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Noise generation by a supersonic leading edge. Part 1: general theory

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Abstract

This paper concerns the calculation of the sound fields which are generated when a vortical gust, convected in a supersonic mean flow, strikes the leading edge of a fan blade or aerofoil. Inviscid linear theory is applied, with the blade modelled as an infinite-span flat rigid plate, and a gust of arbitrary form is considered. By application of Fourier transforms the boundary value problem for the velocity potential is solved, leading to an integral expression for the generated sound field. This expression is applicable everywhere inside a Mach wedge. For gusts localized in the span direction, a farfield approximation is derived which is valid inside a Mach cone, and which is of simple enough form to be evaluated analytically for specified gust shapes. The new feature of this analysis is the consideration of an arbitrary gust form: previous authors have only ever considered the properties of specific gusts, focusing principally on harmonic gusts and jets.

1. Introduction

The aim of this paper is to calculate the sound field generated by the interaction between a semi-infinite flat plate aerofoil and an arbitrary vortical gust convected in a supersonic mean flow. This problem is of practical interest, as it can be related to the calculation of sound fields generated by modern high-performance aeroengines. In such engines an important mechanism of noise generation is the interaction between blades and convected vorticity, as when fan blades encounter ingested turbulence or downstream blades encounter vortices shed by upstream blades. A comprehensive investigation of such a complicated system relies on a great deal of computational and experimental work, and so analytical treatments of simple model problems related to individual components of the system are important in providing physical understanding and

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simple test problems for the aeroacoustic computer codes. The outer regions of the fan blades may move at supersonic speeds relative to the surrounding airflow; thus the study of the supersonic regime is relevant.

In the current study a single fan blade is isolated, and this blade is modelled as a flat plate assumed to lie within a steady mean flow. The case of a subsonic mean flow has been considered in great detail by previous authors. Modelling the aerofoil as a flat half-plane is an approach which has been widely utilized and which leads to analytical results. Various authors have considered such a model, taking a specific incoming gust as the known data for the problem. A brief summary of this work is given by Chapman [1], who goes on to calculate formulae for the sound field generated by an arbitrary gust, thus unifying and generalizing the previous results. Examples of the use of these general formulae are given by Chapman [2].

Fewer results are available for the acoustic problem in the case of a supersonic mean flow. While the aerofoil gust interaction problem has a history of over 50 years, most of the early work is concerned with calculating the magnitude of the forces generated on the aerofoil surface. An excellent review of this work is given by Miles [3]. The early part of his text provides a summary of the linearization of the system, with which this paper assumes the reader is familiar. Later parts of Miles's text contain many results which are of some relevance to the problem considered in the present paper, but while the prime concern here is the pressure field generated throughout the fluid, Miles sets the co-ordinate normal to the aerofoil surface equal to zero at an early stage, so that he only ever calculates the pressure at the surface. This he uses to calculate the values of the variables in which he is primarily interested: the lift and the pitching moments. It appears that in the early literature the acoustic pressure field was of little interest, although the field for one particular gust has been treated: a two-dimensional problem with a sharp-edged gust is considered by several authors, the earliest probably being Strang [4]. The first consideration of the acoustic field in a three-dimensional problem was by Ffowcs Williams and Guo [5], where the gust considered was a cylindrical jet perpendicular to the aerofoil surface. Guo [6–9] considered this problem in great detail, examining not only the generated sound but also the unsteady loading on the aerofoil and the radiated energy. For similar gusts Peake [10] modelled the aerofoil as a flat quarter plane, thus examining corner effects. Peake [11] went on to consider a flat-plate aerofoil of finite span and chord interacting with a harmonic gust.

This paper utilizes the simplest possible model for the blade, the half-plane. In parallel to the subsonic work of Chapman [1], general formulae for the interaction of the aerofoil with an arbitrary gust are derived; these generalize the previous supersonic half-plane results. For ease of comparison the notation adopted is largely identical to that of Chapman [1], though the definition of a set of Doppler-adjusted co-ordinates is slightly different due to the difference between the flow regimes. In Part 1 of this paper the general formulae for the sound fields (which may be fully three dimensional) are derived, while in Part 2 (Ref. [12]) these formulae are applied to a family of gusts which give rise to two-dimensional sound fields, such fields being easily analyzed and understood.

The analysis in this paper begins in Section 2 with a full description of the physical system to be studied and a discussion of the boundary value problem which describes this system. An analytical solution to the problem is derived in Section 3 by the application of Fourier transforms. In Section 4 a simple approximation to this complicated solution is derived, which is valid in the acoustic far field for a certain type of gust. A brief example of the application of the derived results is given in Section 5, followed in Section 6 by a general discussion of the results and possible future work.

2. The physical system

The physical system to be considered is sketched in Fig. 1, which shows part of a flat plate aerofoil of infinite span and semi-infinite chord, located within an inviscid fluid of density ρ_0. A wind tunnel frame of reference is chosen, so that the aerofoil is stationary and the fluid exhibits a supersonic mean flow of speed U in a direction normal to the aerofoil leading edge, such that the aerofoil lies at zero angle of attack to the mean flow. A system of (right-handed) cartesian co-ordinates is defined such that the mean flow is in the positive x direction, the aerofoil leading edge is coincident with the z-axis, and the y-axis is normal to the aerofoil surface. Then the aerofoil lies in the half-plane $y = 0, x \geqslant 0$, and the mean flow velocity is $U\mathbf{e}_x$, where $\mathbf{e}_x$ is a unit vector in the positive x direction. The x, y and z directions may be described as streamwise, vertical and spanwise, respectively.

The mean flow Mach number is defined as $M = U/c_0$, where c_0 is the speed of sound, which is assumed constant throughout the fluid. As the mean flow is supersonic, $M > 1$ throughout the following analysis. However, neither the solution to be derived nor the subsonic solution of Chapman [1] are applicable in the limit $M \to 1$, and the transonic case must be considered separately.

