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The effects of large vibration amplitudes on the axisymmetric mode shapes and natural frequencies of clamped thin isotropic circular plates. Part II: iterative and explicit analytical solution for non-linear coupled transverse and in-plane vibrations

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Abstract

The objective of this paper is to present a more realistic and complete study of the geometrically non-linear free vibrations of clamped immovable circular plates by taking into account the in-plane deformation, which has not been examined in Part I of this series of papers (J. Sound Vibration 265 (2003) 123). The problem is solved by a numerical iterative procedure in order to obtain more accurate results for vibration amplitudes up to twice the plate thickness. The numerical results thus obtained are presented and compared with the available published results, and with the ones calculated when neglecting the in-plane displacement for the first two non-linear axisymmetric mode shapes. An explicit analytical solution is then presented and its range of validity, for the fundamental non-linear mode, is determined via a detailed comparison with the solution based on the iterative procedure. The results obtained by the explicit method show the usefulness of the new approach in comparison with the single mode approach solution frequently used in non-linear vibration analysis of structures, and are expected to be easy to implement in fatigue models in order to make more realistic and secure predictions of the structural fatigue life.

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1. Introduction

The use of thin plates-like structures is extensive in various modern engineering problems and they are often subjected to severe dynamic loading. This may result in large vibration amplitudes of these structures inducing a dynamic behaviour which is different in many ways from that predicted by the classical linear theory. In such situations, it is necessary to include the geometrical non-linearity when investigating the structural dynamic behaviour. The von Kármán type of geometrically non-linear strain–displacement relationships is the most widely used in the literature to derive the governing equations of motion of thin plate-structures. The governing equations are coupled non-linear partial differential equations for which exact solutions are not yet available. Numerical methods are often used for obtaining general solutions. Also, as pointed out in Ref. [1], no general and systematic approach to non-linear problems is available which allows all or at least most of the various non-linear effects to be described in a unified manner. For example, the single mode approach was often adopted as a useful tool for investigating the effect of geometrical non-linearity on resonant phenomena [2]. The single mode assumption permits analytical solutions to be obtained for the amplitude frequency dependence and the non-linear forced response. However, this assumption has been shown both theoretically and experimentally to be insufficient for beams and plates in Refs. [1,3,4], since the mode shape thus assumed is amplitude independent and therefore leads to linear patterns of the bending stress rather than the non-linear patterns. In the study of geometrically non-linear axisymmetric vibrations of clamped circular plates, the common approach has been to use an assumed space or time mode. The different methods of solution used in the literature related to the subject of interest here have been presented and discussed in Part I of this series of papers in which some references were given [5–17]. Very recently, the finite element method has been applied to study non-linear free vibrations of hinged orthotropic circular plates with a concentric rigid mass using von Kármán equations [18], and geometrically non-linear free vibrations of polar orthotropic circular plates with various boundary conditions, using the three-dimensional elasticity theory with all of the non-linear terms retained in the strain expressions [19]. Since the single mode approach is not completely adequate for the study of geometrically non-linear free vibration of thin structures as mentioned above, multimode analyses are needed in order to determine accurately the amplitude-dependent non-linear frequencies and the associated non-linear mode shapes. The latter are of crucial importance in engineering problems because the stresses are related to the first and second derivatives of the mode shape. In Part I of this series of papers [20], a theoretical multimodal model based on Hamilton's principle and spectral analysis, which reduces the non-linear free vibration problem to solution of a set of non-linear algebraic equations, was used in order to study the effect of large vibration amplitudes on the non-linear natural frequencies and mode shapes of the first two non-linear axisymmetric mode shapes of clamped circular plates. This model allows quantitative estimates of non-linear stresses to be obtained in sensible regions of the structure, which may be of crucial importance in the fatigue life prediction of structures working in or exposed to a severe environment. However, the model used in Ref. [20] was restricted in a sense that only the transverse displacement was considered in the formulation. In the present paper, geometrically non-linear free vibrations of clamped immovable isotropic circular plates are investigated by using the multimodal model mentioned above taking into account not only the coupling between the higher vibration modes but also the influence of the in-plane deformation. By assuming harmonic transverse motion, the in-plane and out-of-plane displacements are expanded in the form of finite series of basic functions, namely the linear free vibration modes of the clamped immovable circular plate, obtained in terms of Bessel's functions. The discretized expressions for the total strain and kinetic energies are then derived. The application of Hamilton's principle reduces the large amplitudes free vibration problem to a set of coupled non-linear algebraic equations in terms of the contribution coefficients of the in-plane and out-of-plane basic functions. When the in-plane inertia is neglected, the above set may be reduced, via a simple analytical transformation, to a set of non-linear algebraic equations in terms of the contribution coefficients of the out-of-plane basic functions only. This reduced set is formally similar to that derived in Part I but with a different fourth order tensor, which now also includes the influence of the in-plane radial displacement. This set of non-linear algebraic equations represents a non-linear eigenvalue problem, which reduces to the well-known linear eigenvalue problem derived from Rayleigh–Ritz analysis when the non-linearity is omitted. The non-linear eigenvalue problem needs to be solved iteratively. The non-linear iterative procedure described in Refs. [21,22] is used here as a first approach for accurate determination of the non-linear resonant frequencies, the deflection shapes and the distributions of the associated membrane, bending and total stresses, for the first two clamped circular plate non-linear axisymmetric mode shapes, at various non-dimensional amplitudes. The results obtained from the proposed model are discussed and compared with available published results and also with the results obtained when the in-plane deformation is neglected.

In a second approach, an explicit solution, which may be appropriate for engineering purposes, or for further analytical investigations, is proposed and discussed in the present paper. This method of solution is based on the linearization of the set of non-linear algebraic equations in the neighbourhood of each resonance. The proposed explicit analytical solution is more accurate than the single mode approach solution since it leads also to amplitude-dependent mode shapes and non-linear bending stress patterns, and is as accurate as the iterative method of solution for a certain range of vibration amplitudes to be determined in each case, as illustrated in Section 4.2 in the case of the fundamental non-linear mode shape of a clamped immovable circular plate.

2. General formulation

2.1. Mathematical model

Consider a circular plate of thin uniform thickness h and radius a that is clamped along its edge. The co-ordinate system is chosen such that the middle plane of the plate coincides with the (r, θ) plane, the origin of the co-ordinate system being at the centre of the plate with the z-axis downward, as shown in Fig. 1. The plate is made of an elastic, homogeneous isotropic material.

In large amplitude axisymmetric vibrations of circular plates, the non-vanishing components of the strain tensor are given by [9]

$$\varepsilon_r = \frac{\partial U}{\partial r} + \frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2 - z\frac{\partial^2 W}{\partial r^2}, \quad \varepsilon_\theta = \frac{U}{r} - \frac{z}{r}\frac{\partial W}{\partial r}, \tag{1}$$

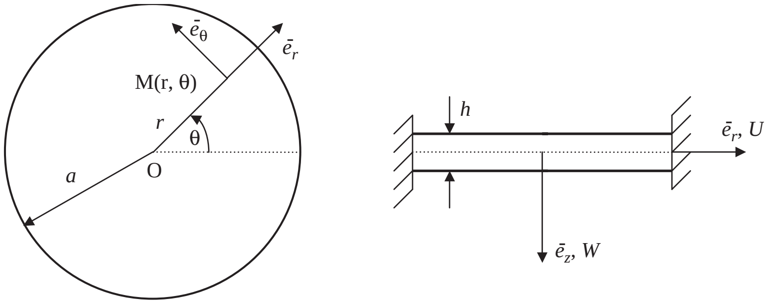

Fig. 1. Clamped circular plate notation.

where U is the middle plane in-plane radial displacement and W is the out-of-plane transverse displacement.

The total strain energy, V, of the circular plate is given as the sum of the strain energy due to bending (V_b) and the membrane strain energy induced by large deflections (V_m): $V = V_b + V_m$. In the case of axisymmetric vibrations, the bending strain energy of the clamped circular plate has been shown to reduce to [20]

$$V_b = \pi D \int_0^a \left[\left(\frac{\partial^2 W}{\partial r^2} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial r} \right)^2 \right] r \, \mathrm{d}r \tag{2}$$

in which $D = Eh^3/(12(1-\nu^2))$ is the bending stiffness of the plate, and E and ν are Young's modulus and the Poisson ratio of the plate material.

In terms of displacements, the expression for the membrane strain energy induced by large deflections for an axisymmetric circular plate is given by [23]

$$V_m = \frac{12\pi D}{h^2} \int_0^a \left[\left(\frac{\partial U}{\partial r} \right)^2 + \frac{U^2}{r^2} + 2\nu \frac{U}{r} \frac{\partial U}{\partial r} + \left(\frac{\partial W}{\partial r} \right)^2 \frac{\partial U}{\partial r} + \frac{1}{4} \left(\frac{\partial W}{\partial r} \right)^4 + \nu \frac{U}{r} \left(\frac{\partial W}{\partial r} \right)^2 \right] r \, \mathrm{d}r. \tag{3}$$

The total strain energy, V, is then given by

$$V = \pi D \int_0^a \left[\left(\frac{\partial^2 W}{\partial r^2} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial r} \right)^2 \right] r \, \mathrm{d}r + \frac{12\pi D}{h^2} \int_0^a \left[\left(\frac{\partial U}{\partial r} \right)^2 + \frac{U^2}{r^2} + 2\nu \frac{U}{r} \frac{\partial U}{\partial r} + \left(\frac{\partial W}{\partial r} \right)^2 \frac{\partial U}{\partial r} + \nu \frac{U}{r} \left(\frac{\partial W}{\partial r} \right)^2 + \frac{1}{4} \left(\frac{\partial W}{\partial r} \right)^4 \right] r \, \mathrm{d}r. \tag{4}$$

The kinetic energy, T, of the circular plate is

$$T = \pi \rho h \int_0^a \left[\left(\frac{\partial W}{\partial t} \right)^2 + \left(\frac{\partial U}{\partial t} \right)^2 \right] r \, \mathrm{d}r \tag{5}$$

in which rotatory inertia is neglected and ρ is the mass per unit volume of the plate material.

As mentioned in the introduction, the most common approach in the seek of approximate solutions of geometrically non-linear vibration of structures is the separation of space and time functions. Furthermore, previous experimental and theoretical studies concerned with beams and plates [1,24,25] have shown that harmonic distortion of the non-linear response of the structure excited harmonically occurs at large vibration amplitudes. In the case of free vibration of thin plates, the temporal function often takes the form of a Fourier cosine series as used in Refs. [26–28] in which only odd harmonics are retained, due to the character of cubic non-linearity. However, as discussed in Ref. [29], the separation of harmonics carried out in Refs. [1,24] on the measured response signals at various points of a plate excited non-linearly, at various excitation levels, has shown that the higher harmonic components remain very small, compared with the first-harmonic component of the response. Also, the theoretical study made in Ref. [28] has shown that the time series converge very rapidly and that there is only a very little difference between using the first harmonic and the first two harmonics in the Fourier series expansion.

An experimental observation corresponding to large amplitude plate vibrations has also been made in Ref. [30], in the case of an annular plate with a free inside edge and a built-in outside edge, in order to justify the steady state harmonic form of the time co-ordinate function used in Ref. [31], for finite amplitude vibrations of thin annular and circular plates. By comparing the bending and membrane strain wave forms generated by two circumferential strain gauges, the authors concluded that the assumed time-mode solutions ($\sin(\omega t)$ for the transverse displacement and $\sin^2(\omega t)$ for the stress function) yield favourable results.

Based on these considerations, the transverse displacement function $W(r,t)$ has been assumed in the present paper, which is mainly concerned with the amplitude dependence of the first-harmonic component spatial distribution, to be given by

$$W(r,t) = w(r)\cos(\omega t). \tag{6}$$

The in-plane radial displacement function $U(r,t)$ is taken in the following form:

$$U(r,t) = u(r)\cos^2(\omega t). \tag{7}$$

A similar assumption has been made in Refs. [9,10] for the stress function and in Ref. [32] for the in-plane displacement function. It is to be noted that in one-term Galerkin procedures in both stress function and displacement approaches, if the transverse function is assumed to be in the form: $W(r,t) = w(r)q(t)$, then the temporal function taken for the stress function [5], or the in-plane displacement function [6] is $q^2(t)$.

