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Letter to the Editor

Prestressed vibration analysis of a cylindrical shell with an oblique end

X.J. Hu, D. Redekop*

Department of Mechanical Engineering, University of Ottawa, Ottawa, Ont., Canada K1N 6N5

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1. Introduction

Cylindrical shells with oblique ends are encountered in such industrial applications as mitred pipe bends, 'hillside' nozzles in pressure vessels, and diagonal tubular members in offshore rigs. Formulating analytical or semi-analytical solutions for shells of such geometry is a major challenge. While solutions have been presented for the elastostatic [1] and free vibration [2] problems for such shells apparently no solution has yet been given for the problem of prestressed vibration.

In the current study the differential quadrature method (DQM) is applied to the problem of the prestressed vibration of a cylindrical shell with an oblique end. The curved shell surface is developed onto a plane, and blending functions are used to map the geometrically irregular domain onto a square parent domain. The blending functions resemble those used earlier by Malik and Bert [3] for the vibration problem of an irregular plate. The governing equations are solved in the parent domain, and special attention is paid to the boundary conditions on the elliptical oblique end of the shell. Results obtained using the method are compared with results found using the finite element method (FEM).

2. Geometry and boundary conditions

A cylindrical shell is considered having a radius r, mean height L', and thickness h (Fig. 1). The position of a typical point P on the shell mid-surface is given by the physical co-ordinates Y, θ. Displacement components u, v, w (respectively in the axial, circumferential and normal directions), and stress resultants are defined in these physical co-ordinates. The base of the shell lies in a plane perpendicular to the shell axis, while the top lies in a plane that is oblique at an

*Corresponding author. Tel.: +1-613-562-5800x6290; fax: +1-613-562-5177.
E-mail address: dredekop@tesla.cc.uottawa.ca (D. Redekop).

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angle of α with the shell axis. At both the base and top of the shell clamped support conditions are assumed. A uniform external pressure p acts on the surface of the shell.

The shell mid-surface, which in the physical co-ordinate system is irregular, and continuous in the circumferential direction, is first considered developed onto a plane (Fig. 2). In the development process two artificial boundary lines are created at the former $\theta = \pm 180^\circ$ line, disrupting the circumferential continuity. Conditions must be enforced on these two lines, comparable to the continuity conditions existing over the $\theta = \pm 180^\circ$ line in the original shell. Planar co-ordinates X, Y which have dimensions of length are used to describe positions on the developed surface. Non-dimensional co-ordinates for this surface are defined as $\theta = X/r$, $\psi = Y/r$. The governing domain equations for the shell are written in these non-dimensional co-ordinates.

A square parent domain is defined in the natural co-ordinates ξ, η with $-1 \leqslant \xi \leqslant 1$, $-1 \leqslant \eta \leqslant 1$. Blending functions [3] are then employed to develop mapping relations between the natural co-ordinates ξ, η and the developed co-ordinates θ, ψ. The blending functions are given by

$$s = \tfrac{1}{2}[(1-\eta)\bar{s}_1(\xi) + (1+\xi)\bar{s}_2(\eta) + (1+\eta)\bar{s}_3(\xi) + (1-\xi)\bar{s}_4(\eta)] \\ - \tfrac{1}{4}[(1-\xi)(1-\eta)s_1 + (1+\xi)(1-\eta)s_2 + (1+\xi)(1+\eta)s_3 + (1-\xi)(1+\eta)s_4], \qquad (1)$$

where $s = \theta, \psi$. The $\bar{\theta}_i(\xi)$, $\bar{\theta}_i(\eta)$, $\bar{\psi}_i(\xi)$, $\bar{\psi}_i(\eta)$ expressions are the parametric equations for the edges of the developed surface, and the θ_i, ψ_i are the non-dimensional Cartesian co-ordinates of the corner points of the developed surface. Using Eq. (1) the relations between the two sets of

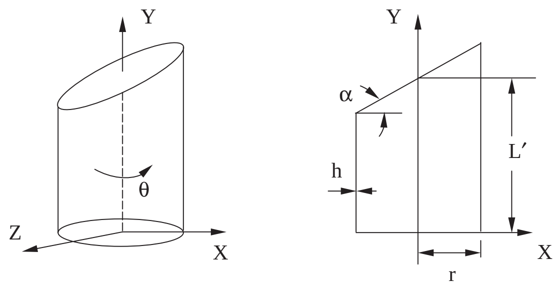

Fig. 1. Circular cylindrical shell with oblique end.

