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Letter to the Editor

Wave propagation analysis of frame structures using the spectral element method

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1. Introduction

Frame structures are representative of basic elements of large-scale structure systems and bridges, for example. When the objective of a large-scale space structure is considered, it is assumed that the dynamic load affects both the structure under construction and the rendezvous docking of the spacecraft. Flexural, longitudinal, and torsional waves applied on such a frame structure propagate through structural members and then reflect at ends or junctions; they finally form standing waves. These propagating waves are observed to constitute the entire dynamic response. Dynamic response analysis and stress analysis become a significant problem when strength evaluation is considered in design.

Doyle proposed a spectral element method using the fast Fourier transform (FFT) to analyze wave propagation in frame structures [1,2]. In this method, the geometrically uniform member can be replaced with only one spectral beam element; this ultimately reduces the total number of degrees of freedom in the system. In addition, the propagation of flexural, longitudinal, and torsional waves in members is accurately expressed because the exact solution of beam, rod, and torsion bars can be used on the frequency domain. However, Doyle solved problems mainly on infinite or semi-infinite beams because the use of the FFT always provides periodicity.

This paper presents a new spectral element method. In this method, the Laplace transform is applied instead of the FFT, thereby avoiding the problem of periodicity. By this proposed method, 3-D frame structures with finite-length beams can be treated practically.

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2. Spectral beam element stiffness matrix

The fundamental equation is derived on a flexural wave of a beam. Consider an elastic beam with a uniform cross-section subjected to dynamic forces at two ends, as shown in Fig. 1. It is also assumed that beam deformation depends on the Bernoulli–Euler beam theory. An equilibrium equation and boundary conditions at the two ends are obtained as the following:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

$$EI\left(\frac{\partial^3 w}{\partial x^3}\right)_1 = N_1, \quad -EI\left(\frac{\partial^2 w}{\partial x^2}\right)_1 = M_1, \quad -EI\left(\frac{\partial^3 w}{\partial x^3}\right)_2 = N_2, \quad EI\left(\frac{\partial^2 w}{\partial x^2}\right)_2 = M_2. \tag{2}$$

Here, E, I, ρ, A, and L are, respectively, Young's modulus, the second-moment inertia of cross-section, the mass density, the area of the cross-section, and the length of beam; w is the transversal displacement, which is the function of co-ordinate x and time t; N_1 and N_2 are external shear forces at the left end and right end; M_1 and M_2 are external moments at the left end and right end; subscripts 1 and 2 denote values at the left end and right end of the beam, respectively.

Next, the Laplace transform is applied to both sides of equilibrium in Eq. (1). In the Laplace transform, the time derivative is replaced with Laplacian operator s. Therefore, Eq. (1) is rewritten as

$$EI\frac{\partial^4 \hat{w}}{\partial x^4} + s^2\rho A\hat{w} = 0. \tag{3}$$

Here, ^ denotes that the function is Laplace transformed. Eq. (3) is a differential equation of order 4 with the co-ordinate x. The general solution of this equation can be expressed using arbitrary constants $C_1, \ldots, C_4$ as

$$\hat{w}(x) = C_1 e^{-ik_b x} + C_2 e^{-k_b x} + C_3 e^{-ik_b(L-x)} + C_4 e^{-k_b(L-x)}, \tag{4}$$

where

$$k_b = \frac{(1+i)}{\sqrt{2}}\sqrt{s}\left(\frac{\rho A}{EI}\right)^{1/4}. \tag{5}$$

From the displacement $\hat{w}(x)$, nodal displacements may be determined as

$$\hat{w}(0) = \hat{w}_1, \quad \frac{\partial \hat{w}}{\partial x}(0) = \hat{\phi}_1, \quad \hat{w}(L) = \hat{w}_2, \quad \frac{\partial \hat{w}}{\partial x}(L) = \hat{\phi}_2. \tag{6}$$

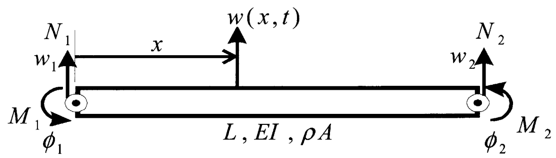

Fig. 1. Beam element, with uniform cross-section, subjected to dynamic forces at two ends.

