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Stability of an elastic rod on a fractional derivative type of foundation

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Abstract

The lateral vibration of an axially loaded elastic rod positioned on a fractional derivative type of foundation is studied. It is shown that the dynamics of the problem is governed by a system of two coupled linear differential equations with fractional derivatives. For this system of equations the questions of existence, regularity and the stability of solution are analysed. The results are compared with the stability bound for an elastic rod on Winkler (elastic) type of foundation.

1. Introduction

The study of stability of elastic and visco-elastic rods is important for both theoretical and pragmatic reasons. While the theory of stability of elastic rods, as well as visco-elastic rods described by integer time derivatives of the deformation is well developed, there are few studies of the stability of visco-elastic rod described by fractional (or better to say real), derivatives of the deformation (see for example Refs. [1–4]).

Many important structures may be modelled as an elastic axially loaded rod positioned on visco-elastic foundation. A railway track being the primary example [5]. Here the stability of an elastic rod, of finite length, positioned on a visco-elastic foundation of fractional derivative type will be analysed. Thus, the results presented here represent a generalization of the classical problem of determining stability boundary of an elastic rod on elastic foundation (for example Winkler type foundation). It will be shown that the use of visco-elastic foundation for a axially loaded rod produces two effects. Firstly, the foundation has a stabilizing effect, and secondly, it

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leads to dissipation of energy in the lateral motion, thus causing quick disappearance of vibrations. Stability bounds obtained here will be compared with the stability bounds that follow from the classical Euler (static) theory.

Consider a simply supported elastic rod, the equations of transversal vibrations for the case when the influence of axis compressibility is taken into account, are (see Ref. [6] pp. 338, 392)

$$\begin{aligned}
&\frac{\partial H}{\partial S} = \rho_0 \frac{\partial^2 x}{\partial t^2} - q_x, \quad \frac{\partial V}{\partial S} = \rho_0 \frac{\partial^2 y}{\partial t^2} - q_y, \\
&\frac{\partial M}{\partial S} = -V(1+\varepsilon)\cos\vartheta + H(1+\varepsilon)\sin\vartheta - m, \\
&\frac{\partial x}{\partial S} = (1+\varepsilon)\cos\vartheta, \quad \frac{\partial y}{\partial S} = (1+\varepsilon)\sin\vartheta, \\
&\varepsilon = \frac{H\cos\vartheta + V\sin\vartheta}{EA}, \quad \frac{\partial\vartheta}{\partial S} = \frac{M}{EI},
\end{aligned} \tag{1}$$

where x and y are coordinates of an arbitrary point on the rod axis, S is the arc length of the rod axis in the undeformed state so that $S \in (0, L)$ where L is the length of the rod in the undeformed state, t is the time, H and V are components of the force in an arbitrary cross-section of the rod along the $\bar{x}$ and $\bar{y}$ axes of a rectangular Cartesian coordinate system $\bar{x} - B - \bar{y}$, respectively, M is the bending moment, ε is the axial strain of the rod axis, i.e., $\varepsilon = (\mathrm{d}s - \mathrm{d}S)/\mathrm{d}S$, where $\mathrm{d}s$ is the length in the deformed state of an element of the rod axis whose length in the undeformed state is $\mathrm{d}S$, and q_x, q_y and m are the intensities of the distributed forces and couples per unit length of the rod axis in the undeformed state, ϑ is the angle between the tangent to the rod axis and $\bar{x}$ axis, ρ_0 is the line density of the rod, EI is the bending rigidity of the rod and EA is the extensional rigidity of the rod. For the rod shown in Fig. 1 the boundary conditions are

$$\begin{aligned}
&y(0,t) = 0, \quad x(0,t) = 0, \quad y(L,t) = 0, \\
&M(0,t) = 0, \quad M(L,t) = 0, \quad H(L,t) = -F.
\end{aligned} \tag{2}$$

Suppose that the rod is positioned on a visco-elastic foundation. It is assumed that the foundation is of fractional derivative type. Such foundation has importance as railpad in the railway track model (see Ref. [5]). If the foundation is made of fractional type visco-elastic material then the force in the foundation Q and deformation $\varDelta$ of the foundation (in the present case $\varDelta = y$) for a generalized Zener model are connected as (also see Ref. [7])

$$Q + \tau_Q Q^{(\alpha)} = E_p(y + \tau_y y^{(\alpha)}), \tag{3}$$

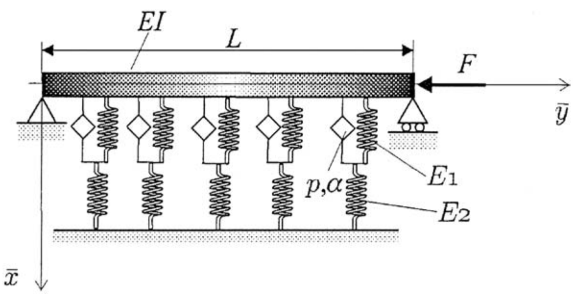

Fig. 1. Coordinate system and load configuration.

