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Letter to the Editor

Energy flow of axisymmetric elastic waves in a three-layered, transtropic–isotropic–transtropic, composite cylinder

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1. Introduction

Markuš and Mead [1] studied physical phenomena connected with elastic wave propagation in anisotropic media. They presented a classification of the possible free wave motions in a three-layered composite thick cylinder. Both the inner and outer layers were made out of transtropic (transversely isotropic) material namely of short-strand fibreglass and polyester resin (GFRP). The middle layer was made of isotropic rubbery material. To obtain a correct and closed-form solution for the propagating waves, Bessel and special Frobenius series were used. Numerical results were given in the form of dispersion curves.

Jing and Tzeng [2] presented an approximate elasticity solution for arbitrarily laminated anisotropic cylindrical closed shells of finite length with simply supported ends. They transformed the coupled partial differential equations with constant coefficients by dividing the shell into several thin coaxial laminae.

Kudlička [3] modified the previous method to obtain dispersion curves for an axisymmetric problem of arbitrarily laminated, orthotropic unclosed cylindrical thick-walled pipes of infinite length. He determined the first five dispersion curves and relative amplitudes of axial and radial displacements for axisymmetric elastic waves propagating along the axis of a boron-epoxy pipe.

Kudlička [4] investigated a problem of the energy flow of elastic waves in an epoxy matrix composite reinforced by continuous boron fibres. For this anisotropic material, mean values of the energy flow densities for axisymmetric elastic waves in a hollow cylinder, as functions of the radial co-ordinate and the wave number, were determined. The results for some wave numbers and phase velocities belonging to the basic dispersion curve were compared with those obtained for an isotropic epoxy cylinder with the same geometry as the anisotropic one.

In this paper, energy flow densities of axisymmetric elastic waves propagating along the axis of a thick-walled, three-layered (transtropic–isotropic–transtropic) composite shell are determined.

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The geometry and material properties of the cylinder analyzed are the same as those used in Ref. [1]. The inner and outer transtropic GFRP (glass fibres, resin polyester) layers of equal thickness surround the middle isotropic rubber layer. It is shown, how the isotropic rubber core influences both the dispersion curves and energy flow densities. The results for some values from the basic dispersion curve are compared with those obtained for a pure transtropic cylinder without the rubber core.

2. Energy flow in an elastic, homogenous and isotropic medium

If a volume element of an elastic medium is disturbed, both the state change and energy needed propagate across the total medium as a *flow* with certain space density. The energy transport can be expressed by some quantity, e.g., by the *energy flow density*. This vector quantity has at some selected point the same direction as the energy transport. Its value is

$$I = cw = c(w_k + w_p), \tag{1}$$

where c is the energy transport velocity, w is the total mechanical, w_k is the kinetic and w_p is the potential energy density.

It can be derived, that both the energy densities for a harmonic wave propagating along x-axis, $u(x,t) = A\cos[K(x-ct)]$, have equal values

$$w_k = w_p = \tfrac{1}{2}\rho K^2 c^2 A^2 \sin^2[K(x-ct)]. \tag{2}$$

Here A is the amplitude, $K = 2\pi/\lambda$ is the wave number, λ is the wavelength, c is the phase velocity of the harmonic wave considered and ρ is the density of the medium. It is evident, that the mean value of the energy flow density belonging to one period of this wave motion

$$\bar{I} = c\bar{w} = \tfrac{1}{2}\rho K^2 c^3 A^2. \tag{3}$$

Note the mean value of $\sin^2[K(x-ct)]$ across one period is equal to 1/2.

3. Energy flow in a hollow anisotropic cylinder

Eq. (3) can be generalized for various wave motions of an anisotropic media. For the axisymmetric wave propagating along the axis of a hollow infinitely long cylinder (Fig. 1), the displacements in the radial (r) and axial (x) co-ordinates in the cylindrical system (r,θ,x) are

$$u_r = U\cos[K(x-ct)], \quad u_x = W\sin[K(x-ct)], \tag{4}$$

where U and W are the amplitudes of the waves and they are functions of r only. The meaning of the other quantities is explained in the previous section. The displacement in the circumferential direction $u_\theta = 0$.