The spatial functions $u(r)$ and $w(r)$ are expanded in the form of finite series of pi and po in-plane $u_i(r)$ and out-of-plane $w_i(r)$ basic functions, respectively, as follows:

$$w(r) = a_i w_i(r), \quad u(r) = b_i u_i(r), \tag{8}$$

where the usual summation convention for repeated indices is used from 1 to po and from 1 to pi for the a_i's and b_i's coefficients respectively.

The discretized forms for the total strain and kinetic energies are respectively given by the following expressions:

$$V = \tfrac{1}{2} a_i a_j k^1_{ij}\cos^2(\omega t) + \tfrac{1}{2}[a_i a_j a_k a_l b^1_{ijkl} + a_i a_j b_k c_{ijk} + b_i b_j k^2_{ij}]\cos^4(\omega t), \tag{9}$$

$$T = \tfrac{1}{2}\omega^2[a_i a_j m^1_{ij}\sin^2(\omega t) + b_i b_j m^2_{ij}\sin^2(2\omega t)]. \tag{10}$$

In these equations, m^1_{ij}, m^2_{ij}, k^1_{ij}, k^2_{ij} are the mass and rigidity tensors associated with W and U, respectively, b^1_{ijkl} and c_{ijk} are, respectively, a fourth order and a third order non-linearity tensors. The general terms of these tensors are given by

$$\begin{aligned}
m^1_{ij} &= 2\pi\rho h \int_0^a w_i w_j r\,\mathrm{d}r; \quad m^2_{ij} = 2\pi\rho h \int_0^a u_i u_j r\,\mathrm{d}r,\\
k^1_{ij} &= 2\pi D \int_0^a \left(\frac{\mathrm{d}^2 w_i}{\mathrm{d}r^2}\frac{\mathrm{d}^2 w_j}{\mathrm{d}r^2} + \frac{1}{r^2}\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}w_j}{\mathrm{d}r} \right) r\,\mathrm{d}r,\\
k^2_{ij} &= \frac{24\pi D}{h^2} \int_0^a \left(\frac{\mathrm{d}u_i}{\mathrm{d}r}\frac{\mathrm{d}u_j}{\mathrm{d}r} + \frac{1}{r^2} u_i u_j + \frac{v}{r}\frac{\mathrm{d}u_i}{\mathrm{d}r} u_j + \frac{v}{r} u_i \frac{\mathrm{d}u_j}{\mathrm{d}r} \right) r\,\mathrm{d}r,\\
c_{ijk} &= \frac{24\pi D}{h^2} \int_0^a \left(\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}w_j}{\mathrm{d}r}\frac{\mathrm{d}u_k}{\mathrm{d}r} + \frac{v}{r}\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}w_j}{\mathrm{d}r} u_k \right) r\,\mathrm{d}r,\\
b^1_{ijkl} &= \frac{6\pi D}{h^2} \int_0^a \left(\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}w_j}{\mathrm{d}r}\frac{\mathrm{d}w_k}{\mathrm{d}r}\frac{\mathrm{d}w_l}{\mathrm{d}r} \right) r\,\mathrm{d}r.
\end{aligned} \tag{11}$$

It appears from Eqs. (11) that the mass and rigidity tensors are symmetric, and the fourth order tensor b^1_{ijkl} and the third order tensor c_{ijk} are such that

$$b^1_{ijkl} = b^1_{klij} = b^1_{jilk} = b^1_{ikjl}, \quad c_{ijk} = c_{jik}. \tag{12}$$

The dynamic behaviour of the structure is governed by Hamilton's principle, which is symbolically written as

$$\delta \int_0^{2\pi/\omega} (V - T)\,\mathrm{d}t = 0. \tag{13}$$

Replacing V and T by their discretized expressions in Eq. (13), integrating the time functions, calculating the derivatives with respect to the a_i's and b_i's, and taking into account the properties of symmetry of the tensors involved, leads to the following set of non-linear algebraic equations:

$$\begin{aligned}
&2a_i k^1_{ir} + 3a_i a_j a_k b^1_{ijkr} + \tfrac{3}{2} a_i b_k c_{irk} - 2\omega^2 a_i m^1_{ir} = 0, \quad r = 1, \ldots, po,\\
&\tfrac{3}{4}(a_i a_j c_{ijs} + 2b_i k^2_{is}) - 2\omega^2 b_i m^2_{is} = 0, \quad s = 1, \ldots, pi.
\end{aligned} \tag{14}$$

To simplify the analysis and the numerical treatment of the set of non-linear algebraic equations, non-dimensional formulation has been considered by putting the spatial displacement functions as

$$w_i(r) = h w^*_i(r^*), \;\; u_i(r) = \lambda h u^*_i(r^*), \tag{15}$$

where $r^* = r/a$ is the non-dimensional radial co-ordinate and $\lambda = h/a$ is a non-dimensional geometrical parameter representing the ratio of the plate thickness to its radius.

Eqs. (14) can be written in a non-dimensional form as

$$\begin{aligned}
&2a_i k^{1*}_{ir} + 3a_i a_j a_k b^{1*}_{ijkr} + \tfrac{3}{2} a_i b_k c^*_{irk} - 2\omega^{*2} a_i m^{1*}_{ir} = 0, \quad r = 1, \ldots, po,\\
&\tfrac{3}{4}(a_i a_j c^*_{ijs} + 2b_i k^{2*}_{is}) - 2\lambda^2 \omega^{*2} b_i m^{2*}_{is} = 0, \quad s = 1, \ldots, pi,
\end{aligned} \tag{16}$$

where ω^* is the non-dimensional non-linear frequency parameter defined by

$$\omega^{*2} = \frac{\rho h a^4}{D}\omega^2. \tag{17}$$

The m_{ij}^{1*}, m_{ij}^{2*}, k_{ij}^{1*}, k_{ij}^{2*}, c_{ijk}^* and b_{ijkl}^{1*} terms are non-dimensional tensors related to the dimensional ones by the following equations:

$$\begin{aligned}(m_{ij}^1, m_{ij}^2) &= 2\pi\rho a^2 h^3 (m_{ij}^{1*}, \lambda^2 m_{ij}^{2*}),\\ (k_{ij}^1, k_{ij}^2, c_{ijk}, b_{ijkl}^1) &= \frac{2\pi D h^2}{a^2}(k_{ij}^{1*}, k_{ij}^{2*}, c_{ijk}^*, b_{ijkl}^{1*}).\end{aligned} \tag{18}$$

These non-dimensional tensors are given by

$$\begin{aligned}
m_{ij}^{1*} &= \int_0^1 w_i^* w_j^* r^* \mathrm{d}r^*, \quad m_{ij}^{2*} = \int_0^1 u_i^* u_j^* r^* \,\mathrm{d}r^*,\\
k_{ij}^{1*} &= \int_0^1 \left(\frac{\mathrm{d}^2 w_i^*}{\mathrm{d}r^{*2}}\frac{\mathrm{d}^2 w_j^*}{\mathrm{d}r^{*2}} + \frac{1}{r^{*2}}\frac{\mathrm{d}w_i^*}{\mathrm{d}r^*}\frac{\mathrm{d}w_j^*}{\mathrm{d}r^*}\right) r^* \,\mathrm{d}r^*,\\
k_{ij}^{2*} &= 12\int_0^1 \left(\frac{\mathrm{d}u_i^*}{\mathrm{d}r^*}\frac{\mathrm{d}u_j^*}{\mathrm{d}r^*} + \frac{1}{r^{*2}} u_i^* u_j^* + \frac{v}{r^*}\frac{\mathrm{d}u_i^*}{\mathrm{d}r^*} u_j^* + \frac{v}{r^*} u_i^* \frac{\mathrm{d}u_j^*}{\mathrm{d}r^*}\right) r^* \,\mathrm{d}r^*,\\
c_{ijk}^* &= 12\int_0^1 \left(\frac{\mathrm{d}w_i^*}{\mathrm{d}r^*}\frac{\mathrm{d}w_j^*}{\mathrm{d}r^*}\frac{\mathrm{d}u_k^*}{\mathrm{d}r^*} + \frac{v}{r^*}\frac{\mathrm{d}w_i^*}{\mathrm{d}r^*}\frac{\mathrm{d}w_j^*}{\mathrm{d}r^*} u_k^*\right) r^* \,\mathrm{d}r^*,\\
b_{ijkl}^{1*} &= 3\int_0^1 \frac{\mathrm{d}w_i^*}{\mathrm{d}r^*}\frac{\mathrm{d}w_j^*}{\mathrm{d}r^*}\frac{\mathrm{d}w_k^*}{\mathrm{d}r^*}\frac{\mathrm{d}w_l^*}{\mathrm{d}r^*} r^* \,\mathrm{d}r^*.
\end{aligned} \tag{19}$$

In the case of thin plates for which λ is very small, the influence of the in-plane inertia involving the term λ^2 can be neglected. This assumption of neglecting in-plane inertia, which is an acceptable assumption in most engineering applications of thin plates [33], is generally adopted in the study of geometrically non-linear vibrations of thin plates. In this case, the second set of equations in Eq. (16) can be solved for the b_i's leading to

$$b_i = a_j a_l d_{jli}^*, \quad i = 1, \ldots, pi, \tag{20}$$

where $d_{ijk}^* = -\frac{1}{2} k_{kl}^{2*^{-1}} c_{ijl}^*$, is a third order tensor expressing the coupling between in-plane and transverse vibrations, in which $k_{ij}^{2*^{-1}}$ represents the inverse of the tensor k_{ij}^{2*}. Substituting Eq. (20) into the first set of Eqs. (16) leads to an uncoupled set of non-linear algebraic equations in terms of the a_i's coefficients only

$$a_i k_{ir}^{1*} + \tfrac{3}{2} a_i a_j a_k b_{ijkr}^* - \omega^{*2} a_i m_{ir}^{1*} = 0, \quad r = 1, \ldots, po. \tag{21}$$

Here, b_{ijkl}^* is a fourth order tensor given by

$$b_{ijkl}^* = b_{ijkl}^{1*} + \tfrac{1}{2} c_{ijn}^* d_{kln}^*. \tag{22}$$

In some studies on non-linear vibrations of circular plates [15,16,20], the in-plane radial displacement is neglected. If this assumption is used in our formulation, a similar set of non-linear algebraic equations, depending on the contribution coefficients a_i $(i = 1 - po)$, is obtained as follows:

$$a_i k_{ir}^{1*} + \tfrac{3}{2} a_i a_j a_k b_{ijkr}^{1*} - \omega^{*2} a_i m_{ir}^{1*} = 0, \quad r = 1, \ldots, po. \tag{23}$$

The effects of neglecting the in-plane displacement on the non-linear vibration behaviour of clamped immovable circular plates will be discussed throughout this paper.

2.2. Numerical details for the clamped immovable circular plate

The chosen out-of-plane basic functions $w_i^*(r^*)$ for a clamped axisymmetric circular plate are given by [20]

$$w_i^*(r^*) = A_i \left[\mathrm{J}_0(\beta_i r^*) - \frac{\mathrm{J}_0(\beta_i)}{\mathrm{I}_0(\beta_i)} \mathrm{I}_0(\beta_i r^*) \right], \tag{24}$$

where β_i is the ith real positive root of the transcendental equation

$$\mathrm{J}_1(\beta)\mathrm{I}_0(\beta) + \mathrm{J}_0(\beta)\mathrm{I}_1(\beta) = 0 \tag{25}$$

Here J_n and I_n are, respectively, the Bessel and the modified Bessel functions of the first kind and of order n. The parameter β_i is related to the ith non-dimensional linear frequency parameter $(\omega_\ell^*)_i$ of the plate by

$$\beta_i^2 = (\omega_\ell^*)_i. \tag{26}$$

Numerical values of β_i are computed numerically by solving Eq. (25) and the first 10 values are given in Table 1.