Superimposed upon the mean flow is some convected vortical disturbance, referred to as the gust. It is assumed that the form of this gust is known, but to obtain general formulae an arbitrary form must be considered. This vortical gust will be associated with a velocity field, which on striking the aerofoil generates a sound field. It is assumed that the pressure fluctuations within this sound field are of small enough amplitude relative to the mean pressure that linear theory may be applied.

It is important to note that the incoming gust is a vorticity field, since the case of an incoming sound field would be rather different. The case of an incoming sound field results in a Sommerfeld half-plane diffraction problem, with the incoming acoustical energy being scattered by the aerofoil, but with no new acoustical energy being generated. By contrast, in the problem treated here, the incident energy is all in the vortical field, and some of this energy is converted to

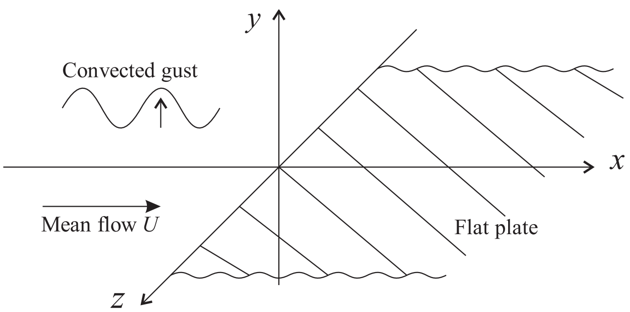

Fig. 1. Co-ordinate system for the study of a flat-plate aerofoil at rest in a mean flow of speed U, with a convected gust present in the fluid. The aerofoil occupies the plane $y = 0, x \geqslant 0$.

acoustical energy at the leading edge. Thus the present problem is one of noise generation, as opposed to scattering. A further physical difference that would make a large impact on the formulation of the problem is that the incoming vorticity field moves at the mean flow velocity, while the velocity of an incoming sound field would be a sum of the mean flow velocity and the propagation velocity of the sound relative to the fluid. Also, in a Sommerfeld problem the incoming field is generally modelled as a plane wave, which by its nature will strike the entire leading edge. However, in the vorticity problem, the nature of the turbulent vorticity field is such that there exist highly localized regions of vorticity, which strike only a small region of the leading edge, leading to a very different form of sound field. Thus in a Sommerfeld problem the scattered field exists in the region within a Mach wedge, while in the vorticity problem realistic gusts should generate acoustic fields which are concentrated within a Mach cone.

The consideration of the gust and the generated sound field as separate independent entities is justified by the "splitting theorem" (see Ref. [13, pp. 220–222]). This states that in any linear system with a constant mean flow velocity, the total velocity perturbation $\mathbf{u}$ may be decomposed into the sum of an irrotational (zero curl) part $\mathbf{u}_1$ and a solenoidal (zero divergence) part $\mathbf{u}_2$. The linearized continuity and momentum equations can be shown to be independent of the solenoidal velocity $\mathbf{u}_2$, and consequentially $\mathbf{u}_1$ is solely responsible for the pressure fluctuations in the fluid. Thus $\mathbf{u}_1$ may be described as the acoustic particle velocity. On the other hand, the vorticity of the fluid is independent of the irrotational velocity $\mathbf{u}_1$, so that $\mathbf{u}_2$ may be described as the vortical velocity. This is the velocity field associated with the gust. Then in the linear model these two components behave independently within the fluid. They are coupled only in the region of solid boundaries, where they must combine in such a way as to satisfy a boundary condition of zero normal velocity. Thus before the vortical velocity field $\mathbf{u}_2$ strikes the aerofoil, there is no sound, but upon interaction with the plate an acoustic velocity field $\mathbf{u}_1$ is generated in order to satisfy the boundary condition. One important consequence of the splitting theorem that becomes apparent in the later analysis is that all sound generation takes place at the leading edge, rather than on the aerofoil surface.

The vortical velocity field $\mathbf{u}_2$ is convected with the mean flow of fluid. Other than this convection, the gust is "frozen" (a result of the linear theory), so that from the perspective of an observer moving with the mean flow the velocity field $\mathbf{u}_2$ is fixed. Given that the natural boundary condition is one of zero normal velocity on the plate surfaces, it is seen that the important information required will be the y component of the vortical velocity $\mathbf{u}_2$, evaluated in the plane $y = 0$. Being frozen but convected, this can be a function of time t and streamwise displacement x only in the combination $t - x/U$. Then the required velocity component is described by a function $f(t - x/U, z)$, which is allowed to be arbitrary so that general formulae are derived. Given a problem where the vorticity field $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}_2$ is specified, rather than the required velocity component, it is a straightforward procedure to calculate this component in terms of a triple Biot–Savart integral (see, for example, Ref. [14]). For present purposes however, it seems most convenient simply to assume the function $f(t - x/U, z)$ is known.

The acoustic disturbance is described by an acoustic particle velocity $\mathbf{u}_1$ and a pressure perturbation p. Since $\mathbf{u}_1$ is irrotational, the disturbance may be described in terms of a velocity potential φ by

$$\mathbf{u}_1 = \nabla\varphi, \quad p = -\rho_0\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\varphi. \tag{1}$$

From the linearized continuity and momentum equations (see Ref. [3]), it is known that this potential must obey the convected wave equation

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\varphi-\nabla^2\varphi=0. \tag{2}$$