with $0<\alpha<1$. In Eq. (3) $(\cdot)^{(\alpha)}$ is used to denote the αth derivative of a function $(\cdot)$ taken in Riemann–Liouville form as (see Refs. [8,9])

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}g(t) = g^{(\alpha)} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g(\xi)\,\mathrm{d}\xi}{(t-\xi)^\alpha} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g(t-\xi)\,\mathrm{d}\xi}{\xi^\alpha}. \tag{4}$$

The dimension of the constants τ_y and τ_Q is [time]$^\alpha$. The constants E_p, τ_Q and τ_y in Eq. (3) are called instantaneous modulus of the pad and the relaxation time of strain and stress, respectively. If one uses a rheological model shown under the rod in Fig. 1 (standard linear solid in which dashpot is replaced by an element called "springpot" characterized by two constants p, and α) then the constants in Eq. (5) are given as (see Ref. [10]): $E_p = E_1E_2/(E_1+E_2)$, $\tau_y = (p/E_2)$, $\tau_Q = p/(E_1+E_2)$. Here E_1 and E_2 are the moduli of elasticity of the springs in both parallel and series connection. It is assumed that the following inequality, as a consequence of the second law of thermodynamics, is satisfied (see Refs. [11,12])

$$E>0, \quad \tau_Q>0, \quad \tau_y>\tau_Q. \tag{5}$$

Suppose that the pads are positioned under the rod continuously so that

$$q_x = 0, \quad q_y = -bQ, \tag{6}$$

where b is a constant depending on the part of the rod's width that is supported by pads. Note that in the case $\alpha = 1$ the foundation becomes a standard visco-elastic solid.

The trivial solution to the system (1), (2), (3) and (5) in which the rod axis is straight reads

$$\begin{aligned}
&H^0(S,t) = -F, \quad V^0(S,t) = 0, \quad M^0(S,t) = 0,\\
&x^0(S,t) = \left(1-\frac{F}{EA}\right)S,\\
&y^0(S,t) = 0, \quad \varepsilon^0(S,t) = -\frac{F}{EA},\\
&\vartheta^0(S,t) = 0, \quad Q^0(S,t) = 0.
\end{aligned} \tag{7}$$

Let the solution to Eqs. (1), (2), (3) and (5) be written in the form $H = H^0 + \Delta H, \ldots, f = f^0 + \Delta f$, where $\Delta H, \ldots, \Delta\vartheta$ are perturbations, assumed to be small. By substituting this in Eq. (8) and neglecting the higher order terms in perturbations $\Delta H, \ldots, \Delta f$

$$\begin{aligned}
&\frac{\partial \Delta H}{\partial S} = \rho_0\frac{\partial^2\Delta x}{\partial t^2}, \quad \frac{\partial \Delta V}{\partial S} = \rho_0\frac{\partial^2\Delta y}{\partial t^2} + b\Delta Q,\\
&\frac{\partial \Delta M}{\partial S} = -\Delta V\left(1-\frac{F}{EA}\right) - F\left(1-\frac{F}{EA}\right)\Delta\vartheta,\\
&\frac{\partial \Delta x}{\partial S} = 0, \quad \frac{\partial \Delta y}{\partial S} = \left(1-\frac{F}{EA}\right)\Delta\vartheta,\\
&\Delta\varepsilon = \frac{\Delta H}{EA}; \quad \frac{\partial\Delta\vartheta}{\partial S} = \frac{\Delta M}{EI},\\
&\Delta Q + \tau_Q\Delta Q^{(\alpha)} = E_p(\Delta y + \tau_y\Delta y^{(\alpha)}),
\end{aligned} \tag{8}$$

is obtained, subject to

$$\Delta H(l,t)=0,\quad \Delta M(0,t)=0,\quad \Delta M(l,t)=0,$$
$$\Delta x(0,t)=0,\quad \Delta y(0,t)=0,\quad \Delta y(l,t)=0. \tag{9}$$

Introducing the dimensionless quantities

$$\lambda=\frac{Fl^2}{EI},\quad \tau=\frac{t}{\sqrt{\rho_0 l^4/EI}},\quad \mu=\sqrt{\frac{EAl^2}{EI}},\quad u=\frac{\Delta y}{l},$$
$$\xi=\frac{\Delta x}{l},\quad \beta=b\frac{E_p l^4}{EI},\quad q=\frac{\Delta Q}{lE_p},$$
$$\tau_q=\tau_Q\left(\frac{EI}{\rho_0 l^4}\right)^{\alpha/2},\quad \tau_u=\tau_y\left(\frac{EI}{\rho_0 l^4}\right)^{\alpha/2}. \tag{10}$$

from Eqs. (8), (9) it follows that

$$\frac{\partial^4 u}{\partial\xi^4}+\lambda\left(1-\frac{\lambda}{\mu^2}\right)\frac{\partial^2 u}{\partial\xi^2}+\left(1-\frac{\lambda}{\mu^2}\right)^2\frac{\partial^2 u}{\partial\tau^2}+\beta\left(1-\frac{\lambda}{\mu^2}\right)^2 q=0,$$
$$q+\tau_q q^{(\alpha)}=(u+\tau_u u^{(\alpha)}), \tag{11}$$

and

$$u(0,\tau)=0,\quad \frac{\partial^2 u}{\partial\xi^2}(0,\tau)=0,\quad u(1,\tau)=0,\quad \frac{\partial^2 u}{\partial\xi^2}(1,\tau)=0. \tag{12}$$