From Eqs. (4) and (6), one can solve for constants $C_1, \ldots, C_4$ in terms of nodal displacement as

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = [\mathbf{B}] \begin{Bmatrix} \hat{w}_1 \\ \hat{\phi}_1 \\ \hat{w}_2 \\ \hat{\phi}_2 \end{Bmatrix}. \tag{7}$$

Matrix $[\mathbf{B}]$ is so complicated that details are omitted here. Considering Eqs. (7) and (4), the displacement $\hat{w}(x)$ is written in terms of nodal displacements as

$$\hat{w}(x) = [\mathrm{e}^{-\mathrm{i}k_b x} \;\; \mathrm{e}^{-k_b x} \;\; \mathrm{e}^{-\mathrm{i}k_b(L-x)} \;\; \mathrm{e}^{-k_b(L-x)}][\mathbf{B}] \begin{Bmatrix} \hat{w}_1 \\ \hat{\phi}_1 \\ \hat{w}_2 \\ \hat{\phi}_2 \end{Bmatrix}. \tag{8}$$

Substituting the above equation into boundary condition of Eq. (2) and expressing the results in a matrix form yields

$$\begin{Bmatrix} \hat{N}_1 \\ \hat{M}_1 \\ \hat{N}_2 \\ \hat{M}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} \alpha & \bar{\gamma}L & -\bar{\alpha} & \gamma L \\ & \beta L^2 & -\gamma L & \bar{\beta}L^2 \\ & & \alpha & -\bar{\gamma}L \\ sym & & & \beta L^2 \end{bmatrix} \begin{Bmatrix} \hat{w}_1 \\ \hat{\phi}_1 \\ \hat{w}_2 \\ \hat{\phi}_2 \end{Bmatrix} \equiv [\mathbf{K}_b(s)] \begin{Bmatrix} \hat{w}_1 \\ \hat{\phi}_1 \\ \hat{w}_2 \\ \hat{\phi}_2 \end{Bmatrix}, \tag{9}$$

where

$$\begin{aligned} &\alpha = (CSh + SCh)(k_bL)^3/det, \quad \bar{\alpha} = (S + Sh)(k_bL)^3/det, \\ &\beta = (-CSh + SCh)(k_bL)/det, \quad \bar{\beta} = (-S + Sh)(k_bL)/det, \\ &\gamma = (-C + Ch)(k_bL)^2/det, \quad \bar{\gamma} = SSh(k_bL)^2/det, \\ &det \equiv 1 - CCh, \quad C \equiv \cos k_bL, \quad S \equiv \sin k_bL, \\ &Ch \equiv \cosh k_bL, \quad Sh \equiv \sinh k_bL. \end{aligned}$$

Eq. (9) is the fundamental equation on the flexural wave in the beam on the Laplace domain; $[\mathbf{K}_b(s)]$ is a stiffness matrix of the spectral beam element.

3. Spectral rod element stiffness matrix and spectral torsion bar element stiffness matrix

Fig. 2 shows a rod subjected to dynamic forces at two ends. The fundamental equation on the longitudinal wave in a rod element is obtained in a way similar to that in the case of the beam element. As a result, the fundamental equation is expressed in matrix form as

$$\begin{Bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{Bmatrix} = \frac{EA}{L}\frac{kL}{\sin kL}\begin{bmatrix} \cos kL & -1 \\ -1 & \cos kL \end{bmatrix}\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix} \equiv [\mathbf{K}_r(s)]\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}, \tag{10}$$

where

$$k_r = \mathrm{i}s\sqrt{\frac{\rho A}{EA}}, \tag{11}$$

and $[\mathbf{K}_r(s)]$ is a stiffness matrix of the spectral rod element on the Laplace domain.