The chosen in-plane basic functions $u_i^*(r^*)$ for an immovable axisymmetric circular plate are given by [13]

$$u_i^*(r^*) = B_i \mathrm{J}_1(\alpha_i r^*), \tag{27}$$

where α_i is the ith real positive root of the equation $\mathrm{J}_1(\alpha) = 0$, from which the first 10 numerical values of α_j are computed and are listed in Table 1. The functions $w_i^*(r)$ and $u_i^*(r)$ are normalized in such a manner that

$$m_{ij}^{1*} = \int_0^1 w_i^* w_j^* r^* \, \mathrm{d}r^* = \delta_{ij},$$

$$m_{ij}^{2*} = \int_0^1 u_i^* u_j^* r^* \, \mathrm{d}r^* = \delta_{ij}. \tag{28}$$

The first six out-of-plane basic functions w_i^* are shown in Ref. [20] and the in-plane basic functions u_i^* ($i = 1$–6) are given in Fig. 2.

The parameters k_{ij}^{1*}, k_{ij}^{2*}, c_{ijk}^* and b_{ijkl}^{1*} involved in the model were computed numerically by using Simpson's rule with 160 steps in the range [0,1].

Table 1
Numerical values of the clamped immovable circular plate parameters α_i and β_i intervening in the expressions of the in- and out-of-plane basic functions, respectively, for $i = 1$–10

i	α_i	β_i
1	3.83170597020751	3.19622061658254
2	7.01558666981562	6.30643704768842
3	10.17346813506272	9.43949913787641
4	13.32369193631422	12.57713064043065
5	16.47063005087763	15.71643852680748
6	19.61585851046824	18.85654552222951
7	22.76008438059277	21.99709515760648
8	25.90367208761838	25.13791540603749
9	29.04682853491686	28.27891310951825
10	32.18967991097441	31.42003344728260

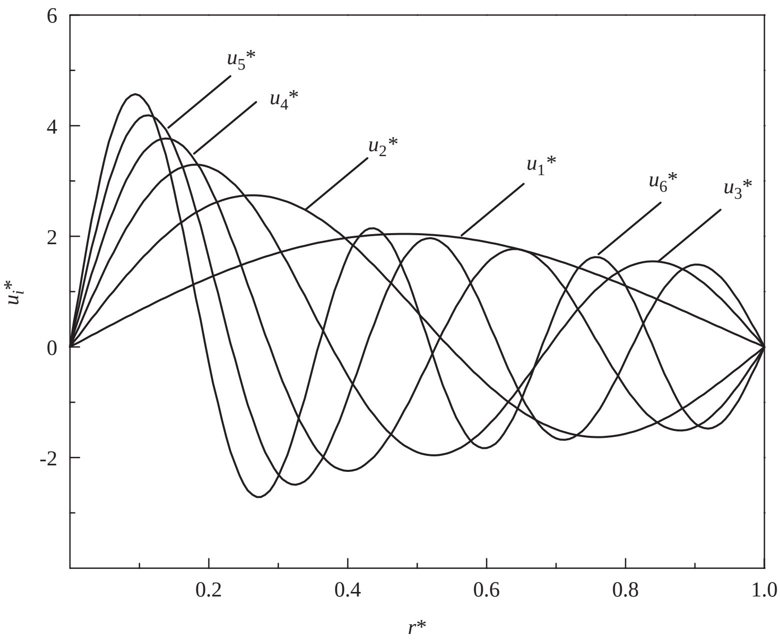

Fig. 2. The first six in-plane basic functions of the clamped immovable circular plate.

3. Methods of solution

The set of non-linear algebraic equations (21), also called the amplitude equation, can be written in a matrix form as

$$([\mathbf{K}^{1*}] + [\mathbf{K}nl^*])\{\mathbf{A}\} - \omega^{*2}[\mathbf{M}^{1*}]\{\mathbf{A}\} = \{\mathbf{0}\}, \tag{29}$$

where $[\mathbf{M}^{1*}]$, $[\mathbf{K}^{1*}]$ and $[\mathbf{K}nl^*]$ are the non-dimensional mass matrix, the non-dimensional linear stiffness matrix and the non-dimensional non-linear geometrical stiffness matrix respectively.

Each term of the matrix $[\mathbf{K}nl^*]$ is a quadratic function of the column matrix of coefficients $\{A\} = [a_1 a_2 \ldots a_{po}]^t$, and is given by $(Knl^*)_{ij} = (3/2)a_k a_l b^*_{ijkl}$. It can be seen that when the non-linear term is neglected, the non-linear eigenvalue problem (29) reduces to the classical eigenvalue problem

$$[\mathbf{K}^{1*}]\{\mathbf{A}\} = \omega^{*2}[\mathbf{M}^{1*}]\{\mathbf{A}\} \tag{30}$$

which is the Rayleigh–Ritz formulation of the linear vibration problem. In the linear case, the eigenvalue equation (30) leads to a series of eigenvalues and corresponding eigenvectors. In the non-linear case, the solution of Eq. (29) should lead to a set of amplitude-dependent eigenvectors, with their amplitude-dependent associated eigenvalues. In this paper, in order to determine the r_0th non-linear axisymmetric mode shape of the clamped immovable circular plate, two methods of solution are used, which are discussed in the next two subsections.

3.1. Iterative method of solution

The iterative method of solution adopted here is that used in Refs. [21,22,34] for fully clamped isotropic and laminated rectangular plates. This method consists of solving successive linear eigenvalue problems by starting from the linear eigenvalue problem (30) until the convergence of the value of the eigenvalue ω^{*2} is achieved, leading also to the normalized eigenvector $\{A\}$, corresponding to the mode considered, according to the specified amplitude of vibration considered. It is to be noted that the non-linear stiffness matrix $[Knl^*]$ is calculated in each iteration from the scaled eigenvector according to the specified amplitude of vibration obtained at the centre of the circular plate. For further details on this numerical iterative procedure, the reader is referred to Refs. [21,22].

For the r_0th ($r_0 = 1$ or 2) non-linear axisymmetric mode and for a given amplitude of vibration w^*_{max}, the numerical iterative procedure determines accurately the non-dimensional non-linear frequency parameter ω^* and the corresponding normalized eigenvector $\{A\}$, which in turn gives the r_0th non-linear axisymmetric mode shape: $w^*(r^*) = a_i w^*_i(r^*)$, $i = 1 - po$. The corresponding in-plane shape function, i.e., $u^*(r^*) = b_i u^*_i(r^*)$, $i = 1 - pi$, is determined by computing the in-plane contribution coefficients b_i from Eq. (20). Also the associated non-linear bending and membrane stresses can be determined quite easily (see Appendix A for stress details).

3.2. Explicit analytical solution

The purpose here is to replace the iterative method of solution of the set of non-linear algebraic equations (21), necessary to obtain the clamped circular plate non-linear axisymmetric mode shapes and associated non-linear resonant frequencies at large vibration amplitudes, by an explicit solution, which may be appropriate for engineering purposes, or for further analytical investigations. The single mode assumption, which consists of neglecting all of the co-ordinates except the single resonant co-ordinate, has been widely used in the study of geometrically non-linear vibration of structures. This is due to the great simplifications it introduces in the theory on one hand, and on the other hand because the error it induces in the estimation of the non-linear frequency remains very small for a wide range of vibration amplitudes [2]. However, the single mode approach is insufficient because it does not give any information about the amplitude dependence of the structure deflection shape, with its practically important effect on the strain and stress distributions, which are quantities of crucial importance with respect to the structural fatigue life prediction [35]. The explicit method of solution remedies this insufficiency by maintaining the multimode character of the solution, and also takes advantage of the accuracy of the single mode approach concerning the amplitude frequency dependence. Thus, rather than neglecting the contributions of all of the non-resonant modes as is the case in the single mode approach, these contributions are regarded as small compared to the r_0th resonant mode contribution a_{r_0}, and denoted in what follows as ε_i $(i \neq r_0)$. To illustrate the method, the fundamental non-linear mode shape is considered here by taking $r_0 = 1$. The analysis for the higher non-linear modes would proceed similarly. A less constraining assumption, compared to the single mode approach, is made by neglecting in the expression $a_i a_j a_k b^*_{ijkr}$, appearing in Eq. (21), the first, second and third order terms with respect to ε_i, i.e. terms of the type $a_1^2 \varepsilon_k b^*_{11kr}$ or of the type $a_1 \varepsilon_j \varepsilon_k b^*_{1jkr}$ or of the type $\varepsilon_i \varepsilon_j \varepsilon_k b^*_{ijkr}$, so that the only remaining term is $a_1^3 b^*_{111r}$. Thus, Eq. (21) becomes

$$a_i k^{1*}_{ir} + \tfrac{3}{2} a_1^3 b^*_{111r} - \omega^{*2} a_i m^{1*}_{ir} = 0, \quad r = 1, \ldots, po \tag{31}$$

in which the repeated index i is summed over the range $[1, po]$. Since the use of linear clamped circular plate mode shapes as basic functions leads to diagonal mass and rigidity matrices, the above system can be rewritten as

$$a_r k^{1*}_{rr} + \tfrac{3}{2} a_1^3 b^*_{111r} - \omega^{*2} a_r m^{1*}_{rr} = 0, \quad r = 1, \ldots, po \tag{32}$$

in which no summation is involved. The first equation of this set corresponding to $r = 1$ leads to

$$\omega^{*2} = \frac{k^{1*}_{11}}{m^{1*}_{11}} + \frac{3}{2}\frac{b^*_{1111}}{m^{1*}_{11}} a_1^2 \tag{33}$$

which is the same formula as that obtained in Ref. [2] by using the single mode approach and the harmonic balance method to the problem of geometrically non-linear beam vibrations. It is to be noted here that formula (33) is slightly different from that used in Part I of this series of papers, based on the principle of conservation of energy, due to the presence of the factor 3/2. Such a difference, which affects slightly the non-linear frequency estimates, has been encountered previously in similar non-linear problems, and discussed for example in Ref. [3]. The $(po - 1)$ remaining equations can be written as

$$(k^{1*}_{rr} - \omega^{*2} m^{1*}_{rr})\varepsilon_r = -\tfrac{3}{2} a_1^3 b^*_{111r}, \quad r = 2, \ldots, po \tag{34}$$

which permits one to obtain explicitly the unknown basic function contributions $\varepsilon_2, \ldots, \varepsilon_{po}$ in terms of the predominant first basic function contribution a_1 as follows:

$$\varepsilon_r = -\frac{3 a_1^3 b^*_{111r}}{2(k^{1*}_{rr} - \omega^{*2} m^{1*}_{rr})}, \quad r = 2, \ldots, po. \tag{35}$$

Inserting Eq. (33) into Eq. (35) and recalling that in the case considered here, the mass matrix is identical to the identity matrix, Eq. (35) may be simplified to

$$\varepsilon_r = \frac{3 a_1^3 b^*_{111r}}{2(k^{1*}_{11} + \frac{3}{2} a_1^2 b^*_{1111} - k^{1*}_{rr})}, \quad r = 2, \ldots, po. \tag{36}$$

Expression (36) is an explicit simple formula, allowing direct calculation of the higher mode contributions to the first non-linear clamped circular plate mode shape, as functions of the predominant first mode contribution a_1 and the known parameters k^{1*}_{rr} and b^*_{111r}. Thus, for each value of the amplitude coefficient a_1, corresponding to a given maximum non-dimensional vibration amplitude, the first non-linear amplitude-dependent clamped circular plate mode shape, $w^*_{nl1}(r^*, a_1)$, can be defined as a series involving the clamped circular plate modal parameters, depending on the sufficient number po of axisymmetric clamped circular plate functions $w^*_1, w^*_2, \ldots, w^*_{po}$

$$w^*_{nl1}(r^*, a_1) = a_1 w^*_1(r^*) + \sum_{r=2}^{po} \frac{3a_1^3 b^*_{111r}}{2(k^{1*}_{11} + \frac{3}{2} a_1^2 b^*_{1111} - k^{1*}_{rr})} w^*_r(r^*) \tag{37}$$

in which the predominant term, proportional to the first linear mode shape, is $a_1 w^*_1(r^*)$, and the other terms, proportional to the higher linear mode shapes $w^*_2(r^*), \ldots, w^*_{po}(r^*)$, are the corrections due to the non-linearity.