Note that Eq. $(5)_{2,3}$ imply

$$\tau_u>\tau_F>0. \tag{13}$$

Suppose that the solution to Eqs. (11), (12) is assumed in the form

$$u(\xi,\tau)=T(\tau)\sin n\pi\xi,\quad F(\xi,\tau)=V(\tau)\sin n\pi\xi,\quad n\in\mathbb{N}. \tag{14}$$

Then by substituting Eq. (14) into Eq. (11) and setting n = 1 (the most interesting is the first mode), it follows:

$$T^{(2)}(\tau)+pT(\tau)+\beta V(\tau)=0,$$
$$V(\tau)+\tau_F V^{(\alpha)}(\tau)=T(\tau)+\tau_u T^{(\alpha)}(\tau), \tag{15}$$

where

$$p=\frac{\pi^2\left[\pi^2-\lambda(1-\lambda/\mu^2)\right]}{(1-\lambda/\mu^2)^2}. \tag{16}$$

Let the initial conditions for $T(\tau)$ be given as $T(0)=T_0, T^{(1)}(0)=T_1$. Then from Eq. $(15)_2$ it follows $V(0)=T_0$. Thus in what follows system (15) will be analysed with restriction (13) and the initial conditions $T(0)=T_0, T^{(1)}(0)=T_1, V(0)=T_0$. In the special case when the slenderness ratio of the rod tends to infinity, i.e., $\mu\to\infty$ Eq. (15) corresponds to the case of an inextensible rod (i.e., $EA \to \infty$). In this case

$$p = \pi^2(\pi^2 - \lambda). \tag{17}$$

In what follows either of the values of p given by Eqs. (16) or (17) may be used.

2. Solution to the system (15)

2.1. Laplace transformation of solutions and their properties

To solve Eq. (15) the Laplace transformation technique will be used. Let $\hat{T}(z)$ denote the Laplace transformation of $T(t)$, i.e., $\hat{T}(z) = \mathscr{L}(T)(z) = \int_0^\infty e^{-z\tau} T(\tau)\, d\tau$. Then from Eq. (15) it follows that

$$\begin{aligned} &z^2\hat{T}(z) + p\hat{T}(z) + \beta\hat{V}(z) = T_1 + zT_0, \\ &\tau_q z^\alpha \hat{V}(z) + \hat{V}(z) = \tau_u z^\alpha \hat{T}(z) + \hat{T}(z). \end{aligned} \tag{18}$$

In Eq. (18) z^α is the principal branch of this multiform function. In writing Eq. $(18)_2$ the fact that $\mathscr{L}(T^{(\alpha)})(z) = z^\alpha \mathscr{L}(T)(z) - \left(1/\Gamma(1-\alpha)\int_0^t T(\tau)d\tau/(t-\tau)^\alpha\right)_{t=0}$ is used. The term in parentheses vanish if $T(t)$ is bounded when $t \to 0$ [9]. By solving Eq. (18)

$$\begin{aligned} \hat{T}(z) &= \frac{(\tau_q z^\alpha + 1)(T_1 + T_0 z)}{\Delta}, \\ \hat{V}(z) &= \frac{(\tau_u z^\alpha + 1)(T_1 + zT_0)}{\Delta}, \end{aligned} \tag{19}$$

is obtained, where

$$\Delta = (z^2 + p)(\tau_q z^\alpha + 1) + \beta(\tau_u z^\alpha + 1).$$

Next the zeros of Δ in the complex plane are estimated. First Δ is written as

$$\Delta = (z^2 + A)(\tau_q z^\alpha + 1) - B, \tag{20}$$

where

$$A = p + \beta\frac{\tau_u}{\tau_q}, \quad B = \beta\left(\frac{\tau_u}{\tau_q} - 1\right) > 0, \tag{21}$$

and $B > 0$ because of Eq. (13), $A \neq 0$ and $\beta > 0$. Then $\Delta = 0$ if and only if

$$(z^2 + A)(\tau_q z^\alpha + 1) = B > 0. \tag{22}$$

Let $z = r\exp(i\gamma)$ with $i = \sqrt{-1}$, $\gamma \neq 0$. Then

$$(z^2 + A) = \rho\exp(i\theta), \tag{23}$$

where $\rho \neq 0$ (because of Eq. (22)) with θ arbitrary. From Eq. (22) one obtains

$$(\tau_q z^\alpha + 1) = \frac{B}{\rho}\exp(-i\theta + i2k\pi), \quad k \in \mathbb{N}. \tag{24}$$