Fig. 3 shows nodal moments and rotations for a torsional bar element. The fundamental equation on the torsional wave in a bar element is obtained as

$$\begin{Bmatrix} \hat{T}_1 \\ \hat{T}_2 \end{Bmatrix} = \frac{GJ}{L}\frac{kL}{\sin kL}\begin{bmatrix} \cos kL & -1 \\ -1 & \cos kL \end{bmatrix}\begin{Bmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{Bmatrix} \equiv [\mathbf{K}_t(s)]\begin{Bmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{Bmatrix}, \tag{12}$$

where

$$k_t = \mathrm{i}s\sqrt{\frac{\rho J}{GJ}}. \tag{13}$$

Here, G and J are the shearing modulus and torsional stiffness factor for the element cross-section.

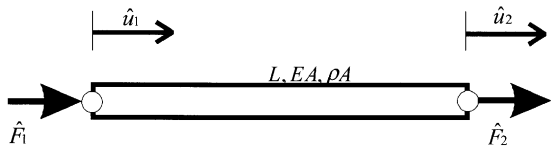

Fig. 2. Nodal forces and displacements for rod element in axial loading.

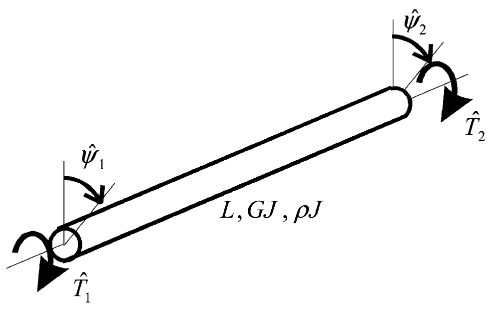

Fig. 3. Nodal moments and rotations for torsional bar element.

4. Global stiffness matrix

In this paper, 3-D frame structures are considered. Therefore, dynamic loading to a member of a frame structure comprises axial loading, bending in each of the two principal planes, and torsion. Fig. 4 shows the total set of forces and displacements.

The stiffness matrix for this case is obtained by superimposing individual cases of bending, axial loading, and torsion, which are given by Eqs. (9), (10), and (12), respectively. As a result, the fundamental equation for a member of frame structures on the Laplace domain is written in matrix form as

$$\left\{ \begin{array}{c} \hat{X}_1 \\ \hat{Y}_1 \\ \hat{Z}_1 \\ \hat{M}_{x1} \\ \hat{M}_{y1} \\ \hat{M}_{z1} \\ \hat{X}_2 \\ \hat{Y}_2 \\ \hat{Z}_2 \\ \hat{M}_{x2} \\ \hat{M}_{y2} \\ \hat{M}_{z2} \end{array} \right\}_e = [\mathbf{K}(s)]_e \left\{ \begin{array}{c} \hat{u}_1 \\ \hat{v}_1 \\ \hat{w}_1 \\ \hat{\theta}_{x1} \\ \hat{\theta}_{y1} \\ \hat{\theta}_{z1} \\ \hat{u}_2 \\ \hat{v}_2 \\ \hat{w}_2 \\ \hat{\theta}_{x2} \\ \hat{\theta}_{y2} \\ \hat{\theta}_{z2} \end{array} \right\}_e . \tag{14}$$

Transforming the stiffness matrix of the individual member to the global reference axes of a frame structure and superimposing the transformed matrices provides the global stiffness matrix, $[\mathbf{K}(s)]$, of a frame structure. Finally, the fundamental equation of the frame structure on the Laplace domain is obtained as

$$\{\hat{\mathbf{F}}\} = [\mathbf{K}(s)]\{\hat{\mathbf{u}}\}, \tag{15}$$

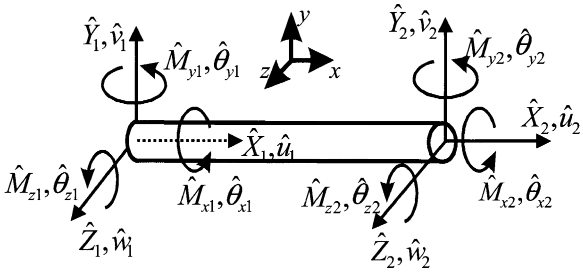

Fig. 4. Total set of forces and displacements of a member of frame structures.

where $\{\hat{\mathbf{F}}\}$ is the Laplace-transformed dynamic nodal force vector and $\{\hat{\mathbf{u}}\}$ is the Laplace-transformed nodal displacement vector. These transforming and superimposing techniques are similar to the case of conventional FEM [3].