This amplitude-dependent mode shape permits the determination of the distribution of the associated non-linear bending stress of the clamped immovable circular plate. To determine the distribution of the associated membrane stress, the in-plane displacement coefficients b_i $(i = 1 - pi)$ have to be determined. Since the main concern here is to seek approximate explicit solutions, two simplified formulations for the determination of the b_i's coefficients with a quite small number of structural modal parameters are presented in the following:

- *First formulation*

  As a first approximation for determination of the b_i's coefficients, one can proceed as above by neglecting in the expression $a_i a_j d^*_{ijk}$ appearing in Eq. (20) both first- and second order terms with respect to ε_i, so that the in-plane contribution coefficients are simply given by

$$b_i = a_1^2 d^*_{11i}, \quad i = 1 - pi. \tag{38}$$

  In this case, the in-plane shape function $u^*(r^*)$ is directly given by the following series:

$$u^*(r^*) = a_1^2 d^*_{11i} u^*_i(r^*) \tag{39}$$

  in which a summation is made over the repeated index i.
- *Second formulation*

  An improvement could be made to Eq. (38) by adding the first order term $a_1 \varepsilon_l d^*_{1li}$ $(l > 1)$, so that the in-plane basic function contribution coefficients are given by

$$b_i = a_1^2 d^*_{11i} + \sum_{l=2}^{po} a_1 \varepsilon_l d^*_{1li}, \quad i = 1 - pi. \tag{40}$$

  In terms of the predominant contribution coefficient a_1 and the known modal parameters of the structure, the in-plane shape function is now given by

$$u^*(r^*) = a_1^2 \left[d^*_{11i} + \sum_{r=2}^{po} \frac{3a_1^2 b^*_{111r} d^*_{1ri}}{(2k^{1*}_{11} + 3a_1^2 b^*_{1111} - 2k^{1*}_{rr})} \right] u^*_i(r^*). \tag{41}$$

It will be shown later, that this second formulation improves significantly the membrane stress estimates, for ranges of vibration amplitudes exceeding those permitted by the first formulation, as discussed below. It is also noted here that the range of validity of this method of solution is quite interesting from the engineering point of view, as will be seen when discussing the numerical results. If more accurate results are needed for higher amplitudes, the so-called second formulation developed in Refs. [35–37] for the transverse displacements can be used to extent the range of validity of the results concerning the amplitude-dependent non-linear mode shapes, and the associated non-linear stress distributions.

4. Numerical results and discussion

The non-linear free axisymmetric vibrations of a clamped immovable circular plate have been examined here by using the two methods of solution discussed above. In order to determine the sufficient number of in- and out-of-plane basic functions which achieve a good accuracy for large vibration amplitudes, a convergence study is made by using the iterative method of solution. This permitted determination of the number of basic functions to be adopted in the explicit solution, and also its range of validity.

4.1. Iterative method of solution

4.1.1. Convergence study of the spectral expansion

The convergence study of the spectral expansions used in the model is discussed here for the first and the second non-linear axisymmetric mode shapes. It is to be noted that the convergence criteria should not be restricted to the non-linear frequency, as was the case in Refs. [21,34,38], but must also involve the non-linear bending and membrane stresses, in order to obtain reliable results with respect to engineering purposes. Figs. 3(a)–(c) show the effects of the number of in-plane and out-of-plane basic functions on the non-linear frequency ratios, on the edge surface bending stress and on the edge membrane stress, associated to the first non-linear axisymmetric mode shape, respectively, for a value of the non-dimensional amplitude obtained at the plate centre equal to 1.5. The numerical results presented in this paper are obtained with the Poisson ratio, $v = 0.3$. The influence of the in-plane displacement is clearly seen, which shows that it has to be taken into account in the case of large vibration amplitudes of clamped circular plates, with a minimum number $pi = 2$. If the single mode approach is used ($po = 1$), it can be seen that the frequency ratio may be well approximated, but the non-linear surface bending stress estimates are not accurate. Also from Fig. 3(a), it can be concluded that $pi = po = 3$ is sufficient to obtain accurate results for the non-linear frequency ratio. However, in this case, the membrane and the non-linear bending stress will not be well approximated. Accurate estimates of these stresses is achieved for at least $pi = po = 5$, as can be seen from Figs. 3(b) and (c). In-order to obtain more accurate results for vibration amplitudes up to twice the plate thickness, the number of in-plane and out-of-plane basic function is taken equal to 6 in the remainder of this paper ($pi = po = 6$). For the second non-linear axisymmetric mode shape, the same procedure has shown that the sufficient number of out-of-plane and in-plane basic functions to be used is $po = 8$ and $pi = 8$, respectively.

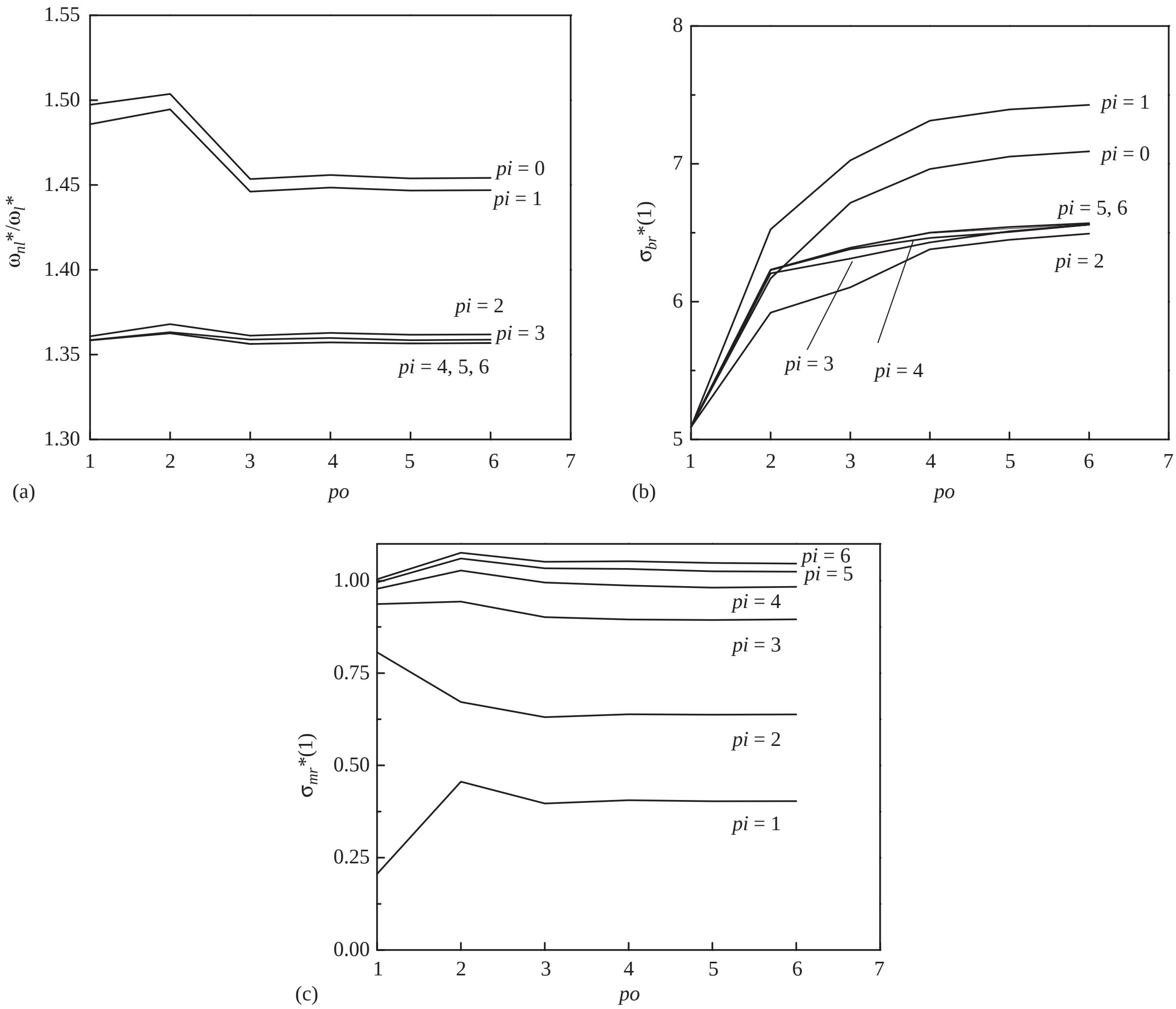

Fig. 3. Effects of the number of in-plane *pi* and out-of-plane *po* basic functions on non-linear frequency ratio (a), non-dimensional radial bending (b) and membrane (c) stresses at the plate edge for the fundamental non-linear mode shape of a clamped immovable circular plate for $w^*_{max} = 1.5$ ($\nu = 0.3$).

4.1.2. Amplitude frequency dependence

Most of the available results in the study of non-linear vibrations of clamped circular plates have been concerned with determination of the so-called back-bone curves, especially for the fundamental non-linear mode shape and only a very little number of references dealt with the higher non-linear mode shapes. The dependence of the non-linear frequency on the non-dimensional vibration amplitude is plotted in Fig. 4, for both the first and second non-linear axisymmetric mode shapes of a clamped immovable circular plate showing a hardening spring effect. The plot shows also that the first non-linear mode shape exhibits less change in frequency with the vibration amplitude than does the second non-linear axisymmetric mode shape. Such a

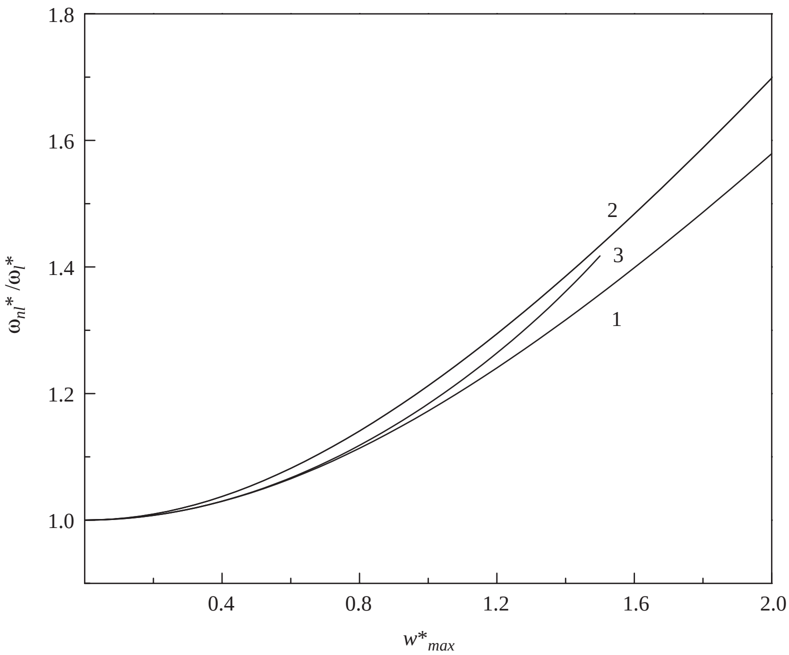

Fig. 4. Effects of large vibration amplitudes on the frequencies of the first (1) and second (2) non-linear axisymmetric mode shapes of a clamped immovable circular plate. Comparison with the explicit analytical solution for the first mode shape (3).