Therefore,

$$\begin{aligned} \operatorname{Im}(z^2 + A) &= \operatorname{Im}(z^2) = r^2 \sin(2\gamma) = \rho \sin\theta, \\ \operatorname{Im}(\tau_q z^\alpha + 1) &= \operatorname{Im}(\tau_q z^\alpha) = \tau_q r^\alpha \sin(\alpha\gamma) = -\frac{B}{\rho}\sin\theta. \end{aligned} \tag{25}$$

If $\gamma \neq 0$ from Eq. (25) it follows that

$$\frac{r^{2-\alpha}}{\tau_q}\frac{\sin(2\gamma)}{\sin(\alpha\gamma)} = -\frac{\rho^2}{B} < 0. \tag{26}$$

Eq. (26) implies

$$|\gamma| > \frac{\pi}{2}. \tag{27}$$

If $\gamma = 0$, then from Eq. (25) one has $\theta = 0$. Suppose that there exists $z = r > 0$ such that $\Delta(r) = 0$. Then, from Eqs. (23), (24) we obtain $r^2 + A = \rho$ and $\tau_q r^\alpha + 1 = B/\rho$. Thus, $r^2 = \rho - A > 0$ so that

$$\rho > A. \tag{28}$$

Also, $\tau_q r^\alpha = B/\rho - 1 > 0$ so that

$$\rho < B. \tag{29}$$

Consequently, if there exists $r > 0$ such that $\Delta(r) = 0$, then $A < \rho < B$, or $B - A > 0$. By using Eq. (21) $B - A > 0$ gives $\beta(\tau_u/\tau_q - 1) - p - \beta(\tau_u/\tau_q) > 0$ or

$$\beta + p < 0. \tag{30}$$

Now it is shown that the condition $B - A > 0$ i.e., condition (30) is not only necessary but also sufficient for the existence of $r > 0$ such that $\Delta(r) = 0$. Thus the condition $\Delta(r) = 0$ is written as

$$\begin{aligned} \Delta(r) &= (r^2 + A)(\tau_q r^\alpha + 1) - B \\ &= \tau_q r^{2+\alpha} + r^2 + A\tau_q r^\alpha + A - B = 0. \end{aligned} \tag{31}$$

Let $y_1 = r^2(\tau_q r^\alpha + 1)$, $y_2 = B - A - \tau_q A r^\alpha$. Then $\Delta(r) = 0$ if $y_1 = y_2$. From Fig. 2 it follows that, independently of the sign of A, the curves $y_1(r)$ and $y_2(r)$ intersect at a point $r = r^* > 0$ if $B - A > 0$.

It is easily seen that $B - A = 0$ is necessary and sufficient condition that $r = 0$ is a solution to $\Delta(r) = 0$. Next, this case is analyzed more closely. Suppose that $B - A = 0$. Then by Eq. (31)

$$\Delta(r) = \tau_q r^{2+\alpha} + r^2 + A\tau_q r^\alpha = 0, \tag{32}$$

hence

$$r^\alpha = (\tau_q r^2 + r^{2-\alpha} + A\tau_q) = 0. \tag{33}$$

If $A > 0$, then $r = 0$ is the unique solution to $\Delta(r) = 0$ with $B - A = 0$.

If $A < 0$ suppose that $r > 0$ exists such that $\tau_q r^2 = (-A)\tau_q - r^{2-\alpha}$. By Fig. 3 such r exists. It is denoted by r^*.

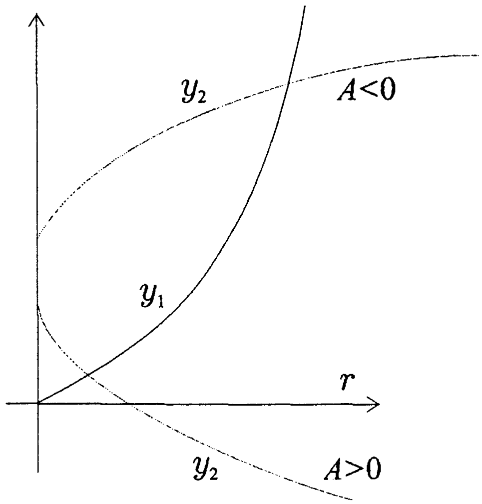

Fig. 2. Solution of Eq. (31).

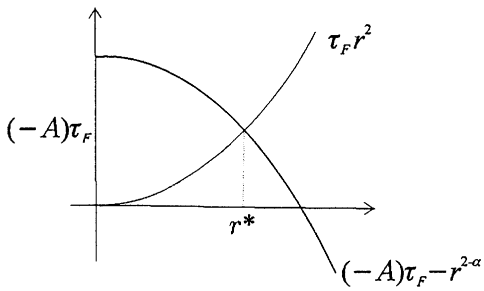

Fig. 3. Solution of $\Delta(r) = 0$ for $A<0$.

From the analysis presented above it follows:

Proposition 2.1. (a) *Let* $z = r\exp(i\gamma)$, $r > 0$, $\gamma \in (-\pi, \pi]$, $\gamma \neq 0$, *be a zero of* $\Delta(z)$. *Then* $|\gamma| > \pi/2$.