It may be deduced that φ must be an odd function of the vertical co-ordinate y. This arises from the symmetry of the system, and may be shown from consideration of the velocity fields. Since the vertical component of the gust velocity is equal on either side of the plate surface, the zero normal velocity condition implies that so too must be the acoustic velocity, and so the resulting acoustic velocity throughout the space will be an even function of y. Thus $\partial\varphi/\partial y$ is even, and so φ is an odd function of y. Then from Eq. (1) the pressure must be an odd function of y. However, since the plate supports a pressure difference between its upper and lower surfaces, the potential may be discontinuous across the plate. The condition of zero normal velocity on the plate surfaces means the sum of the y components of the gust velocity $\mathbf{u}_2$ and the acoustic velocity $\mathbf{u}_1$ must be zero on the half-plane $y=0^{\pm},\ x\geqslant 0$. This boundary condition can be extended to the entire plane $y=0$ by noting that in the region upstream of the leading edge the acoustic velocity is identically zero, since all noise generated at the edge is swept downstream in the supersonic mean flow. Thus, in terms of a Heaviside unit step function, the boundary condition is

$$\frac{\partial\varphi}{\partial y}=-f(t-x/U,z)\mathrm{H}(x)\quad(y=0^{\pm}). \tag{3}$$

To guarantee that a unique and physically sensible solution is obtained, the solution must obey causality and radiation conditions. Thus it is required that no acoustic field exists before the initial interaction between the gust and the aerofoil, and also that the field be outgoing at infinity.

For convenience a set of Doppler adjusted co-ordinates is defined by

$$\bar{x}=x/(M^2-1),\quad \bar{y}=y/(M^2-1)^{1/2},\quad \bar{z}=z/(M^2-1)^{1/2}. \tag{4}$$

These allow the final formulae and also the equations for Mach cones and wedges to be expressed in a compact form. The form of the co-ordinates makes it plain that the solution is not applicable in the limit $M\to 1^+$. In the subsonic work of Chapman [1,2] a similar set of co-ordinates is used, incorporating factors $(1-M^2)^{-1/2}$, which is not applicable in the limit $M\to 1^-$. (The transonic problem must be considered separately, as the transonic governing equation is not the usual convected wave equation.) Chapman [15] argues that his variables are the fundamental scaled co-ordinates for subsonic aeroacoustics, and the fact that the above supersonic counterparts to these appear to account for all the free Doppler factors in the following formulae points to their being just as fundamental, although no detailed discussion of this shall be given here.

3. Solution

The problem is solved by the application of Fourier transforms. A transform from the space variables (x,z) to the wavenumbers (k,m), and from the time t to the frequency ω is effected by the application of a Fourier transform pair, using the definition

$$\Phi(k,y,m,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x,y,z,t)\mathrm{e}^{\mathrm{i}(\omega t - kx - mz)}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t, \tag{5}$$

$$\varphi(x,y,z,t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(k,y,m,\omega)\mathrm{e}^{-\mathrm{i}(\omega t - kx - mz)}\,\mathrm{d}k\,\mathrm{d}m\,\mathrm{d}\omega. \tag{6}$$

To find causal solutions to Eq. (2) the ω contour in the inversion integral (6) must be taken to lie above all singularities in the complex ω plane. This is a standard result in the acoustics literature, and the argument that shows that it gives causal solutions is due to Lighthill [16]. The variables x, z and t are always assumed to be real, while complex values of the variables k, m and ω may be found to be useful. When utilizing complex variables, the real and imaginary parts of a variable shall be denoted by subscripts r and i, respectively.

Transforming the convected wave equation yields the equation

$$\frac{\partial^2 \Phi}{\partial y^2} - \gamma^2\Phi = 0, \tag{7}$$

where $\gamma(k,m,\omega)$ is a function defined by

$$\gamma(k,m,\omega) = \left\{k^2 + m^2 - \frac{(\omega - Uk)^2}{c_0^2}\right\}^{1/2}, \tag{8}$$

with the branch of the square root chosen such that $\mathrm{Re}(\gamma(k,m,\omega)) \geqslant 0$ for real k. Now, since φ is an odd function of y, so too is Φ. Then the solution of Eq. (7) which decays as $|y| \to \infty$ is

$$\Phi(k,y,m,\omega) = A(k,m,\omega)\,\mathrm{sgn}(y)\mathrm{e}^{-\gamma(k,m,\omega)|y|}, \tag{9}$$

where $A(k,m,\omega)$ is a function to be determined.

A transform of the gust term, which takes advantage of the coupling of the x and t dependence of the gust, is defined by

$$F(\omega,m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t',z)\mathrm{e}^{\mathrm{i}(\omega t' - mz)}\,\mathrm{d}t'\,\mathrm{d}z, \tag{10}$$

where $t' = t - x/U$. This is simply the transform of the gust shape as seen at the leading edge, where all the sound is generated. Transforming the boundary condition (3) in terms of the above gust transform gives

$$\frac{\partial \Phi}{\partial y} = \frac{-\mathrm{i}UF(\omega,m)}{\omega - Uk} \quad (y = 0^{\pm}), \tag{11}$$

where in order to ensure the convergence of the integral, ω has been assumed to have a small positive imaginary part, which is consistent with the inversion integral in the ω plane. Differentiating the general solution (9) with respect to y and letting y tend to zero gives a simple condition which may be equated to Eq. (11), leading to an expression for the function $A(k,m,\omega)$. Inserting this into solution (9) gives the particular solution

$$\Phi(k,y,m,\omega) = \frac{\mathrm{i}UF(\omega,m)\,\mathrm{sgn}(y)\mathrm{e}^{-\gamma(k,m,\omega)|y|}}{(\omega - Uk)\gamma(k,m,\omega)}. \tag{12}$$

Inserting this solution into the inverse transform (6) gives the potential φ as a triple integral

$$\varphi(x,y,z,t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}U\,\mathrm{sgn}(y)F(\omega,m)}{(\omega-Uk)\gamma(k,m,\omega)}\mathrm{e}^{-\mathrm{i}(\omega t-kx-mz)}\mathrm{e}^{-\gamma(k,m,\omega)|y|}\,\mathrm{d}k\,\mathrm{d}m\,\mathrm{d}\omega. \tag{13}$$