non-linear effect has been mentioned in Ref. [39] for the first two non-linear mode shapes of fully clamped rectangular plates, and was explained by the fact that the deflection shape associated with the first mode shape produces less induced tensile forces than does that associated with the second mode shape for the same maximum displacement amplitude. Since no exact analytical results are available, a detailed comparison has been made in Ref. [20] between all of the results found in the literature during the period 1961–1986 concerning the amplitude–frequency dependence of the first non-linear mode shape. As these results were based on various analytical assumptions and numerical solution techniques, a general comparison was made by calculating the average and the standard deviation of the non-linear frequency estimates obtained by various methods for each amplitude of vibration. Table 2 shows the averaged non-linear fundamental mode frequency ratios from Ref. [20] and the present results. It can be seen that the results of the present work are in good agreement with the averaged ones. In the same table, the fundamental non-linear mode frequency ratios of the clamped immovable circular plate obtained by using the model without the in-plane displacement u are also given. It is noticed that the neglect of the in-plane displacement in the formulation has an effect of increasing the hardening spring behaviour of the clamped circular plate, as may be expected, since this assumption increases the structural rigidity. For example, at an amplitude of vibration equal to once the plate thickness, the difference between the two formulations is 5%, and increases to about 8% for an amplitude equal to twice the thickness. Therefore, the von Kármán type of geometrically non-linear

Table 2
Comparison of non-linear frequency ratios ($\omega^*_{n\ell}/\omega^*_\ell$) of the fundamental mode of a clamped immovable circular plate with and without the in-plane displacement and with the averaged results from Ref. [20]

w^*_{max}	0.2	0.4	0.6	0.8	1.0	1.5	2.0
Averaged Ref. [20]	1.0073	1.0287	1.0632	1.1095	1.1663	1.3429	1.5499
Present work ($pi = po = 6$)	1.0075	1.0296	1.0654	1.1135	1.1724	1.3568	1.5790
Model with w only ($po = 6$)	1.0108	1.0422	1.0917	1.1560	1.2318	1.4542	1.7025

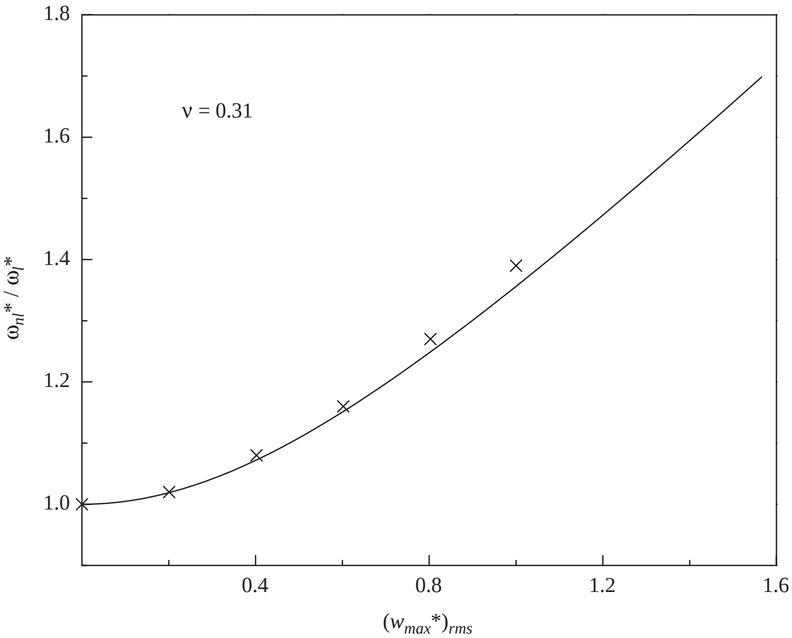

Fig. 5. Comparison of the non-linear frequency ratio of the second non-linear axisymmetric mode shape of a clamped immovable circular plate: (×) values taken from Ref. [40], read from graph; — present work.

strain–displacement relationships including the in-plane displacement u should be used if more accurate estimation of non-linear frequencies of clamped circular plates are needed.

For the second non-linear axisymmetric mode shape, a comparison is made in Fig. 5 between the results obtained here and those obtained in Ref. [40], in which the stress function approach was used and the formulation of the governing equations was based on the Galerkin procedure and the harmonic balance method. The non-linear algebraic equations were solved by using the Newton–Raphson method. The numerical results concerning the effect of the amplitude of vibration on the non-linear frequencies were given by considering the root mean square value (r.m.s.) of the dynamic response at the centre of the plate. It can be seen that the present results are in good agreement with the results of Ref. [40], with a maximum difference of 2.2% for a maximum value of the non-dimensional amplitude of 1.34 corresponding to an r.m.s. value of 1.

The excellent agreement between the non-linear frequencies of the first two non-linear axisymmetric mode shapes of clamped immovable circular plates obtained from the present model and the numerical results published previously demonstrates the usefulness of the non-linear model developed here to analyze the geometrically non-linear free vibration problem of clamped immovable circular plates.

4.1.3. Amplitude dependence of the first and second non-linear axisymmetric mode shapes of clamped immovable circular plates

Previous studies have shown that the mode shapes of beam- and plate-like structures are amplitude dependent [3,4,9,21]. This effect is illustrated in the present case in Figs. 6(a) and (b), in which the normalized non-linear mode shapes of the first two axisymmetric modes of a clamped immovable circular plate are plotted respectively for various values of the maximum non-dimensional amplitudes w^*_{max}. All curves show the amplitude dependence of the first and second axisymmetric non-linear mode shapes and an increase of curvatures near to the clamped edge, which may lead one to expect that the bending stress near to the edge of the plate increases non-linearly with the increase of the vibration amplitude. It can be seen also that the mode shapes become flatter near to the centre of the circular plate with the increase of the vibration amplitude. It is therefore expected that the bending stress near to the plate centre will not increase as much as it does in the linear case. Such a situation has been encountered in Ref. [41] in the case of a clamped–clamped beam, and in Ref. [21] for fully clamped isotropic rectangular plates. All these features will be discussed when analyzing the stresses. The influence of the amplitude of vibration on the normalized in-plane displacement shape functions is also clearly shown in Figs. 7(a)

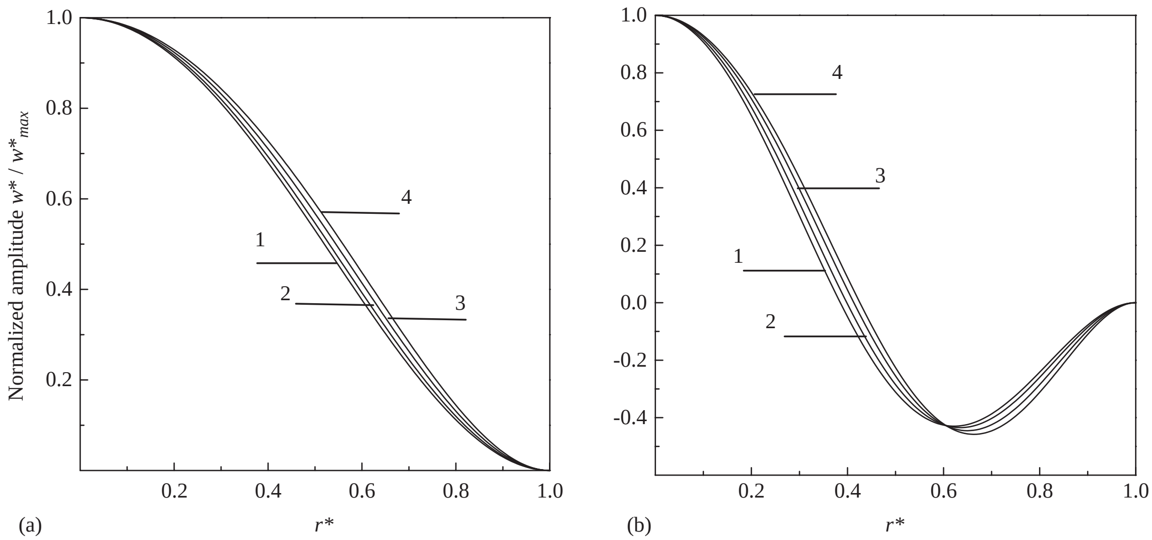

Fig. 6. Normalized radial sections of the first (a) and second (b) non-linear axisymmetric mode shapes of a clamped immovable circular plate at various non-dimensional amplitudes: 1, $w^*_{max} = 0.5$; 2, $w^*_{max} = 1.0$; 3, $w^*_{max} = 1.5$; 4, $w^*_{max} = 2.0$.

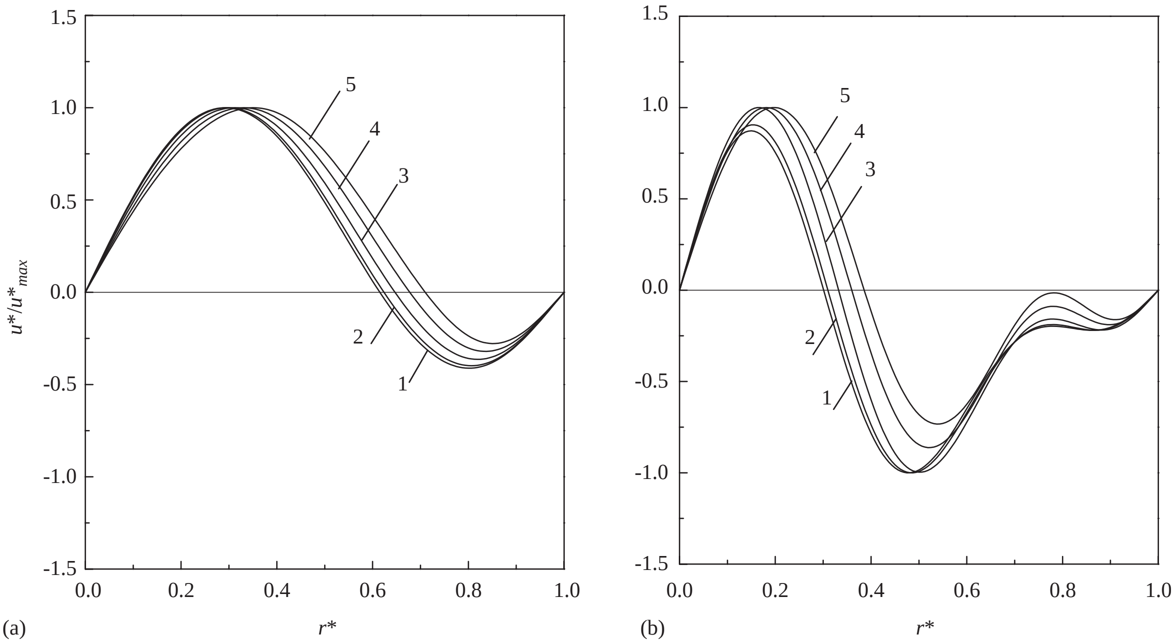

Fig. 7. Normalized in-plane shape functions of the first (a) and second (b) non-linear axisymmetric modes of a clamped immovable circular plate at various non-dimensional amplitudes: 1, $w^*_{max} = 0.001$; 2, $w^*_{max} = 0.5$; 3, $w^*_{max} = 1.0$; 4, $w^*_{max} = 1.5$; 5, $w^*_{max} = 2.0$.

and (b), associated respectively with the first and second non-linear axisymmetric transverse mode shapes.

To investigate the effects of the in-plane displacement on the fundamental non-linear mode shape of a clamped immovable circular plate, a comparison between the normalized mode shape obtained from the present model with and without the in-plane displacement for a non-dimensional maximum amplitude of 2.0 is made in Fig. 8, in which the fundamental linear mode shape is also plotted. The difference between the non-linear mode shapes generated from the present model with and without the in-plane displacement u is clearly seen especially in the central part of the circular plate. The non-linear mode shape obtained by the model without the in-plane displacement approaches the linear mode shape near to the centre of the plate, and the non-linear mode shape obtained by the model with the in-plane displacement near to the clamped edge. It is therefore expected that the non-linear bending stress estimates can be well approximated near to the edge, but will be over-estimated near to the centre of the plate, when the in-plane radial displacement is neglected.

