + x^{1/3} = 0, \quad x(0) = A, \quad x'(0) = 0. \tag{16}$$

In this situation,

$$\int_0^{2\pi} f(A\cos\varphi)\cos\varphi\,\mathrm{d}\varphi = \int_0^{2\pi} A^{1/3}\cos^{4/3}\varphi\,\mathrm{d}\varphi = 4A^{1/3}\int_0^{\pi/2}\cos^{4/3}\varphi\,\mathrm{d}\varphi.$$

Using the relation [7]

$$\int_0^{\pi/2}\cos^n x\,\mathrm{d}x = \int_0^{\pi/2}\sin^n x\,\mathrm{d}x = \frac{\sqrt{\pi}\Gamma\left(n/2+\frac{1}{2}\right)}{2\Gamma(n/2+1)}(n > -1) \tag{17}$$

gives

$$\int_0^{2\pi} f(A\cos\varphi)\cos\varphi\,\mathrm{d}\varphi = 2\sqrt{\pi}A^{1/3}\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{5}{3}\right)}, \tag{18}$$

where $\Gamma(n)$ is the Gamma function [8]. Substituting this equation into Eq. (8) yields

$$\omega = \sqrt{\frac{2\Gamma\left(\frac{7}{6}\right)}{\sqrt{\pi}A^{2/3}\Gamma\left(\frac{5}{3}\right)}}. \tag{19}$$

The approximate solution to Eq. (16) is

$$x = A\cos\left(\sqrt{\frac{2\Gamma\left(\frac{7}{6}\right)}{\sqrt{\pi}A^{2/3}\Gamma\left(\frac{5}{3}\right)}}t\right). \tag{20}$$

For $A = 1$,

$$\omega = \sqrt{\frac{2(0.9260)}{\sqrt{\pi}(0.90330)}} = 1.076. \tag{21}$$

For $A = 1$, the numerical solution and the value obtained by using the method of harmonic balance are, respectively [9],

$$\omega_{num} = 1.054, \quad \omega_{hb} = 1.049. \tag{22, 23}$$

The relative error between approximate solution (21) and numerical solution (22) is 2.09%.

5. Conclusions

A modified method of equivalent linearization has been proposed to solve non-linear oscillations of single-degree-of-freedom systems with odd non-linearity, which works even when the non-linearity is not small. Formula (8) is the main result of this paper. The details of the method have been illustrated by three examples. The first two examples give the same results as those obtained by use of the method of harmonic balance. The disadvantage of this method is that it only gives the first approximations. The possibility of further generalizing the method is now being investigated for the non-linear differential equation $x'' + f(x, x') = 0$.

Acknowledgements

The work was supported by the Natural Science Foundation of Hunan (01JJY3002) and Xiangtan Polytechnic University Scientific Research Fund (E20103).

References

[1] J.-H. He, A new perturbation technique which is valid for large parameters, *Journal of Sound and Vibration* 229 (2000) 1257–1263; doi:10.1006/jsvi.1999.2509.

[2] C.W. Lim, B.S. Wu, A modified Mickens procedure for certain non-linear oscillators, *Journal of Sound and Vibration* 257 (2002) 202–206; doi:10.1006/jsvi.2001.4233.

[3] M. Senator, C.N. Bapat, A perturbation technique that works even when the non-linearity is not small, *Journal of Sound and Vibration* 164 (1993) 1–27; doi:10.1006/jsvi.1993.1193.

[4] R.E. Mickens, Generalization of the Senator–Bapat method to systems having limit cycles, *Journal of Sound and Vibration* 224 (1999) 167–171; doi:10.1006/jsvi.1998.2141.

[5] R.E. Mickens, *Oscillations in Planar Dynamic Systems*, World Scientific, Singapore, 1996.

[6] R.E. Mickens, Oscillations in an $x^{4/3}$ potential, *Journal of Sound and Vibration* 246 (2001) 375–378; doi:10.1006/jsvi.2000.3583.

[7] R.S. Burington, *Handbook of Mathematical Tables and Formulas*, 5th Edition, McGraw-Hill, New York, 1973.
[8] I.N. Bronshtein, K.A. Semendyayev, *Handbook of Mathematics*, Van Nostrand Reinhold, New York, 1985.
[9] R.E. Mickens, Generalized harmonic balance/numerical method for determining analytical approximations to the periodic solutions of the $x^{4/3}$ potential, *Journal of Sound and Vibration* 250 (2002) 951–954; doi:10.1006/jsvi.2001.3782.