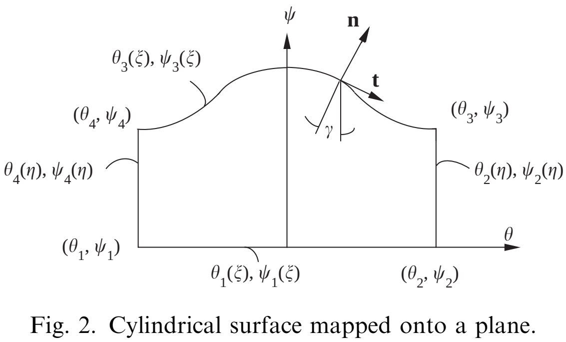

Fig. 2. Cylindrical surface mapped onto a plane.

co-ordinates for the present case are obtained as

$$\theta = \pi\xi, \quad \psi = \zeta(1+\eta), \tag{2}$$

where $\zeta = a + b\cos\phi$, $a = L'/2r$, $b = 0.5\tan\alpha$. The product $\pi\xi$ is represented for clarity by ϕ, and the variables ϕ, η are used subsequently to describe the mapped domain.

Using the chain rule of calculus, the transformation of derivatives from the θ, ψ system to the ϕ, η system can be determined as

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\phi} + \frac{b(1+\eta)\,\sin\phi}{\zeta}\frac{\partial}{\partial\eta}, \quad \frac{\partial}{\partial\psi} = \frac{1}{\zeta}\frac{\partial}{\partial\eta}. \tag{3}$$

These relations correspond to those used by Gill [1] in the static analysis of mitred bends. Relations for higher order and mixed derivatives are readily developed from these basic relations.

For clamped supports at the base ($Y = 0$) the boundary equations are given by

$$u = 0, \quad v = 0, \quad w = 0, \quad \frac{1}{r}\frac{\partial w}{\partial\psi} = 0, \tag{4}$$

where u, v, and w still represent the physical displacement components. For the clamped support conditions at the oblique top of the shell the boundary relations are given as

$$u = 0, \quad v = 0, \quad w = 0, \quad \frac{1}{r}\left(\frac{\partial w}{\partial\theta}\sin\gamma + \frac{\partial w}{\partial\psi}\cos\gamma\right) = 0, \tag{5}$$

where γ is the angle between the normal $\mathbf{n}$ to the shell boundary and the axial co-ordinate line (Fig. 2).

3. Budiansky shell theory

To determine the static stress states and the frequencies of prestressed vibration for the shell the Budiansky shell theory [4] is employed. This theory is an extension of the Sanders linear shell bending theory and is applicable to static, buckling, and prestressed vibration problems. The governing equations are given by

$$K[L]\{U\} + \lambda[\hat{L}]\{U\} + \beta[\bar{L}]\{U\} = \{Q\}, \tag{6}$$

where the symmetric arrays $[L]$ and $[\hat{L}]$ and $[\bar{L}]$ are given by

$$\begin{aligned}
L_{11} &= \frac{\partial^2}{\partial\psi^2} + k_1\frac{\partial^2}{\partial\theta^2}, \quad L_{12} = k_2\frac{\partial^2}{\partial\psi\partial\theta}, \quad L_{13} = v\frac{\partial}{\partial\psi} + k_3\frac{\partial^3}{\partial\psi\partial\theta^2},\\
L_{22} &= k_4\frac{\partial^2}{\partial\psi^2} + k_5\frac{\partial^2}{\partial\theta^2}, \quad L_{23} = \frac{\partial}{\partial\theta} - k_6\frac{\partial^3}{\partial\psi^2\partial\theta} - k\frac{\partial^3}{\partial\theta^3},\\
L_{33} &= 1 + k\left(\frac{\partial^4}{\partial\psi^4} + 2\frac{\partial^4}{\partial\psi^2\partial\theta^2} + \frac{\partial^4}{\partial\theta^4}\right),
\end{aligned} \tag{7}$$