4.1.4. *Analysis of the radial bending, membrane and total stress distributions associated with the first and second non-linear axisymmetric mode shapes*

As shown previously, the present multimodal model enables not only determination of the amplitude–frequency dependence, but also the deformation of the mode shapes due to the geometrical non-linearity. From the latter result, it was expected that the effect of the amplitude

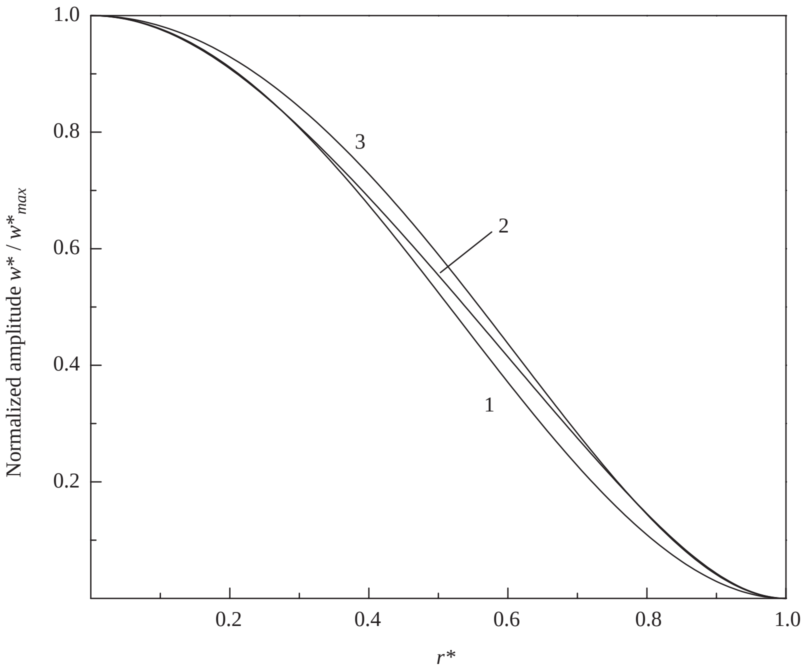

Fig. 8. Comparison of the normalized fundamental linear and non-linear ($w^*_{max} = 2.0$) mode shape from different models: 1, linear; 2, non-linear mode shape from model with w only; 3, non-linear mode shape from model with w and u.

of vibration on the distribution of the associated bending stress would be of greater significance, since the bending stress is related to the derivatives of the amplitude-dependent transverse mode shape. Figs. 9(a) and (b), in which the non-dimensional radial bending stress distributions associated with the first and second non-linear axisymmetric mode shapes are plotted, respectively, for various values of the vibration amplitude, show the amplitude dependence of the bending stress distribution. It can be seen also from Figs. 10 and 11 that the non-linear bending stress exhibits a higher increase near to the clamped edge, compared with that expected in linear theory, but behaves in an opposite manner near to the plate centre. Figs. 10(a) and (b) show that the results obtained here for the non-dimensional surface radial bending stress associated with the first and second axisymmetric non-linear mode shapes at the clamped edge of the plate increase very rapidly with the increase of the vibration amplitude. The rate of increase in the radial bending stress is about twice the rate of increase expected in linear theory for the second mode, and about 1.8 for the first mode, when the maximum non-dimensional amplitude increases from 1.0 to 2.0. Figs. 12(a) and (b) display the radial membrane stress results associated respectively with the first and second non-linear axisymmetric mode shapes at the centre and at the edge of the circular plate. Examination of these figures shows a rapid increase of the membrane stress with increasing amplitude of vibration, especially at the centre of the plate. The non-dimensional radial membrane and total stress distributions associated with the first and second axisymmetric non-linear mode shapes are plotted in Figs. 13 and 14, respectively, for various values of the non-dimensional vibration amplitude. It can be seen that the membrane

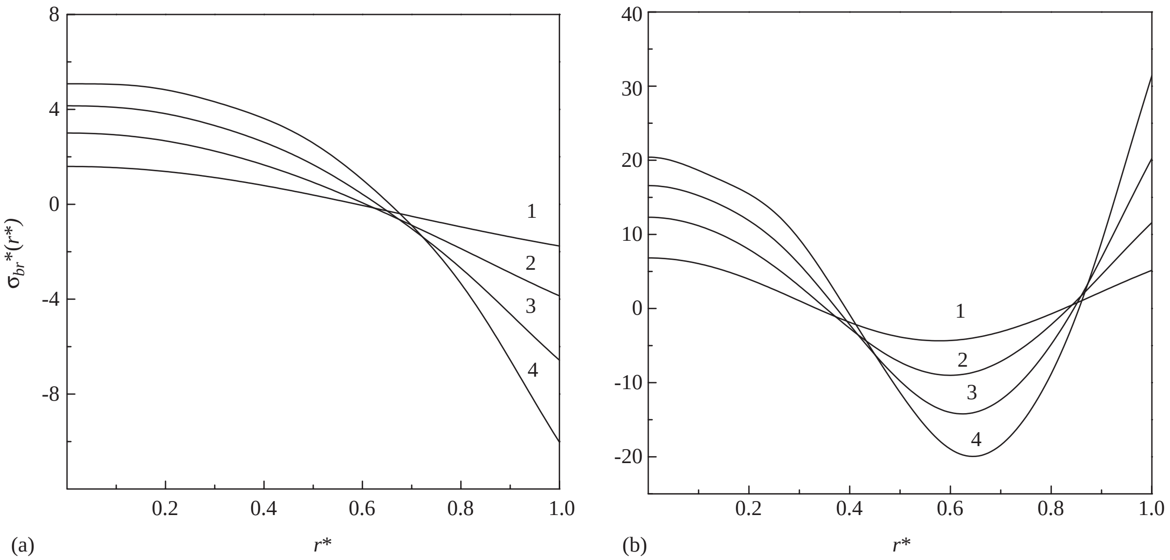

Fig. 9. Non-dimensional radial bending stress distribution associated with the clamped immovable circular plate first (a) and second (b) non-linear axisymmetric mode shapes at various non-dimensional amplitudes: 1, $w^*_{max} = 0.5$; 2, $w^*_{max} = 1.0$; 3, $w^*_{max} = 1.5$; 4, $w^*_{max} = 2.0$.

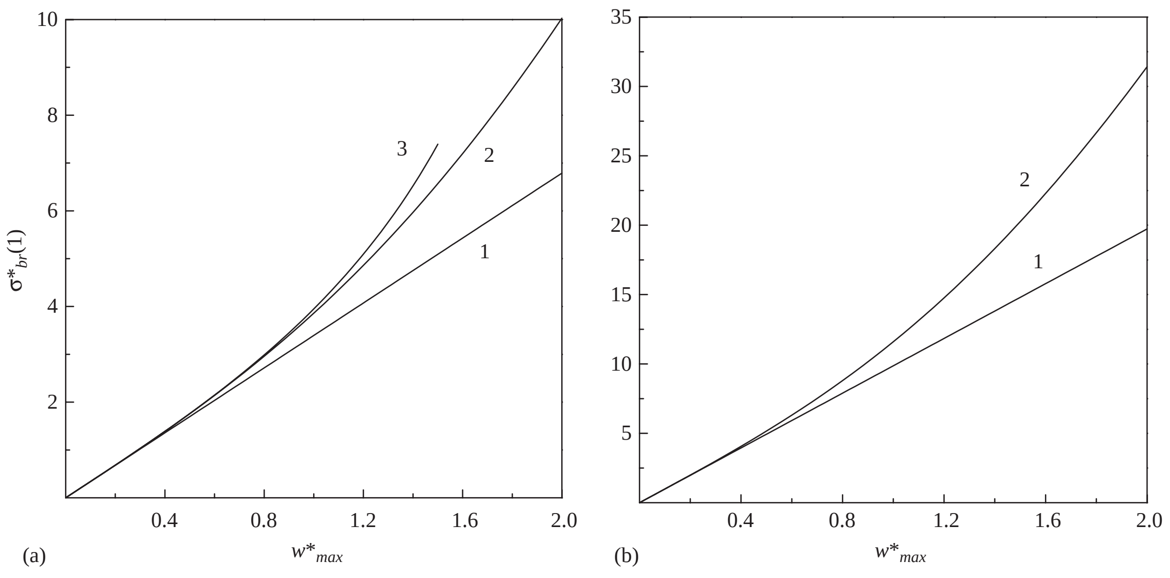

Fig. 10. Effect of large vibration amplitudes on the non-dimensional radial bending stress associated with the first (a) and second (b) non-linear axisymmetric mode shapes at the edge of the clamped immovable circular plate: 1, linear solution; 2, non-linear iterative solution; 3, explicit analytical solution.

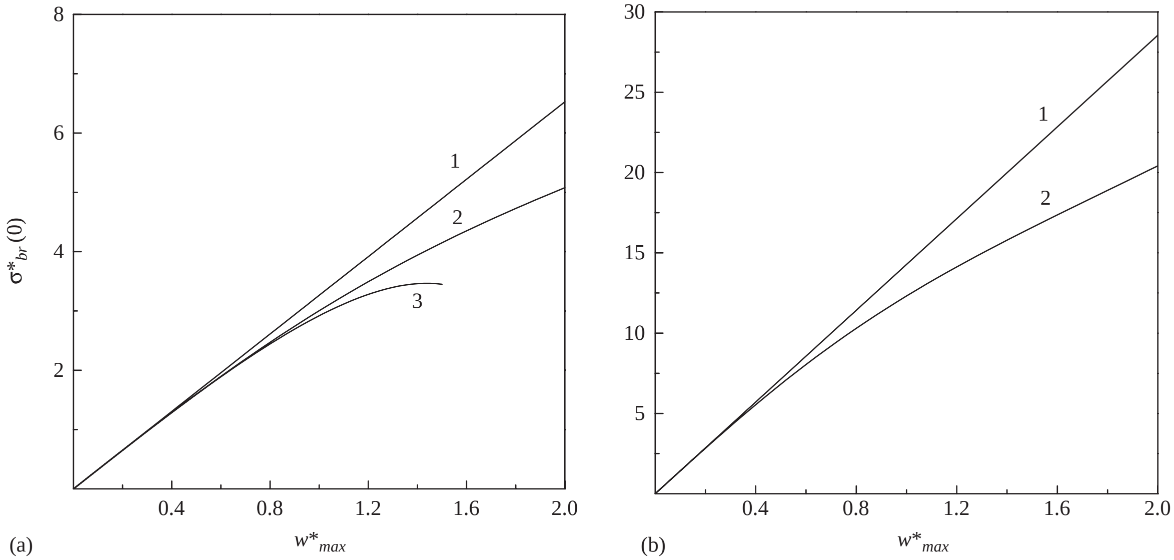

Fig. 11. Effect of large vibration amplitudes on the non-dimensional radial bending stress associated with the first (a) and second (b) non-linear axisymmetric mode shapes at the centre of the clamped immovable circular plate: 1, linear solution; 2, non-linear iterative solution; 3, explicit analytical solution.