The kinetic and potential energy densities are

$$\begin{aligned} w_k &= \tfrac{1}{2}\rho K^2 c^2 \{U^2 \sin^2[K(x-ct)] + W^2\cos^2[K(x-ct)]\}, \\ w_p &= \tfrac{1}{2}(\sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta + \sigma_x\varepsilon_x + \tau_{rx}\gamma_{rx}). \end{aligned} \tag{5}$$

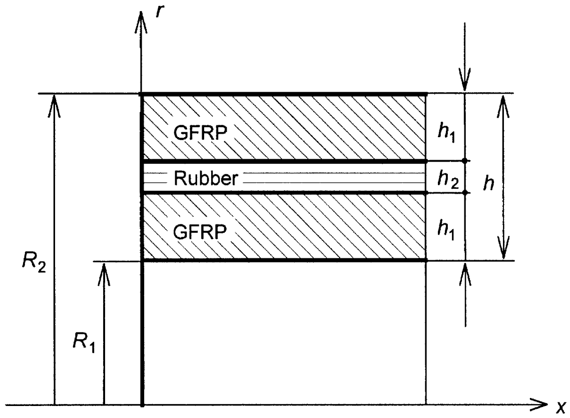

Fig. 1. The geometry and co-ordinate system of the cylinder.

Here the indexed quantities σ, τ, ε and γ express the stresses and strains in the usual notation in a cylindrical co-ordinate system.

By using the procedure presented in Ref. [3], that is by dividing the thick wall of the cylinder into several laminae and by determining the quantities U, V, ε, γ, σ and τ on the boundaries of the laminae, the energy densities w_k and w_p can be determined.

The generalized Hooke's law equations (constitutive equations) for each lamina are given in the matrix form

$$\begin{bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{r\theta} \\ \tau_{xr} \\ \tau_{x\theta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \times \begin{bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \\ \varepsilon_{r\theta} \\ \gamma_{xr} \\ \gamma_{x\theta} \end{bmatrix}. \tag{6}$$

The strains can be expressed as

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_\theta = \frac{u_r}{r}, \quad \varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \gamma_{xr} = \frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}. \tag{7}$$

The differential equations of motion are

$$\begin{aligned} &\frac{1}{r}\frac{\partial(r\sigma_r)}{\partial r} + \frac{\partial \tau_{xr}}{\partial x} - \frac{\sigma_\theta}{r} = \rho\frac{\partial^2 u_r}{\partial t^2}, \\ &\frac{1}{r}\frac{\partial(r\tau_{xr})}{\partial r} + \frac{\partial \sigma_x}{\partial x} = \rho\frac{\partial^2 u_x}{\partial t^2}, \end{aligned} \tag{8}$$

where ρ is the density of the material and t is time.

After substituting Eqs. (6) and (7) into Eqs. (8), the governing equations in terms of the displacements for each lamina become

$$\begin{aligned} &C_{33}\frac{\partial^2 u_r}{\partial r^2} + C_{33}\frac{\partial u_r}{r\partial r} + C_{55}\frac{\partial^2 u_r}{\partial x^2} - C_{22}\frac{u_r}{r^2} + (C_{13} - C_{12})\frac{\partial u_x}{r\partial x} \\ &\quad + (C_{13} - C_{12})\frac{\partial^2 u_x}{\partial x\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}, \\ &(C_{12} + C_{55})\frac{\partial u_r}{r\partial x} + (C_{13} + C_{55})\frac{\partial^2 u_r}{\partial r\partial x} + C_{11}\frac{\partial^2 u_x}{\partial x^2} \\ &\quad + C_{55}\frac{\partial^2 u_x}{\partial r^2} + C_{55}\frac{\partial u_x}{r\partial r} = \rho\frac{\partial^2 u_x}{\partial t^2}. \end{aligned} \tag{9}$$

These equations are coupled partial differential equations with variable coefficients. It is not possible to solve them in a closed form. Introducing the local radial co-ordinate $\xi_k = r - R_k$ located at the centre of kth lamina with the radius R_k ($k = 1, 2, \ldots, M$, M is the lamina's count) and making the approximation $\xi_k/R_k \ll 1$, the following equations for the variable parts of the coefficients are assumed:

$$\frac{1}{r} \approx \frac{1}{R_k}(1 - \eta_k), \quad \frac{1}{r^2} \approx \frac{1}{R_k^2}(1 - 2\eta_k), \tag{10}$$

where $\eta_k = \xi_k/R_k$. Each of the two thick cylinders is being split into N laminae, each of thickness h_1/N. This thickness must satisfy the approximation $h_1/(NR_k) \ll 1$.

Inserting Eqs. (10) and (4) into Eqs. (9), with using index k for the lamina's number, yields for each lamina

$$\begin{aligned} &(C_{22} + C_{55}K^2R_k^2)U_k + (C_{12} - C_{13})KR_kW_k - C_{33}U_k' \\ &\quad - (C_{13} + C_{55})KR_kW_k' - C_{33}U_k'' = c^2\rho K^2R_k^2U_k, \\ &(C_{12} + C_{55})KR_kU_k + C_{11}K^2R_k^2W_k + (C_{13} + C_{55})KR_kU_k' \\ &\quad - C_{55}W_k' - C_{55}W_k'' = c^2\rho K^2R_k^2W_k. \end{aligned} \tag{11}$$