Eq. (15) uses exact solutions for the beam, rod, and torsion bar on the Laplace domain; they are, for example, given by Eq. (4). Therefore, a geometrically uniform member can be replaced by a single spectral beam element.

5. Numerical Laplace transformation

The procedure for the calculation of the dynamic nodal displacement from fundamental Eq. (15) is presented next. This procedure uses a numerical Laplace transformation.

From Eq. (15) and the definition of the inverse Laplace transform, nodal displacement on the time domain is given as

$$\{\mathbf{u}\} = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} [\mathbf{K}(s)]^{-1} \{\hat{\mathbf{F}}(s)\} \mathrm{e}^{st}\, \mathrm{d}s. \tag{16}$$

On the integral of the above equation, changing the variable of $s = \sigma + \mathrm{i}\omega$ provides the following equation wherein σ is some positive real constant:

$$\{\mathbf{u}\} = \mathrm{e}^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\mathbf{K}(\sigma + \mathrm{i}\omega)]^{-1} \{\hat{\mathbf{F}}(\sigma + \mathrm{i}\omega)\} \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{17}$$

On the other hand, the definition of the Laplace transform yields the Laplace-transformed dynamic force as

$$\{\hat{\mathbf{F}}(s)\} = \int_0^\infty \{\mathbf{F}(t)\} \mathrm{e}^{-st}\, \mathrm{d}t. \tag{18}$$

Letting $s = \sigma + \mathrm{i}\omega$ in the above equation gives the following:

$$\{\hat{\mathbf{F}}(\sigma + \mathrm{i}\omega)\} = \int_0^\infty \{\mathbf{F}(t)\} \mathrm{e}^{-\sigma t}\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t. \tag{19}$$

The right side of Eq. (19) is similar to the Fourier integral of $\{\mathbf{F}(t)\}\mathrm{e}^{-\sigma t}$. Therefore, using the FFT, Eq. (19) is calculated numerically. By substituting the result into Eq. (17) and using the inverse FFT, the right side of Eq. (17) is integrated numerically. Then, the nodal displacement is numerically obtained.

6. Numerical result

The wave propagation of 3-D frame structures with finite-length beams is calculated numerically according to the above-mentioned method. The result is compared with the numerical solution obtained by conventional FEM using the Newmark method.

6.1. Wave propagation analysis of a finite-length beam with two free ends

Table 1 shows the sizes and material properties of a beam with two free ends, which is subjected to vertical dynamic force (shown in Fig. 5) at its center. Flexural wave propagation of this beam is analyzed. Three nodes, which are two free ends and the loading point, are considered in spectral element analysis; then, two spectral elements are used. The numerical Laplace transform here uses the sampling rate $\Delta T = 5$ ms, the sampling point number $N = 2048$, and the positive real constant $\sigma = 2\pi/N\Delta T$ [4].

Fig. 6 shows the time history of total vertical displacement in the beam. Displacements at arbitrary locations are calculated by Eq. (13). A flexural wave of a beam is sometimes called a dispersive wave. Features of a dispersive wave are shown in Fig. 6, where the higher frequency waves reach to the end sooner than the lower frequency waves. In addition, Fig. 6 depicts the wave reflection at the free end and superimposition of waves. As a result, results by the spectral element method very well represent the features of the flexural waves of a beam.