and

$$
\begin{aligned}
\hat{L}_{11} &= n_\psi \frac{\partial^2}{\partial\psi^2} + n_\theta \frac{\partial^2}{\partial\theta^2} + 2n_{\psi\theta} \frac{\partial^2}{\partial\psi\partial\theta}, \\
\hat{L}_{12} &= 0, \quad \hat{L}_{13} = -rp \frac{\partial^2}{\partial\psi}, \\
\hat{L}_{22} &= n_\psi \frac{\partial^2}{\partial\psi^2} + n_\theta \left(\frac{\partial^2}{\partial\theta^2} - 1 \right) + 2n_{\psi\theta} \frac{\partial^2}{\partial\psi\partial\theta} + rp, \\
\hat{L}_{23} &= 2n_\theta \frac{\partial}{\partial\theta} + 2n_{\psi\theta} \frac{\partial}{\partial\psi} + rp \frac{\partial}{\partial\theta}, \\
\hat{L}_{33} &= -n_\psi \frac{\partial^2}{\partial\psi^2} + n_\theta \left(-\frac{\partial^2}{\partial\theta^2} + 1 \right) - 2n_{\psi\theta} \frac{\partial^2}{\partial\psi\partial\theta} - rp,
\end{aligned} \tag{8}
$$

and

$$
\bar{L}_{11} = \bar{L}_{22} = 1, \quad \bar{L}_{33} = -1, \quad \bar{L}_{ij} = 0, \quad i \neq j. \tag{9}
$$

The factor K is given by $K = Eh/(1-v^2)$, where v and E are Poisson's ratio and Young's modulus. The constants k_i are given by

$$
\begin{aligned}
k_1 &= \frac{1-v}{2}\left(1 + \frac{k}{4}\right), \quad k_2 = \frac{1+v}{2} - \frac{1-v}{2}\frac{3k}{4}, \quad k_3 = \frac{1-v}{2}k, \\
k_4 &= \frac{1-v}{2}\left(1 + \frac{9k}{4}\right), \quad k_5 = 1 + k, \quad k_6 = \frac{3-v}{2},
\end{aligned} \tag{10}
$$

where the geometric factor k is defined as $k = \frac{1}{12}(h/r)^2$. The parameter λ is related to the prestressed state. The parameter β is related to the frequency of prestressed vibration ω (rad/s) by $\beta = r^2 \rho h \omega^2$, where ρ is the mass density of the shell. The vector of displacements $\{U\}$ represents $\{u\, v\, w\}^{\mathrm{t}}$, while the vector of loads $\{Q\}$ is given by $\{0\, 0\, r^2 p\}^{\mathrm{t}}$. The quantities n_ψ, n_θ, and $n_{\psi\theta}$ represent, respectively, the axial, circumferential, and in-plane shear membrane stress resultants in the shell.

For the determination of the prestressed vibration stress state relations between the stress resultants and displacement components are required. In the Budiansky theory these relations are given by

$$
n_\psi = \frac{K}{r}\left[\frac{\partial u}{\partial\psi} + v\left(\frac{\partial v}{\partial\theta} + w\right)\right], \quad n_\theta = \frac{K}{r}\left[\frac{\partial v}{\partial\theta} + w + v\frac{\partial u}{\partial\psi}\right], \quad n_{\psi\theta} = K\,\frac{1-v}{2r}\left[\frac{\partial v}{\partial\psi} + \frac{\partial u}{\partial\theta}\right]. \tag{11}
$$

Substitution of the co-ordinate and derivative transfer relations (2)–(3) into Eqs. (6)–(9), leads to domain equations in the parent co-ordinate system. These equations together with the boundary conditions (4)–(5) govern the problem.

4. Differential quadrature method

Following the DQM approach [5,6] a grid of sampling points is first defined in the parent domain. The derivatives which appear in the domain and boundary equations are replaced by linear series involving the displacements at the sampling points of the grid and known weighting coefficients. For the axial (ψ) direction the well known Chebyshev–Gauss–Lobatto spacing of sampling points with δ points is used, and a series of polynomial trial functions is selected. For the circumferential (θ) direction equally spaced sampling points are used, and a series of trigonometric trial functions is selected. Continuity over the artificial boundary lines created in the mapping is then automatically satisfied. Explicit formulas for the weighting coefficients are available for the series in both directions [6].

At each sampling point of the DQM grid either the DQM analogues of the boundary or domain equations are represented. For shells there are four boundary conditions, while there are only three governing equations. One of the boundary conditions is then enforced at an adjacent domain point instead of a domain equation. Such a point, labelled a 'δ point', is taken a short distance ($\delta \cong 10^{-5}$) from the boundary point [5].