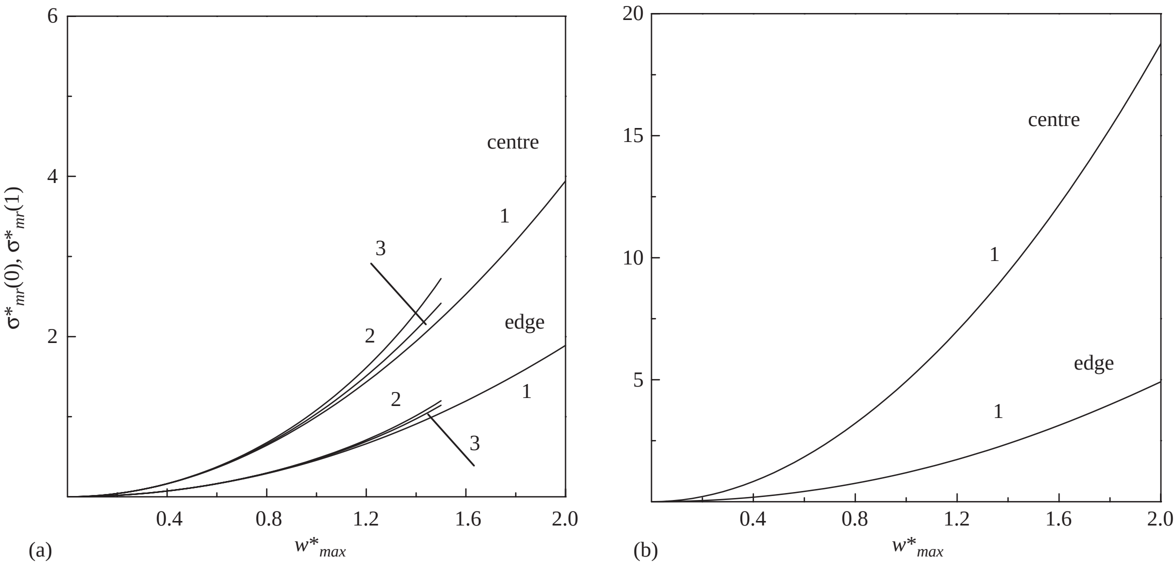

Fig. 12. Effect of large vibration amplitudes on the non-dimensional radial membrane stress associated with the first (a) and second (b) non-linear axisymmetric mode shapes at the centre and at the edge of a clamped immovable circular plate: 1, iterative solution; 2, explicit analytical solution: first formulation; 3, explicit analytical solution: second formulation.

stress can be neglected at small vibration amplitudes. For example, the maximum membrane stress, obtained at the centre of the plate, for a maximum non-dimensional amplitude $w^*_{max} = 0.5$ is only about 6.4% of the membrane stress at the same location at $w^*_{max} = 2.0$ for the first

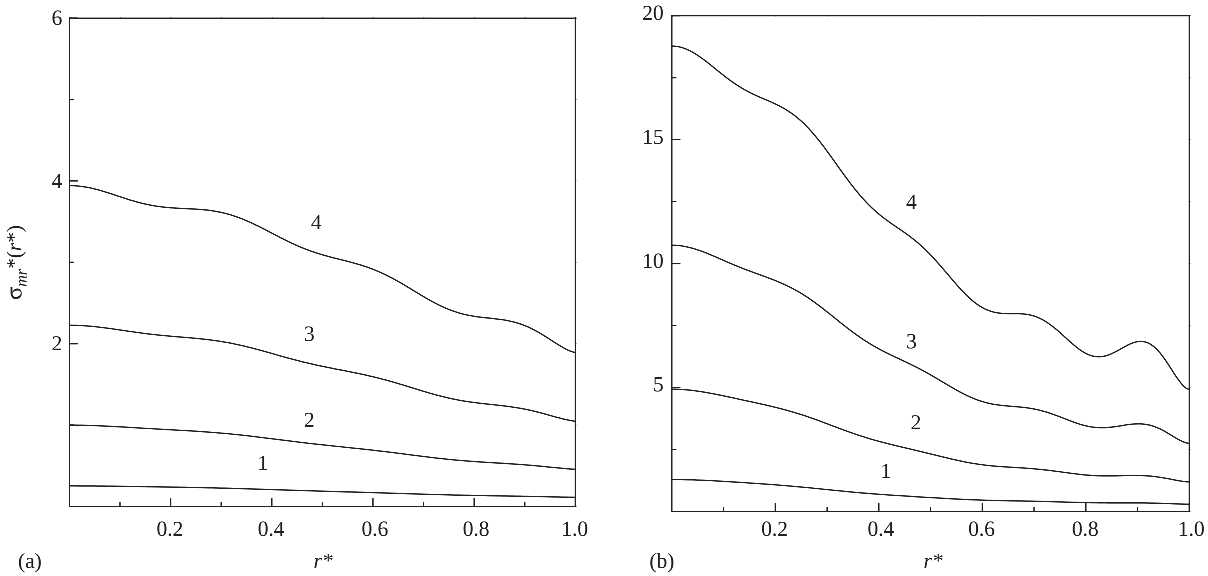

Fig. 13. Non-dimensional radial membrane stress distribution associated with the clamped immovable circular plate first (a) and second (b) non-linear axisymmetric mode shapes for various non-dimensional amplitudes: 1, $w^*_{max} = 0.5$; 2, $w^*_{max} = 1.0$; 3, $w^*_{max} = 1.5$; 4, $w^*_{max} = 2.0$.

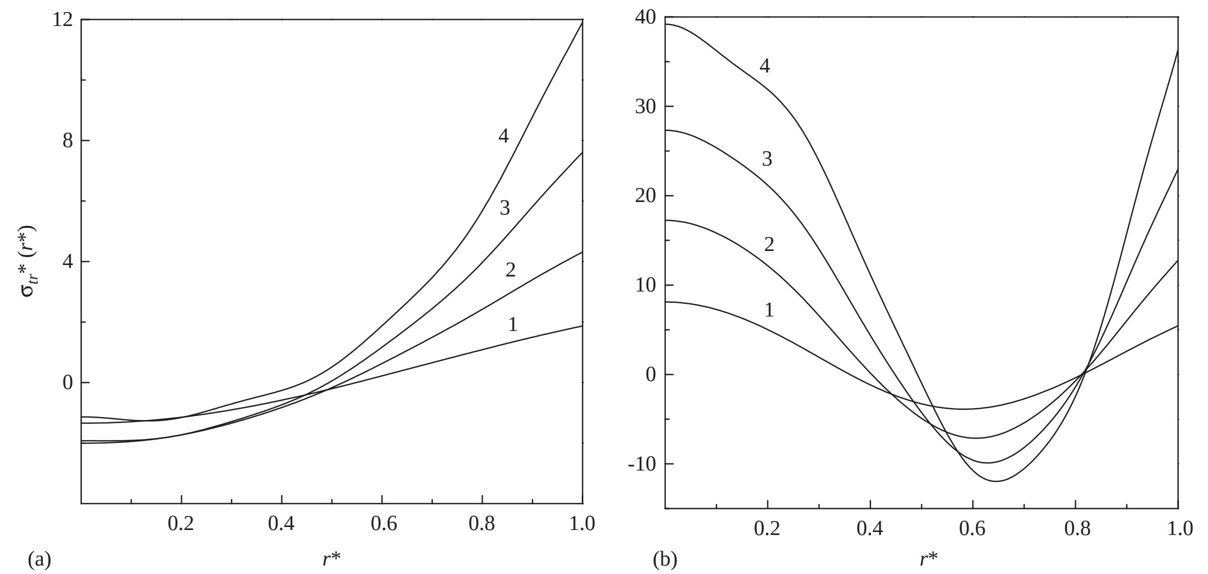

Fig. 14. Non-dimensional radial total stress distribution associated with the clamped immovable circular plate first (a) and second (b) non-linear axisymmetric mode shapes at various non-dimensional amplitudes: 1, $w^*_{max} = 0.5$; 2, $w^*_{max} = 1.0$; 3, $w^*_{max} = 1.5$; 4, $w^*_{max} = 2.0$.

non-linear mode shape and less than 7% for the second non-linear axisymmetric mode shape. Furthermore, the maximum membrane stress at an amplitude of twice the plate thickness is about 33% of the maximum total stress for the first non-linear mode shape and about 48% for the second non-linear axisymmetric mode shape. This indicates that the membrane stress is very important in stress analysis and should not be neglected in engineering design of large deflected structures.

It is to be noted here that the present non-linear iterative results for the bending and membrane stresses depicted in Figs. 10–12 corresponding to the clamped immovable circular plate fundamental non-linear mode shape coincide exactly with those obtained in Ref. [10], in which von Kármán equations and the Kantorovich method were used, and results were obtained by solving numerically a two-point boundary value problem. However, the present model, which leads to numerical solution of a non-linear eigenvalue problem, is quite interesting due to its simplicity.

4.2. Explicit analytical solution

The purpose of this section is to replace the iterative method of solution of the set of non-linear algebraic equations (21), necessary to obtain the clamped circular plate non-linear axisymmetric mode shapes and non-linear resonance frequencies at large vibration amplitudes, by the explicit approximate solution presented in Section 3.2. A comparison is then made between the two solutions, i.e. iterative and analytical, in order to determine the range of validity of the new approach. In this paper, the investigation is restricted to the fundamental non-linear mode shape of the clamped immovable circular plate. The numerical values of the clamped immovable circular plate modal parameters required in the analysis are given in Appendix B. These modal parameters are computed by using a number of in- and out-of-plane basic functions equal to six.

The amplitude–frequency relationship is given in terms of the preponderant contribution coefficient a_1 of the first out-of-plane basic function by

$$\frac{\omega^*_{n\ell}}{\omega^*_{\ell}} = [1 + 4.10905a_1^2]^{1/2}. \tag{42}$$

If only one out-of-plane basic function is used, Eq. (42) can be rewritten as

$$\frac{\omega^*_{n\ell}}{\omega^*_{\ell}} = [1 + 0.375638(w^*_{max})^2]^{1/2}, \tag{43}$$

where w^*_{max} is related here to a_1 by $w^*_{max} = a_1(w^*_1)_{max}$, with $(w^*_1)_{max} = 3.307396$. Eq. (43) differs slightly from that obtained in Ref. [42] for which the coefficient of $(w^*_{max})^2$ was 0.35344. This is due to the different choices of the spatial shape function. However, it is thought that the present solution is more accurate since it is based on the exact linear mode shape, instead of the shape function used in Ref. [42], i.e. $w(r) = h[1 - (r/a)^2]^2$, which was shown in Ref. [5] to yield a fundamental linear frequency 1% larger than the exact one. It can be seen from Fig. 4, in which the amplitude–frequency dependence obtained by the single mode equation (43) is also plotted, that the explicit method gives a good estimate of the non-linear frequency parameter ω^* for maximum plate displacement amplitudes up to the plate thickness, with a percentage error below 1% compared with the exact one given by the iterative method.

The first non-linear amplitude-dependent clamped circular plate mode shape $w^*_{n\ell 1}(r^*, a_1)$, for a maximum non-dimensional amplitude w^*_{max} corresponding to a given value of the contribution coefficient a_1, is defined as a series involving the first six axisymmetric clamped circular plate out-of-plane basic functions by

$$\begin{aligned} w^*_{n\ell 1}(r^*, a_1) = a_1 w^*_1(r^*) &+ \frac{375.570 a_1^3}{(104.363 + 428.833 a_1^2 - 1581.744)} w^*_2(r^*) \\ &- \frac{131.088 a_1^3}{(104.363 + 428.833 a_1^2 - 7939.547)} w^*_3(r^*) \\ &+ \frac{111.438 a_1^3}{(104.363 + 428.833 a_1^2 - 25022.239)} w^*_4(r^*) \\ &- \frac{92.122 a_1^3}{(104.363 + 428.833 a_1^2 - 61012.166)} w^*_5(r^*) \\ &+ \frac{73.639 a_1^3}{(104.363 + 428.833 a_1^2 - 126429.553)} w^*_6(r^*). \end{aligned} \tag{44}$$

The resulting in-plane shape function $u^*(r^*, a_1)$, for a maximum non-dimensional amplitude w^*_{max} corresponding to a given value of the contribution coefficient a_1, is defined as a series involving the first six in-plane basic functions. In the case of the first formulation for the in-plane basic function contribution coefficients, i.e., Eq. (38), the in-plane shape function is explicitly given by

$$\begin{aligned} u^*(r^*, a_1) = a_1^2[&0.1681 u^*_1(r^*) + 0.4514 u^*_2(r^*) \\ &- 0.0401 u^*_3(r^*) + 0.0097 u^*_4(r^*) \\ &- 0.0033 u^*_5(r^*) + 0.0014 u^*_6(r^*)]. \end{aligned} \tag{45}$$

Explicit analytical expressions for the non-dimensional bending and membrane stresses can also be determined quite easily by replacing the explicit formula (44) for the transverse non-linear mode shape and the corresponding ones for the in-plane shape function, i.e., Eq. (45) or its equivalent based on the second formulation, in the expressions for the stresses given in Appendix A. The analytical method of solution preserves the overall trends of the membrane and bending stress changes with the increase of the amplitude of vibration as may be seen in Figs. 10(a), 11(a) and 12(a), in which comparisons are made between the stresses at the centre and at the edge of the plate obtained by the iterative and explicit methods of solutions. This represents an important improvement, compared to the single mode approach solution for which the bending and membrane stresses are respectively linear and parabolic functions of the amplitude of vibration, as obtained at the centre of a clamped circular plate for example in Ref. [8].