(b) $B - A > 0$ $(\beta + p < 0)$ *is a necessary and sufficient condition for the existence of* $r^* > 0$ *such that* $\Delta(r^*) = 0$.

(c) $B - A = 0$ $(\beta + p = 0)$ *is the necessary and sufficient condition that* $r = 0$ *is a solution to* $\Delta(r) = 0$. *If in addition* $A > 0\,(p + \beta(\tau_u/\tau_q) > 0)$, *then* $r = 0$ *is the unique solution to* $\Delta(r) = 0$, *but if* $A < 0\,(p + \beta(\tau_u/\tau_q) < 0)$, *then there exists additionally* $r^* > 0$ *such that* $\Delta(r^*) = 0$.

2.2. *The existence of solutions and the procedure to find them*

It will be shown now that there exist functions $T(t)$ and $V(t)$ such that their Laplace transforms $\hat{T}(z)$ and $\hat{V}(z)$ satisfy Eq. (19). The analysis is present for $T(t)$ only because the method for finding $V(t)$ is just the same.

First $\hat{T}(z)$ given by Eq. $(19)_1$ is written in another form which will allow use of a theorem for the existence of $T(t)$ and which will be more useful in the sequel. Thus, $\hat{T}(z)$ is transformed as follows:

$$\begin{aligned}\hat{T}(z) &= \frac{T_0 z(\tau_q z^\alpha + 1) + T_1(\tau_q z^\alpha + 1)}{\Delta} \\ &= \frac{T_0}{z}\frac{z^2(\tau_q z^\alpha + 1)}{\Delta} + T_1\frac{(\tau_q z^\alpha + 1)}{\Delta} \\ &= \frac{T_0}{z}\left(1 - \frac{\beta(\tau_u z^\alpha + 1) + p(\tau_q z^\alpha + 1)}{\Delta}\right) + T_1\frac{(\tau_q z^\alpha + 1)}{\Delta} \\ &= \frac{T_0}{z} - \frac{T_0}{z}\frac{\beta(\tau_u z^\alpha + 1) + p(\tau_q z^\alpha + 1)}{\Delta} + T_1\frac{z(\tau_q z^\alpha + 1)}{z\Delta} \\ &= \frac{T_0}{z} - \frac{(T_0 p - T_1 z)(\tau_q z^\alpha + 1)}{z\Delta} - \frac{T_0\beta(\tau_u z^\alpha + 1)}{z\Delta} \\ &= \frac{T_0}{z} - \hat{T}_3(z),\end{aligned} \tag{34}$$

where

$$\hat{T}_3(z) = \frac{(T_0 p - T_1 z)(\tau_q z^\alpha + 1)}{z\Delta} + \frac{T_0\beta(\tau_u z^\alpha + 1)}{z\Delta}. \tag{35}$$

It is proved now that every addend in Eq. (34) is the Laplace transform on $\mathbb{R}_+$ of exponential growth by using Ref. [13, I, Chapter 7, Theorem 3]. Consequently, $\hat{T}(z)$ has the same property.

To find $T(t)$ which corresponds to $\hat{T}(z)$ the tables of Laplace transforms could be used, the integral representation

$$T(t) = \lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{x_0 - i\omega}^{x_0 + i\omega}\exp(tz)\hat{T}(z)\,dz, \tag{36}$$

where x_0 belongs to the half plane $\{z; \operatorname{Re} z > \alpha\}$ in which $\hat{T}(z)$ is analytic, the method of residue or to give $T(t)$ by a series. Next, the analytic expression for $T(t)$ is given by a series.

Consider

$$\begin{aligned}\frac{1}{\Delta} &= \frac{1}{(z^2 + A)(\tau_q z^\alpha + 1) - B} \\ &= \frac{1}{(z^2 + A)(\tau_q z^\alpha + 1)}\frac{1}{1 - \frac{B}{(z^2+A)(\tau_q z^\alpha + 1)}} \\ &= \left(\frac{1}{z^2 + A}\right)\left(\frac{1}{\tau_q z^\alpha + 1}\right) \\ &\quad\times\sum_{k=0}^{\infty}\left(\frac{B}{\tau_q}\right)^k\left(\frac{1}{z^2 + A}\right)^k\left(\frac{1}{z^\alpha + 1/\tau_q}\right)^k.\end{aligned} \tag{37}$$

Since

$$\mathscr{L}^{-1}\left(\frac{1}{z^2+A}\right) \equiv S_1(t) = \begin{cases} \dfrac{1}{\sqrt{A}}\sin\sqrt{A}t & \text{if} \quad A>0, \\ \dfrac{1}{\sqrt{-A}}\sinh\sqrt{-A}t & \text{if} \quad A<0, \end{cases}$$

$$\mathscr{L}^{-1}\left(\frac{1}{z^\alpha+1/\tau_q}\right) \equiv S_2(t)$$
$$= \alpha t^{\alpha-1}F_\alpha^{(1)}(z), \quad \text{where } z=-\left(\frac{1}{\tau_q}t^\alpha\right), \tag{38}$$

$F_\alpha(u)$ is Mittag–Leffler's function (cf. Ref. [14]). By Eqs. (37), (38) it follows that

$$\mathscr{L}^{-1}\left(\frac{1}{\Delta}\right) = S_1(t)*S_2(t)*\sum_{k=0}^{\infty}\left(\frac{B}{\tau_q}\right)^k S_1^{*k}(t)*S_2^{*k}(t)$$
$$= S_1(t)*S_2(t)*S_3(t), \tag{39}$$

where

$$S_3(t) = \sum_{k=0}^{\infty}\left(\frac{B}{\tau_q}\right)^k S_1^{*k}(t)*S_2^{*k}(t), \tag{40}$$

and S^{*k} is used to denote $\underbrace{S*\ldots*S}_{k\text{times}}$.