Figs. 7 and 8 show a comparison of the spectral element and conventional FEM results for displacements at the loading point and the free end, respectively. In using conventional FEM, the

Table 1
Size and material steel properties of a beam with two free ends

Thickness (m)	2.50E–04
Width (m)	1.30E–02
Length (m)	10
E (Pa)	2.09E + 11
ρ (kg/m^3)	7.80E + 03
v	0.3

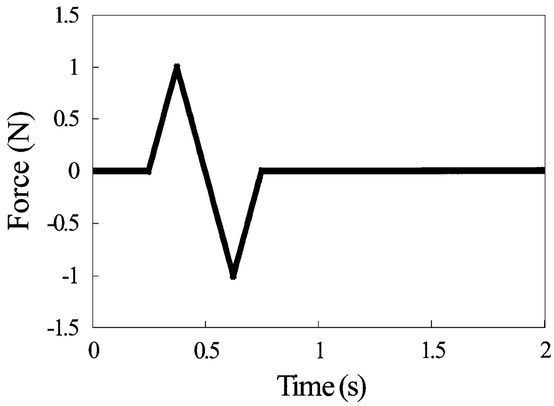

Fig. 5. Time history of dynamic force loading on the beam.

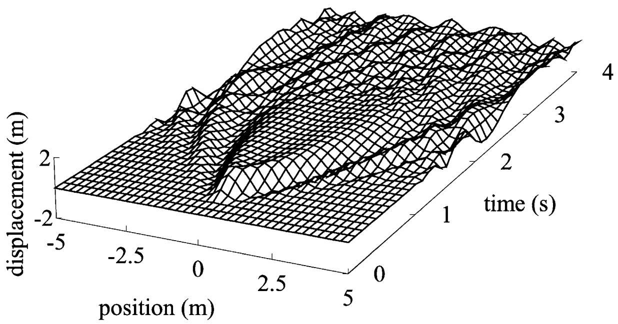

Fig. 6. Flexural wave propagation in the beam.

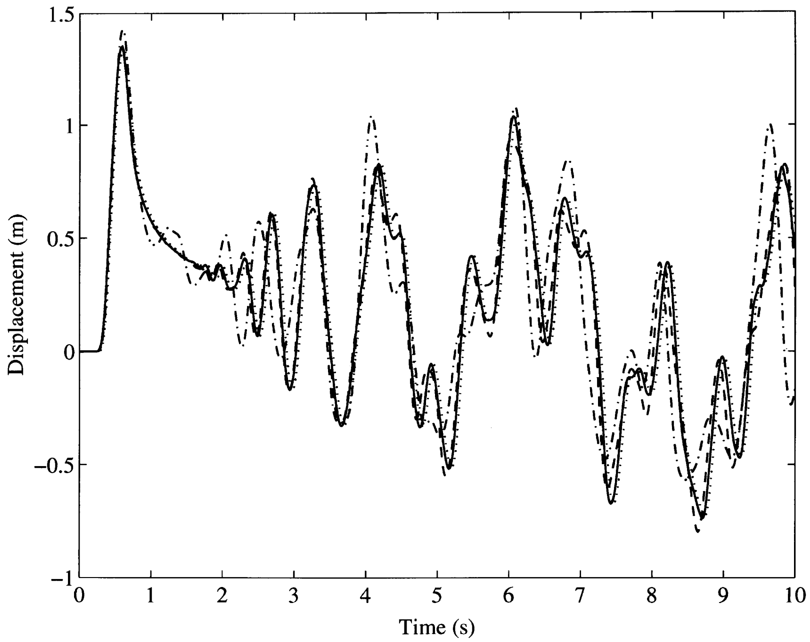

Fig. 7. Comparison of spectral element and conventional FEM results for displacements in the finite-length beam at the loading point. Spectral element method: ——, FEM 16 elements: -·-·-, FEM 32 elements: - - - -, FEM 96 elements: ⋯⋯.

beam is divided into 16, 32, and 96 elements. As the number of elements increase, numerical results of conventional FEM agree with those of the spectral element method.

6.2. Dynamic response analysis of a 3-D frame structure with one end fixed

Table 2 shows the sizes and material properties of a frame member; Fig. 9 shows a 3-D frame structure with one end fixed and the other end subjected to a dynamic force shown in Fig. 10. Figs. 11 and 12 show a comparison of results for spectral element analysis and conventional FEM