This integral is complicated somewhat by the presence of the pole term $(\omega - Uk)^{-1}$. However, the primary interest in this paper is in the acoustic pressure rather than the potential. Then Eq. (1) for the pressure in terms of the potential is applied to the above expression for φ. The differential operators are transformed and taken inside the integral, giving a term $\omega - Uk$. Thus the pole term is cancelled by the transformed operators, leaving a simpler expression for the acoustic pressure:

$$p(x,y,z,t) = \frac{-\rho_0 Mc_0\,\mathrm{sgn}(y)}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(\omega,m)}{\gamma(k,m,\omega)}\mathrm{e}^{-\mathrm{i}(\omega t-kx-mz)}\mathrm{e}^{-\gamma(k,m,\omega)|y|}\,\mathrm{d}k\,\mathrm{d}m\,\mathrm{d}\omega. \tag{14}$$

This pressure integral may be partially integrated for an arbitrary gust, since the gust transform term $F(\omega,m)$ is independent of the streamwise wavenumber k. The k-dependence of the triple integral is

$$K = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kx-\gamma(k,m,\omega)|y|}}{\gamma(k,m,\omega)}\,\mathrm{d}k. \tag{15}$$

This integral is of a form which is fairly common in the solution of wave diffraction problems (see for example Ref. [17]). The essential feature of the integrand is the location of the branch points, which occur at points k for which $\gamma(k,m,\omega)=0$. These points are the roots of a quadratic equation, which are straightforwardly calculated and labelled $k_\pm$, where

$$k_\pm = \frac{M\omega/c_0}{M^2-1} \pm \frac{(\omega^2/c_0^2 + m^2(M^2-1))^{1/2}}{M^2-1}. \tag{16}$$

For definiteness it is specified that for real m and ω the positive root is taken.

For real m and ω the branch points both lie on the real k-axis. To determine the required deformations to the integration contour and the choice of branch cuts, a method due to Landau and Lighthill for finding the causal solution of wave problems (see, for example, Ref. [18, pp. 36–37]) is applied. This method is based upon the frequency ω being taken to be positive with a small positive imaginary part ε (consistent with the choice of contour in the ω plane), and the new locations of the branch points in the complex k-plane are determined. Their locations relative to the real axis then dictate the choice of contour, which must be deformed off the real axis when ε is set to zero, returning the branch points to the real axis. Then when the required contours are known, useful branch cuts may be determined.

For $\omega = \omega_r + \mathrm{i}\varepsilon$ then, the branch point equation (16) is studied. For the special case $m = 0$ it is found that $k_\pm = (\omega/c_0)/(M \mp 1)$, so both branch points are in the first quadrant of the complex k-plane. Then increasing $|m|$, the branch points move off in opposite directions, on curves which asymptote to the horizontal line $k_i = (M\varepsilon/c_0)/(M^2-1)$, which is in the upper half-plane. The crucial point is that the branch points always lie in the upper half of the k-plane. Thus the integration contour, lying on the real k-axis, runs below the branch points. This indicates that when ε is allowed to tend to zero, returning the branch points to the real axis, the integration contour must be deformed in such a way that it continues to run below k_+ and k_-.

The two possible factorizations of $\gamma(k,m,\omega)$ are

$$\gamma(k,m,\omega) = \pm\mathrm{i}(M^2-1)^{1/2}(k-k_+)^{1/2}(k-k_-)^{1/2}. \tag{17}$$

The square roots in the above expression are defined such that for real $k>k_+$ the roots are positive. Then given that $k_\pm$ lie in the upper half-plane, the phase of each of the square roots in the above factorizations is calculated at various locations along the real k-axis, and thus the phase of both possible values of $\gamma(k,m,\omega)$ for all real k is found. It is found from application of the condition $\mathrm{Re}(\gamma(k,m,\omega))\geqslant 0$ for real k that only one of the possible factorizations is appropriate, namely the one with the + sign. Then the integral is

$$K = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx-\mathrm{i}(M^2-1)^{1/2}(k-k_+)^{1/2}(k-k_-)^{1/2}|y|}}{\mathrm{i}(M^2-1)^{1/2}(k-k_+)^{1/2}(k-k_-)^{1/2}}\,\mathrm{d}k. \tag{18}$$

Since the integrand has no branch point at infinity, a single branch cut is chosen, namely a straight line between the branch points.

Consider first evaluating the solution by closing the path of integration with the addition of a semicircle in the lower half of the k-plane (see Fig. 2a). If k is denoted by $|k|\mathrm{e}^{\mathrm{i}\kappa}$ along this semicircle, the above choice of phase for γ implies that for large $|k|$,

$$|\mathrm{e}^{\mathrm{i}kx-\gamma|y|}| \sim |\mathrm{e}^{\mathrm{i}k(x-(M^2-1)^{1/2}|y|)}| \sim \mathrm{e}^{-|k|\sin\kappa(x-(M^2-1)^{1/2}|y|)}. \tag{19}$$

Now since κ takes values between $-\pi$ and 0, the integrand in Eq. (18) decays exponentially as $|k|\to\infty$ so long as $x-(M^2-1)^{1/2}|y|$ is negative. In this case the inclusion of the contour at infinity leaves the value of the integral unchanged. Then the fact that the integrand is regular throughout the lower half-plane means that by Cauchy's theorem the entire contour integral takes the value zero. Thus, the acoustic pressure is found to be zero in the region $x-(M^2-1)^{1/2}|y|<0$. This is a direct result of the fact that there is a supersonic mean flow: it is a statement that there is a large upstream region which the radiated acoustic field does not enter, because the sound is being convected downstream from the source at the leading edge. Thus a Mach wedge is generated, with its apex at the leading edge. The equation of this wedge is readily expressed in