The proof that Eqs. (39), (40) holds can be found in Ref. [1]. Also with the results of this reference the series $S_3(t)$ can be approximated and the approximation's error estimated.

2.3. *Expression for the solution by the $\mathfrak{M}$ transformation*

The result present in this section can be used to find the analytic expressions for $T(t)$ and $V(t)$, and to analyze the asymptotic behaviour of these functions.

For the $\mathfrak{M}$-transformation one can consult [13, II, Kapitel 7]. First it is proved that

$$T(t) = \lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{x_0-i\omega}^{x_0+i\omega}\exp(tz)\hat{T}(z)\,dz$$
$$= \frac{1}{2\pi i}\int_\Sigma \exp(tz)\hat{T}(z)\,dz = \mathfrak{M}(\hat{T})(t). \tag{41}$$

The curve Σ is given in Fig. 4. The point $z_0(\operatorname{Re} z_0>0$ or $z_0=0)$ is a singular point of $\hat{T}(z)$, but $\hat{T}(z)$ is analytic between two curves Σ and $\{z; \operatorname{Re} z = x_0\}$.

By Proposition 2.1 all singularities of the function $\hat{T}(z)$ are of the form $z=0$, $z=r>0$ and $z=r\exp(i\gamma)$, $|\gamma|>\pi/2$. Then there exists a $\delta>0$ such that $\hat{T}_3(z)$ and $\hat{T}(z)$ are analytic on the right of the curve Σ (cf. Fig. 4) which consists of two straight lines having angles $\pi/2+\delta$ and $-(\pi/2+\delta)$ and an arc of a circle with centre in z_0 and of arbitrary small radius.

Let Ω denote the closed domain with the border $\partial\Omega$ consisting of the straight line "1" the arcs "2" and "3" with any $R>0$ and a part of the curve Σ (cf. Fig. 4).

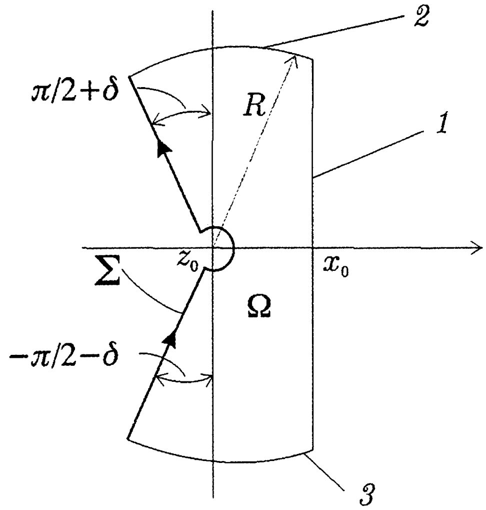

Fig. 4. Integration path for Eq. (41).

It is well known (see Ref. [13, I, p.226]) that

$$\frac{1}{2\pi \mathrm{i}}\int_{\Sigma} \exp(tz) z^{\lambda}\, \mathrm{d}z = \frac{t^{-\lambda-1}}{\Gamma(-\lambda)}, \quad t>0, \quad \lambda \in \mathbb{C}. \tag{42}$$

By using Eq. (42) for $\lambda = -1$, it follows that

$$\frac{1}{2\pi \mathrm{i}}\int_{\Sigma} \exp(tz) z^{-1}\, \mathrm{d}z = 1. \tag{43}$$

Consider now the integral

$$\frac{1}{2\pi \mathrm{i}}\int_{\partial\Omega} \exp(tz) \hat{T}_3(z)\, \mathrm{d}z = 0. \tag{44}$$

By the properties of $\hat{T}_3$, when the radius R of the arcs "2" and "3" tend to infinity, the integral (42) becomes

$$\frac{1}{2\pi \mathrm{i}}\int_{\Sigma} \exp(tz) \hat{T}_3(z)\, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}}\int_{x_0-\mathrm{i}\infty}^{x_0+\mathrm{i}\infty} \exp(tz) \hat{T}_3(z)\, \mathrm{d}z. \tag{45}$$

Now from Eqs. (43) and (45) it follows

$$\begin{aligned} T(t) = \mathscr{L}^{-1}(\hat{T})(t) &= T_0 - \frac{1}{2\pi \mathrm{i}}\int_{x_0-\mathrm{i}\infty}^{x_0+\mathrm{i}\infty} \exp(tz) \hat{T}_3(z)\, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}}\int_{\Sigma} \exp(tz) \hat{T}(z)\, \mathrm{d}z = \mathfrak{M}(\hat{T})(t). \end{aligned} \tag{46}$$

Therefore, the following proposition holds;

Proposition 2.2. *Let $z_0 = r \geqslant 0$ be a singular point of $\hat{T}(z)$ such that for all other singular points z_i of $\hat{T}(z)$ we have* $\mathrm{Re}\, z_i < r$. *Then Eq.* (41) *is true with Σ given in Fig* 4.