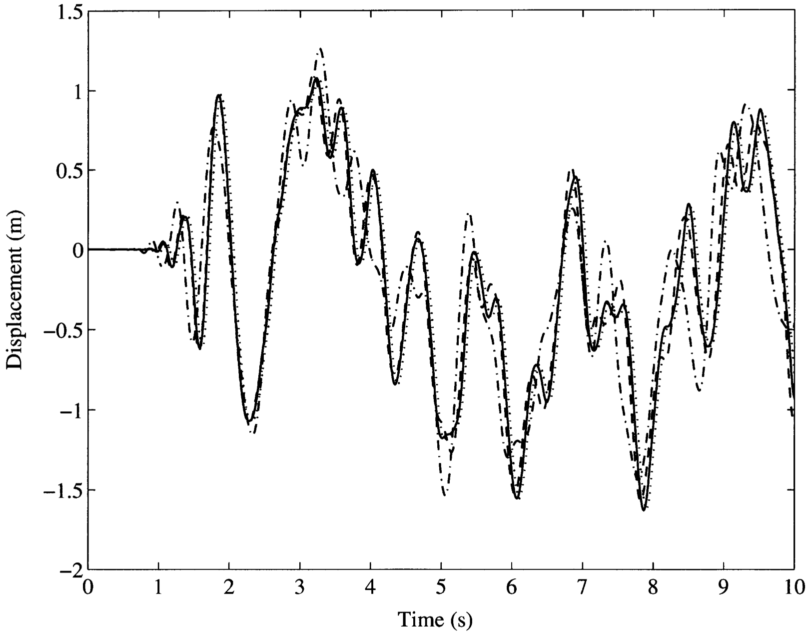

Fig. 8. Comparison of spectral element and conventional FEM results for displacements in the finite-length beam at the free end. Spectral element method: ——, FEM 16 elements: -·-·-, FEM 32 elements: - - - -, FEM 96 elements: ·······.

Table 2
Size and material (aluminum) properties of a frame member

Radius (m)	2.00E–03
E (Pa)	7.03E + 10
ρ (kg/m^3)	7.69E + 03
ν	0.345

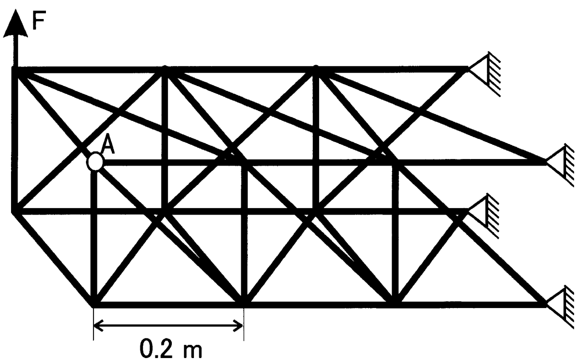

Fig. 9. Shape and size of a 3-D frame structure with one end fixed and the other end subjected to a dynamic force.

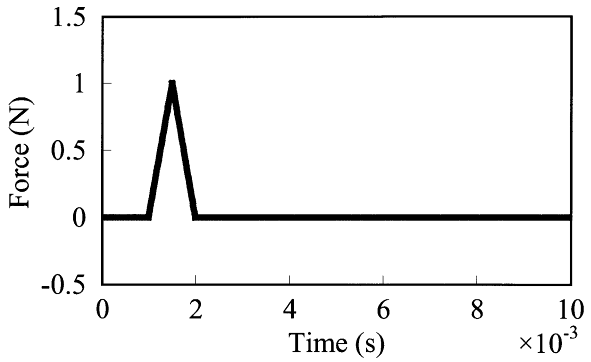

Fig. 10. Time history of dynamic force loading on the frame structure.