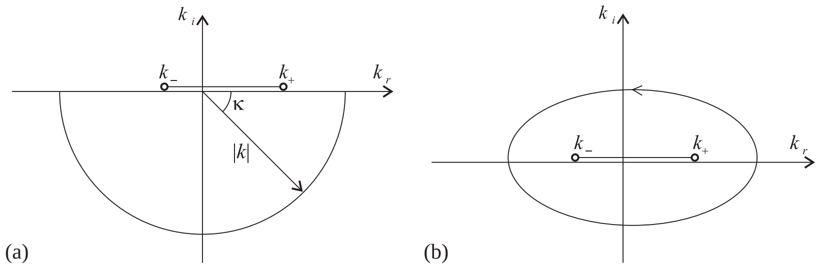

Fig. 2. Integration contours in the complex k-plane. The branch points k_+ and k_- are represented by small circles, and the line between these points is the branch cut: (a) closure of the contour in the lower half-plane with a semicircle $k=|k|\mathrm{e}^{\mathrm{i}\kappa}$, for the region upstream of the Mach wedge; (b) closure of the contour in the upper half-plane; the contour is deformed onto an elliptical path around the branch points, for the region downstream of the Mach wedge.

terms of the adjusted co-ordinates as $\bar{x} = |\bar{y}|$. In the (x, y) co-ordinates it meets the plane $y = 0$ at an angle $\mu = \sin^{-1}(1/M) = \tan^{-1}((M^2 - 1)^{-1/2})$, referred to as the Mach angle. In the $(\bar{x}, \bar{y})$ co-ordinates the Mach angle is fixed at $\pi/4$ radians regardless of Mach number, as these co-ordinates are defined in such a way as to incorporate Doppler factors.

In the region downstream of the wedge, an argument similar to that above shows that the integration contour may be closed with an arc at infinity in the upper half-plane. Now since there are no singularities in the k-plane other than the branch points, the contour may be deformed on to any closed curve (integrated anti-clockwise) which encloses both branch points. A change of space variables is introduced, and then an appropriate elliptical contour in the complex k-plane is defined. Then it is found that integral (18) may be reduced to a standard integral.

A useful change of co-ordinates is

$$x = r\cosh\theta, \quad (M^2 - 1)^{1/2} y = r\sinh\theta, \tag{20}$$

where

$$r = (x^2 - (M^2 - 1)y^2)^{1/2}, \quad \theta = \tanh^{-1}((M^2 - 1)^{1/2} y/x). \tag{21}$$

Initially the region $y > 0$ (i.e., $\theta > 0$) is considered, and the result is then extended by ones knowledge of the symmetry of the problem.

In the complex k-plane the point k_c is defined as being halfway along the branch cut, i.e.,

$$k_c = \tfrac{1}{2}(k_+ + k_-) = \frac{M\omega/c_0}{M^2 - 1}. \tag{22}$$

The appropriate substitution is then defined by

$$k = k_c + (k_+ - k_c)\cosh(\theta + \mathrm{i}s) \quad (-\pi \leqslant s \leqslant \pi). \tag{23}$$

This is a parameterization of an ellipse in the complex k-plane, centered at k_c with its major axis along the branch cut, of sufficient size to enclose the branch points (see Fig. 2b). Thus, as discussed above, it is a suitable integration contour for the integral.

Changing the variables in the integrand gives

$$K = \frac{\mathrm{e}^{\mathrm{i}k_c r\cosh\theta}}{(M^2 - 1)^{1/2}} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}r(k_+ - k_c)\cos s}\, \mathrm{d}s = \frac{\mathrm{e}^{\mathrm{i}k_c r\cosh\theta}}{(M^2 - 1)^{1/2}} 2\pi\, \mathrm{J}_0((k_+ - k_c)r). \tag{24}$$

Here J_0 is the Bessel function of the first type, of order zero. The final step in the above calculation is facilitated by a standard integral for the Bessel function, as given in, for example, Ref. [19, Eq. 9.1.21]. Returning to the original variables gives

$$K = \frac{2\pi \mathrm{e}^{\mathrm{i}M(\omega x/c_0)/(M^2 - 1)}}{(M^2 - 1)^{1/2}} \mathrm{J}_0\left(\left\{\frac{\omega^2}{c_0^2} + m^2(M^2 - 1)\right\}^{1/2} \frac{(x^2 - (M^2 - 1)y^2)^{1/2}}{M^2 - 1}\right). \tag{25}$$

The advantage of the Doppler-adjusted co-ordinates (Eq. (4)) can now be seen: in the above formula both the argument of the Bessel function and the exponent are more simply expressed in terms of these co-ordinates.

Inserting expression (25) into the general pressure formula (14) gives the general solution for the pressure field inside the Mach wedge as

$$p(x,y,z,t) = \frac{-\rho_0 M c_0 \,\mathrm{sgn}(y)}{(2\pi)^2 (M^2-1)^{1/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega,m) \mathrm{e}^{-\mathrm{i}\omega(t-M\bar{x}/c_0)} \mathrm{e}^{\mathrm{i}mz} \times \mathrm{J}_0\left(\left\{\omega^2/c_0^2 + m^2(M^2-1)\right\}^{1/2} (\bar{x}^2-\bar{y}^2)^{1/2}\right) \mathrm{d}m\,\mathrm{d}\omega. \tag{26}$$

Recall the conditions that the above integral is applicable only inside the Mach wedge $\bar{x} > |\bar{y}|$, and that when evaluating this integral for specific gusts, the causality condition dictates that the integration contour in the complex ω-plane should run above any singularities.

This is the most general expression available for a gust of arbitrary form. The complexity of the term in m and ω inside the Bessel function means that the actual evaluation of this integral for specific gusts can be very difficult, though it is possible in several cases which are of considerable interest. There are two notable cases in which at least one of the integrals may be performed exactly. The first of these is the case of a two-dimensional gust, i.e., a gust with a velocity profile which is independent of the spanwise variable z. The other case is that of a gust with a velocity profile which has a Dirac delta function dependence in either the streamwise or the spanwise direction. Such gusts are mathematical idealizations, but they are useful because they relate to simple physical situations which are easily understood, yet which are revealing about the behaviour of more general gusts. For more complex gust profiles however, it is found that evaluation of the pressure integral (26) can be difficult. Thus it is beneficial to derive an asymptotic approximation which will be easier to evaluate for a certain class of physically interesting gusts.