The explicit analytical method of solution applied here to the fundamental non-linear mode shape of a clamped immovable circular plate seems to be very appropriate for the analysis of geometrically non-linear free vibration problems. To have an accurate conclusion concerning the limit of validity of the explicit solution in engineering applications, a criterion was adopted, as in

Refs. [20,35–37], based on the effects of the assumptions made regarding physical quantities, such as the non-linear frequency and the maximum membrane and bending stresses obtained respectively at the centre and at the edge of the circular plate. It was found, as may be seen in Figs. 4 and 10(a), that for vibration amplitudes up to the plate thickness, the error induced by the approximate explicit solution does not exceed 1% for the non-linear frequency, and is about 2.32% for the maximum bending stress. It appears from these results that the non-linear frequency and the bending stress estimates are well approximated for amplitudes up to the thickness of the plate. Concerning the membrane stress results, it can be seen from Fig. 12(a) that the second formulation gives more accurate estimates. For example, at an amplitude of vibration equal to the plate thickness, the error induced for the maximum membrane stress when using the first formulation is about 8.35%, but this error does not exceed 3.7% when the second formulation is used. In engineering design of large deflected structures, the maximum total stress is more important than the individual maximum membrane and bending stresses. In Fig. 15 the maximum total stress obtained at the clamped edge of the plate by different methods is plotted against the maximum non-dimensional amplitude of vibration. It can be seen that the two formulations of the explicit analytical method give identical results and are in good agreement with the iterative method of solution results for a wide range of vibration amplitudes. For example, the error induced in the maximum total stress estimates is only about 2.5%, for a non-dimensional amplitude of vibration equal to the plate thickness.

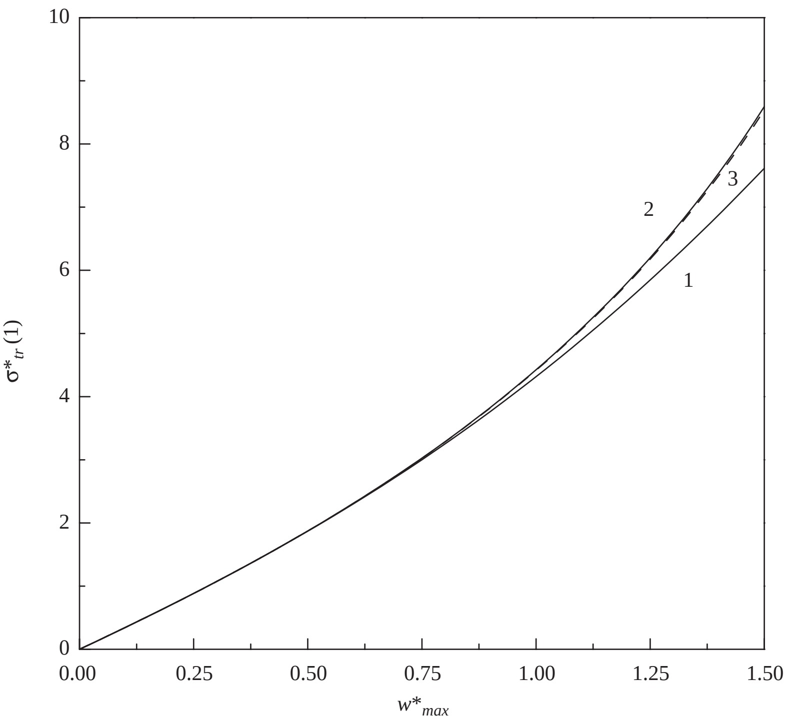

Fig. 15. Comparison of the maximum non-dimensional radial total stress associated with the fundamental non-linear mode shape of a clamped immovable circular plate obtained by: 1, iterative solution; 2, explicit analytical solution: first formulation; 3, explicit analytical solution: second formulation.

5. Conclusions

The non-linear axisymmetric free vibration of a clamped immovable thin isotropic circular plate has been examined theoretically in order to determine the effects of large vibration amplitudes on the first and second axisymmetric mode shapes and their corresponding natural frequencies and associated membrane and bending stress distributions. The governing equations have been derived by assuming transverse harmonic motion and using Hamilton's principle and spectral analysis. When the in-plane inertia is neglected, the theory reduces the non-linear free vibration problem to solution of a set of non-linear algebraic equations, in terms of the contribution coefficients of the transverse displacement only. This set, which represents a non-linear eigenvalue problem, is solved by using an iterative numerical procedure. The validity of the present formulation has been established through comparison of the present results with existing alternative solutions. Accurate non-linear resonant frequency estimates have been obtained for a wide range of vibration amplitudes. The bending and membrane stresses are presented and may be useful in predicting the fatigue life of clamped immovable circular plates undergoing large amplitude vibrations. The stress analysis has shown the inadequacy of the linear theory (or the single mode approach) and also the importance of the membrane stress.

The proposed explicit analytical method of solution, which is expected to be very useful in engineering applications and in further analytical developments, has been applied here in order to obtain analytical results concerning the fundamental non-linear mode shape of a clamped immovable circular plate. The numerical results obtained by the explicit method show that the approximate theory is as accurate as the iterative solution at least up to the plate thickness, and is better than the single mode approach, since it yields also accurate amplitude-dependent mode shapes and distributions of the associated membrane and bending stresses.

Appendix A. Stress expressions

The present model including the in-plane displacement in the formulation, allows one to determine the non-linear bending, membrane and total stresses as opposed to the simplified formulation neglecting the in-plane deformation for which only the bending stress can be calculated with an acceptable accuracy [20].

The non-dimensional surface radial bending stress σ^*_{br} can be defined by

$$\sigma^*_{br} = -\frac{1}{2(1-\nu^2)}\left[\left(\frac{\mathrm{d}^2 w^*}{\mathrm{d}r^{*2}}\right) + \nu\left(\frac{1}{r^*}\frac{\mathrm{d}w^*}{\mathrm{d}r^*}\right)\right] \tag{A.1}$$

and the non-dimensional radial membrane stress σ^*_{mr} is defined by

$$\sigma^*_{mr} = \frac{1}{1-\nu^2}\left[\frac{\mathrm{d}u^*}{\mathrm{d}r^*} + \frac{1}{2}\left(\frac{\mathrm{d}w^*}{\mathrm{d}r^*}\right)^2 + \nu\frac{u^*}{r^*}\right]. \tag{A.2}$$

The relationship between the dimensional and non-dimensional bending and membrane stresses is

$$\sigma^* = \frac{\sigma a^2}{Eh^2}. \tag{A.3}$$

The non-dimensional maximum radial total stress is defined as

$$\sigma^*_{tr} = \sigma^*_{mr} \pm \sigma^*_{br}. \tag{A.4}$$

The appropriate sign is chosen depending on the location of the maximum total stress on the upper or down surfaces of the circular plate.

Appendix B. Modal parameters used in the explicit analytical method for the fundamental mode of a clamped immovable circular plate

B.1. Numerical values of the modal parameters k^{1*}_{ii} and b^*_{111i}

i	k^{1*}_{ii}	b^*_{111i}
1	104.36311062	285.88891499
2	1581.74415330	250.38026291
3	7939.54664268	−87.39203901
4	25022.23904511	74.29229956
5	61012.16607457	−61.41480222
6	126429.55253003	49.09297163

B.2. Numerical values of the modal parameters d^*_{1ij}

$$[d^*_{1ij}] = \begin{bmatrix} 0.16809881 & 0.45136661 & -0.04012269 & 0.00971334 & -0.00329169 & 0.00136092 \\ -1.60520282 & 0.54701817 & 0.63662388 & -0.07867129 & 0.02381642 & -0.00946948 \\ -0.78285908 & -1.60714541 & 0.60086572 & 0.72891760 & -0.10233632 & 0.03456412 \\ 0.54309456 & -0.45263477 & -1.55806294 & 0.61821792 & 0.78385332 & -0.11755908 \\ -0.43543641 & 0.27857495 & -0.35552524 & -1.52053028 & 0.62595505 & 0.82034889 \\ 0.36663070 & -0.21630309 & 0.19664790 & -0.31282160 & -1.49347459 & 0.63007159 \end{bmatrix}.$$

Appendix C. Nomenclature

r, θ, z	cylindrical co-ordinates
U, W	in-plane and out-of-plane displacements of the middle plane point (r, θ) respectively
$\varepsilon_r, \varepsilon_\theta$	radial and circumferential strains
V_b, V_m, V	bending, membrane and total strain energy respectively
E	Young's modulus
v	the Poisson ratio of the plate material
ρ	mass per unit volume of the plate material
a, h	radius and thickness of the circular plate respectively

D	bending stiffness of the plate, $D = Eh^3/(12(1-\nu^2))$
T	kinetic energy
$w(r)$	transverse shape function, $W(r,t) = w(r)\cos(\omega t)$
$u(r)$	in-plane shape function, $U(r,t) = u(r)\cos^2(\omega t)$
a_i, b_i	contribution coefficient of the ith out-of- and in-plane basic function respectively: $w(r) = a_i w_i(r)$, $u(r) = b_i u_i(r)$
pi, po	number of in- and out-of-plane basic functions respectively
$k^1_{ij}, m^1_{ij}, b^1_{ijkl}$	general terms of the rigidity tensor, the mass tensor and the fourth order non-linearity tensor, respectively, associated with the transverse displacement
k^2_{ij}, m^2_{ij}	general terms of the rigidity tensor and the mass tensor, respectively, associated with the in-plane displacement
c_{ijk}	general term of the third order non-linearity rigidity tensor representing the coupling between the in- and the out-of-plane displacements
b^*_{ijkl}	general term of the fourth order non-linearity rigidity tensor taking into account the influence of the in-plane displacement
d^*_{ijk}	general term of the third order tensor allowing the calculation of the kth in-plane contribution coefficient
$u^*_i(r^*), w^*_i(r^*)$	the ith in- and out-of-plane basic functions respectively
r^*	non-dimensional radial co-ordinate, $r^* = r/a$
λ	thickness to radius ratio of the circular plate, $\lambda = h/a$
ω	non-linear frequency parameter
β_i	the ith transverse eigenvalue parameter for a clamped axisymmetric circular plate
$(\omega^*_\ell)_i$	the ith non-dimensional linear natural frequency of axisymmetric vibrations of clamped circular plates: $(\omega^*_\ell)_i = \beta_i^2$
α_i	the ith in-plane eigenvalue parameter for a clamped immovable axisymmetric circular plate
$\{\mathbf{A}\}$	column matrix of out-of-plane contribution coefficients, $\{\mathbf{A}\} = [a_1\ a_2\ \ldots\ a_{po}]$
$[\mathbf{K}nl^*]$	the non-dimensional non-linear geometrical stiffness matrix
w^*_{max}	maximum non-dimensional vibration amplitude
$\varepsilon_i\ (i \neq 1)$	contribution coefficient of the ith transverse basic function
$w^*_{nl1}(r^*, a_1)$	the first clamped immovable circular plate non-linear mode shape for a given assigned value a_1 of the first basic function
σ^*_{br}	non-dimensional surface radial bending stress
σ^*_{mr}	non-dimensional radial membrane stress
σ^*_{tr}	non-dimensional surface radial total stress
*	star exponent indicates non-dimensional parameters

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