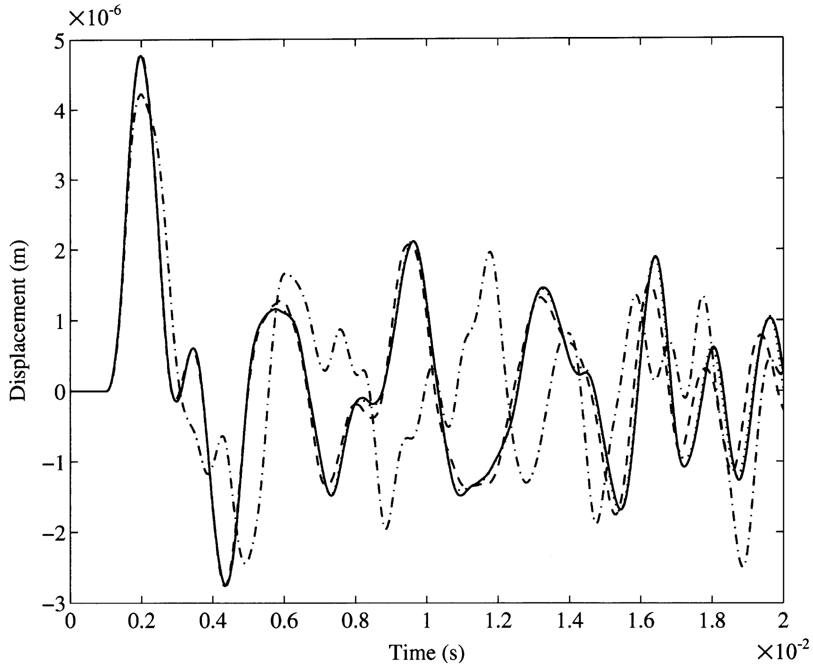

Fig. 11. Comparison of spectral element and conventional FEM results for displacements of the 3-D frame structure at the loading point. Spectral element method: ——, FEM 1 element: -·-·-, FEM 3 elements: - - - -, FEM 5 elements: ······.

for displacements at the loading point and point A (shown in Fig. 9), respectively. In spectral element analysis, each frame's member is replaced with one spectral element; the sampling rate $\Delta T = 20$ μs, the sampling point number $N = 1024$, and the positive real constant $\sigma = 2\pi/N\Delta T$. In using conventional FEM, a frame member is divided into 1, 3, and 5 elements. Just as in the case for the finite-length beam with two free ends, the results of conventional FEM gradually approach the results of the spectral element method with an increased number of elements.

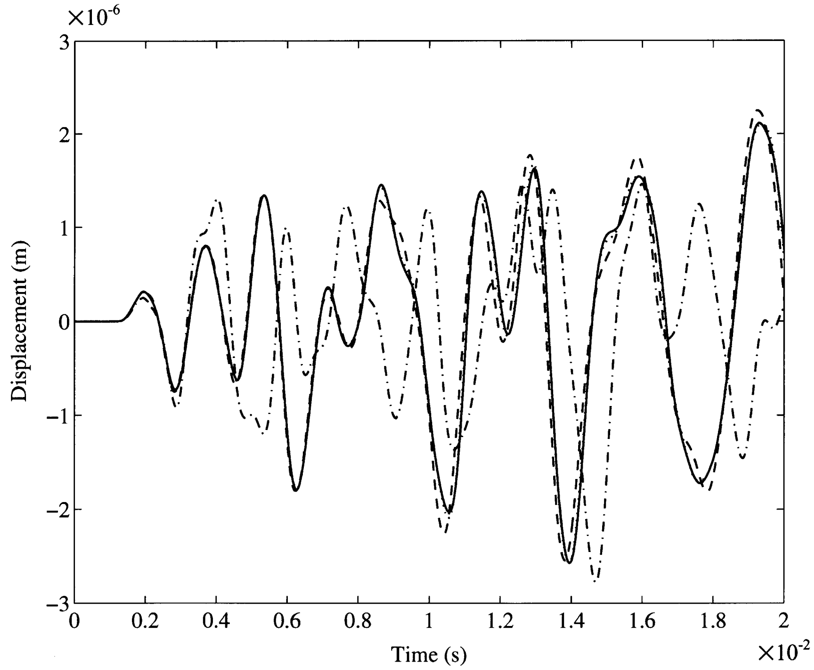

Fig. 12. Comparison of spectral element and conventional FEM results for displacements of the 3-D frame structure at point A. Spectral element method: ——, FEM 1 element: -·-·-, FEM 3 elements: - - - -, FEM 5 elements: ·······.

7. Conclusion

This paper presented a new spectral element method using the numerical Laplace transform for the wave propagation analysis of frame structures. Numerical results for a finite length beam and a 3-D frame structure showed that the proposed method is practical.

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