4. Approximation for gusts localized in the span direction

In the well-studied case of a point source in supersonic flow a Mach cone is generated. A source which is distributed over some finite localized area produces a region of disturbance which at a large distance from the source appears largely conical. Gusts which are localized in the span direction lead to such a localized source at the leading edge, and thus the acoustic field far from the leading edge may be modelled as conical. In aeroengine applications, such localization of the incident gusts is expected. Without loss of generality it is assumed that the centre of the vortical disturbance is located at $z = 0$, so that the field is concentrated within the cone $\bar{x} > (\bar{y}^2 + \bar{z}^2)^{1/2}$. Then for such localized gusts a simple approximation valid inside this cone may be derived.

The first step is to introduce a change of variable, which shall simplify the argument of the Bessel function with relation to the frequency ω. Then an appropriate approximation to the Bessel function is introduced, which is valid for large values of the argument. Manipulation of the resulting double integral leads to an integral in the new variable which may be approximated by standard stationary phase methods, leaving an integral only over ω, which is of a far simpler form than the exact result (26).

Changing the variables then, the appropriate substitution is

$$m = \frac{\omega}{c_0}(M^2-1)^{-1/2} \sinh\chi. \tag{27}$$

The application of this to the pressure integral (26) is straightforward, but leads to a rather large expression which is not given here. The Bessel function in this expression, which has argument $(\omega/c_0)(\bar{x}^2 - \bar{y}^2)^{1/2}\cosh\chi$, is replaced with the standard Hankel approximation, given by Abramowitz and Stegun [19, Eq. (9.2.1)] as

$$\mathrm{J}_0(\alpha) \simeq \left(\frac{2}{\pi\alpha}\right)^{1/2} \cos(\alpha - \pi/4). \tag{28}$$

For the Bessel function of order zero, this is accurate throughout the range $\alpha > 1$. Then looking at the argument of the present Bessel function, one sees that this approximation is good for $(\omega/c_0)(\bar{x}^2 - \bar{y}^2)^{1/2} > 1$, which means the present approximation shall be least accurate near the Mach wedge. Since one is deriving an expression valid inside the Mach cone, this is a small part of the overall field.

For positive real ω the integration contour in the complex χ-plane can be taken as the real χ-axis, traversed from $-\infty$ to ∞. The extension to the case of negative ω is by analytic continuation, by which method it can be shown that the same integration contour is valid for the case of negative real ω. Thus in the resulting double integral both integrations are again from $-\infty$ to ∞.

For convenience a function $G(\omega, \chi)$ is defined, given by

$$G(\omega, \chi) = F(\omega, (\omega/c_0)(M^2 - 1)^{-1/2}\sinh\chi)\cosh^{1/2}\chi. \tag{29}$$

The χ dependence of the double integral is isolated, and the χ integral is labelled $I(\omega)$. The cosine, with which the Bessel function was approximated, is equated with a sum of two complex exponentials, and the integral $I(\omega)$ is thus expressed as a sum of two integrals denoted $I_\pm(\omega)$, defined by

$$I_\pm(\omega) = \frac{\mathrm{e}^{\mp \mathrm{i}\pi/4}}{2(\bar{x}^2 - \bar{y}^2)^{1/4}} \int_{-\infty}^{\infty} G(\omega, \chi)\mathrm{e}^{\mathrm{i}(\omega\bar{z}/c_0)\sinh\chi}\mathrm{e}^{\pm \mathrm{i}(\omega/c_0)(\bar{x}^2 - \bar{y}^2)^{1/2}\cosh\chi}\,\mathrm{d}\chi. \tag{30}$$

Each of the above integrals is related to a pressure, the sum of which is the total acoustic pressure perturbation. The two component pressures are labelled p_1 and p_2, corresponding to $I_+(\omega)$ and $I_-(\omega)$, respectively. These pressures are given by

$$p_{1,2}(x, y, z, t) = \frac{-\rho_0 M c_0\,\mathrm{sgn}(y)}{(2\pi)^2(M^2 - 1)}\left(\frac{2}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \left(\frac{\omega}{c_0}\right)^{1/2} \mathrm{e}^{-\mathrm{i}\omega(t - M\bar{x}/c_0)} I_\pm(\omega)\,\mathrm{d}\omega. \tag{31}$$

The purpose of approximating the Bessel function was to lead to integrals of the form of Eq. (30). To reduce these integrals to a standard form one must simplify the exponents, which may be written $\mathrm{i}(\omega/c_0)(\bar{z}\sinh\chi \pm (\bar{x}^2 - \bar{y}^2)^{1/2}\cosh\chi)$. Within the Mach cone, $(\bar{x}^2 - \bar{y}^2)^{1/2} > |\bar{z}|$, and thus the second bracketed term may be expressed as a phase shifted cosh function of the form $\pm \bar{R}_h(\bar{x}, \bar{y}, \bar{z})\cosh(\chi \mp \beta(\bar{x}, \bar{y}, \bar{z}))$. Here $\bar{R}_h$ and β are real functions of the space variables and are uniquely defined by the fact $(\bar{x}^2 - \bar{y}^2)^{1/2} > 0$. The required functions are easily derived as

$$\bar{R}_h(\bar{x}, \bar{y}, \bar{z}) = (\bar{x}^2 - \bar{y}^2 - \bar{z}^2)^{1/2}, \tag{32}$$

$$\beta(\bar{x}, \bar{y}, \bar{z}) = \cosh^{-1}\left(\left(\frac{\bar{x}^2 - \bar{y}^2}{\bar{x}^2 - \bar{y}^2 - \bar{z}^2}\right)^{1/2}\right) = \sinh^{-1}\left(\frac{-\bar{z}}{(\bar{x}^2 - \bar{y}^2 - \bar{z}^2)^{1/2}}\right). \tag{33}$$