3. Asymptotic behaviour of the solution to Eq. (15)

Following Proposition 2.1 in the asymptotic analysis, consider three cases: $\beta + p < 0$, $\beta + p = 0$ with $A < 0$, $\beta + p < 0$ with $A > 0$ and $\beta + p > 0$. The analysis is based on the following theorem.

Theorem (Doetsch [13, II, p. 157]). *Let the function* $f(z)$ *be analytic in the sector* $|\arg(z - z_0)| \leqslant \Psi, \pi/2 < \Psi < \pi$, *of a neighbourhood of* z_0, *except in* z_0. *Let* $f(z)$ *be locally integrable on the rays* $\arg(z - z_0) = \pm\psi$. *The function* $F(t)$ *is given by* $\mathfrak{M}(f)$, *for* $t \geqslant T$, *where* T *is a positive number. Suppose that for a* $\lambda \in \mathbb{C}$ *one has*

$$f(z) \sim M(z - z_0)^{\lambda}, \quad z \to z_0, \text{ uniformly in } |\arg(z - z_0)| \leqslant \Psi, \tag{47}$$

then

$$F(t) \sim M \exp(z_0 t) \frac{t^{-\lambda-1}}{\Gamma(-\lambda)}, \quad t \to \infty. \tag{48}$$

If $\lambda = 0, 1, 2, \ldots$ *we take* $1/\Gamma(-\lambda) = 0$ *and*

$$F(t) = o(\exp(z_0 t) t^{-\lambda-1}). \tag{49}$$

To find the asymptotic behaviour of $T(t)$ the cited Theorem and Proposition 2.2 are used. The following cases will be distinguished:

Case I: Suppose that $\beta + p < 0$ or $\beta + p = 0$ and $p + \beta(\tau_u/\tau_q) < 0$. By Proposition 2.1 there exists and $r^* > 0$ such that $\Delta(r^*) = 0$. Let M denote

$$M = \lim_{z \to r^*} (z - r^*)\hat{T}(z). \tag{50}$$

Then by Eq. (48)

$$T(t) \sim M \exp(r^* t). \tag{51}$$

Case II: Suppose that $\beta + p > 0$. Since $z = 0$ is a point of ramification and $\lim_{z \to 0} \hat{T}(z) = T_1/(p + \beta)$, then by Eq. (49)

$$T(t) = o(t^{-1}), \quad t \to \infty. \tag{52}$$

Case III: Suppose that $\beta + p = 0$ with $A > 0$. Now $\hat{T}(z) \sim T_1 z^{-\alpha}, z \to 0$. Hence

$$T(t) = \sim T_1 \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t \to \infty. \tag{53}$$

The results are summarized as:

Proposition 3.1. *The asymptotic behaviour of the solution* $T(t)$ *to Eq.* (15) *when* $t \to \infty$ *is*

$$\begin{aligned} &T(t) \sim M \exp(r^* t) \textit{ if } \beta + p < 0, \textit{ or } \beta + p = 0 \textit{ with } p + \beta \frac{\tau_u}{\tau_q} < 0, \\ &T(t) \sim o(t^{-1}) \textit{ if } \beta + p > 0, \\ &T(t) \sim T_1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} \textit{ if } \beta + p = 0 \textit{ with } p + \beta \frac{\tau_u}{\tau_q} > 0. \end{aligned} \tag{54}$$

4. Conclusions

In this work a stability analysis of an elastic rod positioned on visco-elastic foundation of fractional derivative type is presented. In order to examine the influence of the visco-elastic foundation on stability note that the rod without foundation, i.e., $b = 0$ in Eq. (6) is stable according to Liapunov stability criteria, when the value of the dimensionless force λ satisfies (see Ref. [6, p. 363])

$$\lambda < \pi^2, \tag{55}$$

while for the case when

$$\lambda > \pi^2, \tag{56}$$

the instability of a rod without foundation is obtained.

For the elastic rod with visco-elastic foundation of the fractional derivative type, the following cases are distinguished:

1. Suppose that Eq. (55) is satisfied so that, from Eq. (17), $p > 0$. Then, since $\beta > 0$ (see Eq. $(10)_6$) in Proposition 3.1 the case $(54)_2$ applies and the rod is asymptotically stable, i.e. the vibrations of the rod die out.

2. Suppose that Eq. (56) is satisfied, i.e., $p < 0$ and that $\beta + p > 0$. Then according to Proposition 3.1 the rod is stable, although without foundation ($\beta = 0$) the rod would be unstable. Thus, the foundation *stabilizes* the rod.