A two-step procedure is used for the prestressed vibration analysis. In the first step a static analysis is conducted to find the resultants n_ψ, n_θ, and $n_{\psi\theta}$ for a uniform unit normal pressure. In the use of Eq. (6) the parameters λ and β are set to zero, and the displacements are found for the specified pressure. Resultants are then found from the displacements using Eqs. (11). In the second step, the prestressed vibration analysis, Eq. (6) with λ unity and $\{Q\}$ zero, are solved for β and $\{U\}$. The full details of the procedure are given by Hu [7].

5. Results

Results for the prestressed vibration characteristics of cylindrical shells with oblique ends were not found in the literature. For validation it was decided to rely on comparisons DQM and FEM results for each case. Sample results are given for cases where the length ratio L'/r is varied from 1.5 to 2.5, the thickness ratio r/h from 50 to 200, and the obliquity angle α from 0° to 45°. All results are given for shells with material properties of $E = 0.2\text{e}12$ Pa, $\nu = 0.3$, and $\rho = 7770$ kg/m^2. The value assumed for the shell radius in the analyses was 1 m.

Comparisons for the fundamental frequencies are given in Tables 1 and 2. The results labelled FEM correspond to converged FEM values, found using flat four-noded 24 degree-of-freedom elements. The DQM results correspond to a grid size of 22×22.

In Table 1 results for fundamental frequencies are given for cases of non-zero external pressure. The pressure level p_i, was changed according to the r/h ratio with respective values of $p_1 = -5\text{e}5$ Pa, $p_2 = -2.5\text{e}5$ Pa, $p_3 = -1.25\text{e}5$ Pa used for the $r/h = 50, 100, 200$ cases. These pressures represent significantly large values, which however lie below the shell buckling pressures. The results for all cases indicate very close agreement between the DQM and FEM, with maximum differences of the order of 1%. For each r/h case the fundamental frequencies decrease as the length ratio increases. The frequencies also decrease as the obliquity angle α is increased. The most significant dependence on the obliquity angle occurs for short thin shells.

Table 1
Fundamental frequencies for cases with non-zero prestress (rad/s $\times 10^2$)

L'/r	α	$r/h = 50$ ($p = p_1$)		$r/h = 100$ ($p = p_2$)		$r/h = 200$ ($p = p_3$)	
		FEM	DQM	FEM	DQM	FEM	DQM
1.5	0°	14.16	14.14	10.14	10.10	6.436	6.382
	15°	13.12	13.12	9.150	9.152	5.614	5.582
	30°	11.87	11.89	8.148	8.155	4.749	4.757
	45°	10.86	10.64	7.125	7.142	3.874	3.904
2.0	0°	10.83	10.81	7.498	7.471	4.392	4.352
	15°	10.29	10.30	7.071	7.080	3.949	3.959
	30°	9.612	9.618	6.502	6.505	3.443	3.446
	45°	8.869	8.872	5.863	5.868	2.855	2.869
2.5	0°	8.761	8.746	6.092	6.073	3.080	3.052
	15°	8.467	8.464	5.738	5.732	2.844	2.832
	30°	8.052	8.050	5.365	5.361	2.482	2.480
	45°	7.583	7.580	4.939	4.938	2.041	2.050

Table 2
Comparison of fundamental frequencies for cases of internal pressure, zero pressure, external pressure for shells with $r/h = 100$, $L'/r = 2.0$ (rad/s $\times 10^2$)

α	Internal pressure		Zero pressure		External pressure	
	$p = 2.5e5$ Pa		$p = 0$		$p = -2.5e5$ Pa	
	FEM	DQM	FEM	DQM	FEM	DQM
0°	8.826	8.777	8.189	8.150	7.498	7.471
15°	8.302	8.317	7.740	7.735	7.071	7.080
30°	7.749	7.723	7.174	7.161	6.502	6.505
45°	7.115	7.085	6.541	6.528	5.863	5.868

In Table 2 results for fundamental frequencies are given for internal pressure, zero pressure, and external pressure loading cases. The geometric cases of $r/h = 100$, $L'/r = 2.0$, $\alpha = 0°, 15°, 30°, 45°$ are covered. Application of an internal pressure leads to an increase in the fundamental frequency with respect to the zero pressure case, while application of an external pressure leads to a decrease. These trends agree with those mentioned by Redekop et al. [8] in relation to a pressurized toroidal panel.

6. Conclusion

The method using blending functions discussed herein has been successfully applied to the prestressed vibration problem of a cylindrical shell with an oblique end. The method is useful in parametric studies, and results obtained show excellent agreement with finite element results. Further work is planned, involving additional shells of irregular shape.

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