The subscript h in the definition of the first of these variables is in reference to the fact that the level surfaces of this variable are hyperbolic. They lie within the Mach cone, and asymptote to the cone at large x. It emerges that $\bar{R}_h$ is one of the fundamental variables in the supersonic flow regime. In terms of the above variables, the integrals are

$$I_{\pm}(\omega) = \frac{e^{\mp i\pi/4}}{2(\bar{x}^2 - \bar{y}^2)^{1/4}} \int_{-\infty}^{\infty} G(\omega, \chi) e^{\pm i(\omega \bar{R}_h/c_0)\cosh(\chi \mp \beta)} \, d\chi. \tag{34}$$

These are now in a standard form for an asymptotic analysis by the method of stationary phase.

The method of stationary phase gives asymptotic approximations to integrals with oscillatory integrands. The method is discussed in detail by, for example, Borovikov [20, Theorem 1.2], the basic result being the approximation

$$\begin{aligned} I(\lambda) &= \int_{-\infty}^{\infty} \exp[i\lambda\phi(x)] f(x) \, dx \\ &\approx \exp[i\lambda\phi(x_0)] f(x_0) \sqrt{\frac{2\pi}{\lambda|\phi''(x_0)|}} \exp\left[i\frac{\pi}{4}\operatorname{sgn}(\phi''(x_0))\right], \end{aligned} \tag{35}$$

where x_0 is the stationary phase point, defined by $\phi'(x_0) = 0$. This approximation is valid for large values of λ.

This result may be applied to integral (34) for $I_{\pm}(\omega)$, leading to an approximation valid for large $\omega\bar{R}_h/c_0$. Calculation of the result is straightforward, and the details shall not be given here. The result is

$$I_{\pm}(\omega) = \left(\frac{\pi}{2\omega/c_0}\right)^{1/2} e^{\pm i(\omega \bar{R}_h/c_0)} \frac{1}{\bar{R}_h} F(\omega, \pm(\omega/c_0)(M^2-1)^{-1/2}\sinh\beta). \tag{36}$$

Inserting this into Eq. (31) for the acoustic pressure terms gives the general farfield approximation as

$$\begin{aligned} p_{1,2}(x, y, z, t) \simeq & \frac{-\rho_0 M c_0 \operatorname{sgn}(y)}{(2\pi)^2(M^2-1)} \frac{1}{\bar{R}_h} \\ & \times \int_{-\infty}^{\infty} e^{-i\omega(t - M\bar{x}/c_0 \mp \bar{R}_h/c_0)} F\left(\omega, \pm\frac{\omega}{c_0}(M^2-1)^{-1/2}\tan\bar{\theta}_h\right) d\omega. \end{aligned} \tag{37}$$

This expression utilizes the new variable $\bar{\theta}_h$, an angle, in place of the somewhat cumbersome variable β. It is defined by

$$\tan\bar{\theta}_h = -\bar{z}/\bar{R}_h. \tag{38}$$

Approximation (37) is not valid for small $\bar{R}_h$, and so fails very close to the Mach cone, and from the earlier discussion of the approximation made to the Bessel function it is known that (37) is also invalid near the wedge. For large x however, it is found from application of the result to specific examples that the approximation describes most of the acoustic field well. A full discussion of such examples is deferred to a later paper, but now these equations are used to analyze a single simple example, which exhibits behaviour typical of the generated fields, and one important result is pointed out.

5. Example of application of formulae

To show the use of the results derived above, a single example of the calculation of a fully three-dimensional sound field is now given. The gust to be considered is of a type which may be classed "separable normal": the velocity function $f(t - x/U, z)$ is the product of a function of $t - x/U$ and a function of the streamwise variable z. The gust to be considered is

$$f(t - x/U, z) = v_0 \mathrm{e}^{-\mathrm{i}\omega_0(t - x/U)} \mathrm{e}^{-(z/a)^2/2}, \tag{39}$$

where v_0 is a vertical velocity, ω_0 is a frequency and a is a parameter specifying the width of the gust. The Doppler adjusted versions of these variables are defined to match those of the space variables in the appropriate directions, that is $\bar{v}_0 = v_0/(M^2 - 1)^{1/2}$, $\bar{a} = a/(M^2 - 1)^{1/2}$. This gust form represents a harmonic wave, which has a Gaussian distribution in the span direction, so that the source is localized. The Fourier transform (10) of the gust is

$$F(\omega, m) = (2\pi)^{3/2} v_0 a \delta(\omega - \omega_0) \mathrm{e}^{-(ma)^2/2}, \tag{40}$$

where $\delta(\alpha)$ denotes a Dirac delta function. Inserting this into the exact pressure integral (26) gives an m integral for which no solution could be found. However, for the farfield integral (37), the presence of the delta function allows the integration to be carried out explicitly. The sum of the two terms gives the farfield pressure as

$$\begin{aligned} p(x, y, z, t) \simeq \frac{-\rho_0 M c_0 \bar{v}_0 \,\mathrm{sgn}(y)}{(2\pi)^{1/2}} \frac{\bar{a}}{\bar{R}_h} \mathrm{e}^{-\mathrm{i}\omega_0(t - M\bar{x}/c_0)} \\ \times \cos(\omega_0 \bar{R}_h/c_0) \mathrm{e}^{-((\omega_0 \bar{a}/c_0) \tan \bar{\theta}_h)^2/2}, \end{aligned} \tag{41}$$

where it is seen that the adjusted co-ordinates have incorporated all of the Doppler terms. From the derivation of the general farfield equation (37), it is found that the approximation to the Bessel function is valid in this case for $\omega_0(\bar{x}^2 - \bar{y}^2)^{1/2}/c_0 > 1$, and that the final expression was valid for large $\omega_0 \bar{R}_h/c_0$.