3. If $\beta + p = 0$ and the thermodynamic restriction is satisfied, i.e., $\tau_u > \tau_q$ (see Eq. (13)) then $p + \beta(\tau_u/\tau_q) > 0$ and according to Proposition 3.1 the case $(54)_3$ applies so that the rod is stable.

4.If, as in the previous case, $\beta + p = 0$ but the thermodynamic restriction (see Eq. (13)) is violated, then $\tau_u < \tau_q$ and accordingly $p + \beta(\tau_u/\tau_q) < 0$ and the rod is according to Proposition 3.1 the case $(54)_1$ unstable. The same is true if $\beta + p < 0$ independently of the thermodynamic restriction.

On the basis of the analysis presented here it follows that the value of the sum of parameters p and β determines the stability of the lateral vibration of the rod: if $p + \beta > 0$ the rod is stable and if $p + \beta < 0$ the rod is unstable. The case $p + \beta = 0$ corresponds to stable vibrations if the thermodynamic restriction $\tau_u > \tau_q$ is satisfied, otherwise the vibrations are unstable.

Now compare results obtained here with the results of static stability analysis of an elastic rod on *elastic* foundation. Note that the case $\tau_u = \tau_q$ in Eq. (11) corresponds to elastic foundation. For such a foundation Euler stability criteria leads to the critical force $F_{cr} = (\pi^2 EI/l^2) + \mu l^2/\pi^2$ (see Ref. [15, p. 109]) where μ is, in our notation $\mu = bE_p$. Thus

$$F < \frac{\pi^2 EI}{l^2} + \frac{bE_p l^2}{\pi^2}, \tag{57}$$

guarantees (static) stability of a rod on elastic foundation. The dynamic analysis presented in this paper as the conditions for stability requires $p + \beta > 0$, or $\pi^2(\pi^2 - (Fl^2/EI)) + b(E_p l^4/EI) > 0$. This condition is satisfied if Eq. (57) holds. Thus, the condition for stability of a rod on an elastic foundation can be obtained, as a special case, from the results presented here.

Finally results, summarized in Proposition 3.1 could be applied to determine the stability boundary of the rod without foundation, i.e., for $b = 0$. The case $(54)_1$ for stability requires $p > 0$ and this is equivalent to Eq. (55).

The results presented in this work could be generalized either by analyzing more complicated fractional derivative models as in Refs. [16–18] or other boundary conditions.

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References

[1] B. Stankovic, T.M. Atanackovic, On a model of a visco-elastic rod, *Fractional Calculus and Applied Analysis* 4 (2001) 501–522.

[2] T.M. Atanackovic, B. Stankovic, Dynamics of a visco-elastic rod of fractional derivative type, *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)* 82 (2002) 377–386.

[3] G.-G. Li, Z.-Y. Zhu, C.-J. Cheng, Dynamical stability of viscoelastic column with fractional derivative constitutive relation, *Applied Mathematics and Mechanics* 22 (2001) 294–303.

[4] B. Stankovic, T.M. Atanackovic, Dynamics of a rod made of generalized Kelvin–Voigt material, *Journal of Mathematical Analysis and Applications* 268 (2002) 550–563.

[5] A. Fenander, A fractional derivative railpad model included in a railway track model, *Journal of Sound and Vibration* 212 (1998) 889–903.

[6] T.M. Atanackovic, *Stability Theory of Elastic Rods*, World Scientific, Singapore, 1997.

[7] T. Pritz, Analysis of four-parameter fractional derivative model of real solid materials, *Journal of Sound and Vibration* 195 (1996) 103–115.

[8] K.B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[9] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach, Amsterdam, 1993.

[10] A. Schmidt, L. Gaul, Implementation von Stoffgesetzen mit fraktionalen Ableitungen in die Finite Elemente Methode, *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)* 83 (2003) 26–37.

[11] R.L. Bagley, P.J. Torvik, On the fractional calculus model of viscoelastic behaviour, *Journal of Rheology* 30 (1986) 133–155.

[12] T.M. Atanackovic, A modified Zener model of a viscoelastic body, *Continuum Mechanics and Thermodynamics* 14 (2002) 137–148.

[13] G. Doetsch, *Handbuch der Laplace-Transformation*, Vols. 1 and II, Basel, Birkhäuser, 1950, 1955.

[14] A. Erdélyi, *Higher Transcendental Functions*, McGraw-Hill, New York, 1955.

[15] W.F. Chen, T. Atsuta, *Theory of Beam-Columns I*, McGraw-Hill, New York, 1976.

[16] Yu. Rossikhin, M.V. Shitikova, Analysis of dynamic behavior of viscoelastic rods whose rheological models contain fractional derivatives of two different orders, *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)* 81 (2001) 363–376.

[17] T. Pritz, Five-parameter fractional derivative model for polymeric damping materials, *Journal of Sound and Vibration* 265 (2003) 935–952.

[18] B. Stankovic, T.M. Atanackovic, On a Viscoelastic rod with constituive equation containing fractional derivatives of two different orders, *Mathematics and Mechanics of Solids*, in press.