The pressure field described by Eq. (41) exhibits several interesting features, some of which are typical of fields created by other localized gusts. First, note the decay factor $\bar{R}_h^{-1}$, which is typical of all three-dimensional supersonic problems. It is comparable to the familiar R^{-1} decay of spherical fields in subsonic problems, but in the supersonic case the field decays not with distance from the source, but rather with distance from the Mach cone. Also, note the complex exponent with argument $\omega_0(t - M\bar{x}/c_0)$: this factor is seen in all cases where the gust is harmonic in nature. This factor depends on the linear co-ordinate x, yet it comes about from the interference of spherically spreading waves. At any point within the Mach cone, two wavefronts are incident at any given time, as some parts of the generated spherically spreading wavefronts attempting to propagate upstream will meet others generated at a later time propagating downstream. That these should combine in such a way as to give rise to a function with argument $t - M\bar{x}/c_0$ is not immediately obvious.

The variable $\bar{\theta}_h$ takes the value zero in the vertical plane $z = 0$, and tends to $\pm\pi/2$ as the Mach cone is approached. Then the Gaussian term with argument $(\omega_0 \bar{a}/c_0) \tan \bar{\theta}_h$ shows that despite the R^{-1} decay term, the field does not diverge on the Mach cone, but rather decays to zero. The exception is in the plane $z = 0$, where near the Mach wedge pressure appears to become arbitrarily large, but this is in the very small region where the approximation is not valid. The Gaussian shows that for large $\omega_0\bar{a}/c_0$ the field becomes highly directional, having a single lobe around the vertical plane $z = 0$, and decaying exponentially away from this plane. A comparison of this field with exact results (obtained using numerical integration routines in Matlab) shows that the approximation is largely accurate.

The exact integral (26) may also be used to investigate briefly the region near the Mach wedge, where the above approximation is not valid. It is of use to define the variable $\bar{r}_h = (\bar{x}^2 - \bar{y}^2)^{1/2}$, which is a characteristic variable of two-dimensional sound fields (see Ref. [12]). Inserting the Fourier transform (40) into the pressure integral (26), the delta function is used to calculate the ω integral first. In the resulting m integral, the Bessel function may be replaced by a Taylor series, following which the integration is fairly trivial. This gives the expression for the field near the Mach wedge as

$$\begin{aligned} p(x,y,z,t) = &-\rho_0 M c_0 \bar{v}_0 \operatorname{sgn}(y) \mathrm{e}^{-\mathrm{i}\omega_0(t-M\bar{x}/c_0)} \mathrm{e}^{-(\bar{z}/\bar{a})^2/2} \\ &\times \left[1 - \frac{1}{4}\left(\frac{\omega_0 \bar{r}_h}{c_0}\right)^2 - \frac{1}{4}\frac{\bar{r}_h^2}{\bar{a}^2}\left(1 - \frac{\bar{z}^2}{\bar{a}^2}\right) + O(\bar{r}_h^4)\right]. \end{aligned} \tag{42}$$

From this result it can be seen that at leading order in $\bar{r}_h$, i.e., very close to the Mach wedge, the sound field is unattenuated, and the spanwise shape of the sound field is simply the gust shape function. Thus the high pressures seen near the Mach angle in the farfield expression (41) are in fact the points in the sound field where the pressure is greatest. These peak pressures near the Mach angle turn out to be a general feature of the generated sound fields.

The presence of an unattenuated pressure peak on the Mach wedge can be shown to be an important general feature of all of the generated sound fields. In the limit $\bar{r}_h \to 0$, the argument of the Bessel function in the exact pressure integral (26) becomes small, and a power series may be taken, the leading order term of which is simply 1. Thus in the limiting case the pressure integral simply becomes a shifted Fourier inversion of $F(\omega, m)$, so that at the Mach wedge

$$p(x, \pm x/(M^2-1)^{1/2}, z, t) = \frac{-\rho_0 M c_0 \operatorname{sgn}(y)}{(M^2-1)^{1/2}} f(t - M\bar{x}/c_0, z). \tag{43}$$

Thus we see the field in this direction does not spread out or attenuate, and the pressure field shape is a simple (stretched) multiple of the gust shape. The above equation instantly gives the peak pressure due to any gust.

6. Summary and further work

To summarize, the above formulae are now sufficient to describe the acoustic field generated when a convected vortical gust strikes a flat plate in supersonic flow. An acoustic pressure field is generated at the leading edge when the velocity field due to the gust strikes the aerofoil, and this generated sound field spreads through the fluid at the speed of sound. Since the fluid itself is moving supersonically the field all moves downstream, and so is contained within the Mach wedge $\bar{x} = |\bar{y}|$. For any given gust, the pressure field is described exactly by Eq. (26). This solution is in the form of a complicated double integral, which for a given gust form can be computed numerically, but which may also be evaluated analytically for certain simple problems. In the physical problem of interest, the gusts are generally highly localized in space, so that they strike only a small section of the leading edge and thus generate a field only within a cone within the wedge. For such gusts the exact integral is often too complicated to yield an exact analytical solution, but the farfield is in this case well described by the asymptotic approximation given by the simple integral (37), in terms of variables defined in Eqs. (32) and (38). In all cases, the pressure peak is on the Mach wedge, where the pressure field does not attenuate as it propagates to the farfield.

Work is ongoing in the calculation of a number of examples of the fields due to specific gusts. Both near and far fields are being studied, and formulae (26) and (37) allow a study to be made of the way in which different gust features, such as sharp edges, affect the generated field. In those instances where analytical results can not be obtained numerical evaluation is generally straightforward. A study of two-dimensional sound fields is presented in Part 2 (Ref. [12]).

The linear inviscid theory presented here could be extended beyond the very simple model geometry to take account of the effects of a side edge or a trailing edge; also, the effects of mean loading on the blade could be considered.

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