The prime denotes the derivative with respect to η_k. These equations are coupled ordinary differential equations with constant coefficients for the amplitudes U_k and W_k. The general solution of Eqs. (11) for kth lamina is assumed in the form

$$U_k = \sum_{i=1}^{4} A_{ki}\exp(\alpha_{ki}\eta_k), \quad W_k = \sum_{i=1}^{4} A_{ki}\Psi_{ki}\exp(\alpha_{ki}\eta_k). \tag{12}$$

The procedure for obtaining the unknown coefficients α_{ki}, A_{ki} and Ψ_{ki} is described in Ref. [3]. It is not possible to explain in detail here due to its mathematical complication. It can be noted that the boundary conditions on the inner and outer surfaces and the continuity conditions for stresses and displacements on the boundaries between coincident laminae must be satisfied.

After determining the amplitudes U_k and W_k given by Eqs. (12), the displacements u_r and u_x can be computed from Eqs. (4). Here $U = U_k$ and $W = W_k$ for $r = R_k(1 + \eta_k)$. The strains ε_x, ε_θ, ε_r and γ_{xr} can be determined from Eqs. (7) and the stresses σ_r, σ_x, σ_θ, σ_r, $\tau_{r\theta}$, τ_{xr}, and $\tau_{x\theta}$ from Eqs. (6). The kinetic and potential energy densities w_k and w_p are given by Eqs. (5). The mean value of the total energy flow density is their sum. An explicit expression of this quantity is very complicated and therefore it is not written here. All required computations were made by means of a computational program created by the author in the language of the Wolfram's software system, Mathematica [5].

4. Numerical results

As it was mentioned, the energy flow densities were determined for the axisymmetric elastic wave propagation along the axis of a three-layered, transtropic–isotropic–transtropic, cylinder with the same geometry and material properties as were considered in Ref. [1].

The three-layered cylinder, Fig. 1, has the inner and outer transtropic layers (GFRP—glass fibres, resin polyester) of equal thickness h_1. The thickness of the middle isotropic layer (rubber) is $h_2 = h_1/5$. The outer radius to the total thickness ratio $R_2/h = 4$, $h = 2h_1 + h_2$. The mechanical properties of the GFRP layers are: the in-plane Young's moduli $E_x = E_\theta = 15.65$ GPa, the transverse Young's modulus $E_r = 7.7$ GPa, the shear modulus $G_{xr} = 5.0$ GPa, the Poisson ratios $v_{x\theta} = 0.31$, $v_{xr} = 0.315$, $v_{\theta r} = 0.215$, and the density $\rho_1 = 1576$ kg m^{-3}. The corresponding elastic modules are

$$\mathbf{C}_{GFRP} = \begin{bmatrix} 21 & 9 & 7 & 0 & 0 & 0 \\ 9 & 21 & 7 & 0 & 0 & 0 \\ 7 & 7 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \text{GPa}. \tag{13}$$

The mechanical properties of the rubber layer are: Young's modulus $E = 0.279$ GPa, the shear modulus $G = 0.094$ GPa, the Poisson ratio $v = 0.478$, the density $\rho_2 = 1056$ kg m^{-3}. The corresponding elastic modulae are

$$\mathbf{C}_{rubb} = \begin{bmatrix} 2.24 & 2.05 & 2.05 & 0 & 0 & 0 \\ 2.05 & 2.24 & 2.05 & 0 & 0 & 0 \\ 2.05 & 2.05 & 2.24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.094 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.094 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.094 \end{bmatrix} \text{GPa}. \tag{14}$$

For simplicity, the non-dimensional wave number K' and the non-dimensional phase velocity c' were defined by these formulas:

$$K' = \frac{R_M}{\lambda}, \quad c' = \frac{c}{\sqrt{C_{44}/\rho_1}}, \tag{15}$$

where $R_M = (R_1 + R_2)/2$ is the middle radius of the cylinder, C_{44} is the elastic module of the GFRP, λ and ρ_1 are already known quantities.

The phase velocity c' was computed for some values of the wave number K' from the interval $(0;3\rangle$ for $M=22$ laminae (2 times 10 for the two transtropic layers and 2 for the isotropic layer in case of the three-layered cylinder). The thickness of each the lamina satisfied the approximation $h_1/(NR_k) \ll 1$ (0.015 for $k=1$ and 0.011 for $k=22$; $N=10$). The first (lowest) dispersion curve for the three-layered cylinder is shown in Fig. 2 (the top curve). For comparison, there is also plotted the first dispersion curve for the pure transtropic GFRP material (the bottom curve). The count of the laminae used was sufficient for the relative error of the phase velocity less than 1%. This error was determined by means of computing the phase velocities for 33 laminae.

The curves have similar shape. They distinguish themselves by maxima for $K' \to 0$ (low frequencies). They take approximately equal values for the greater values (higher frequencies).

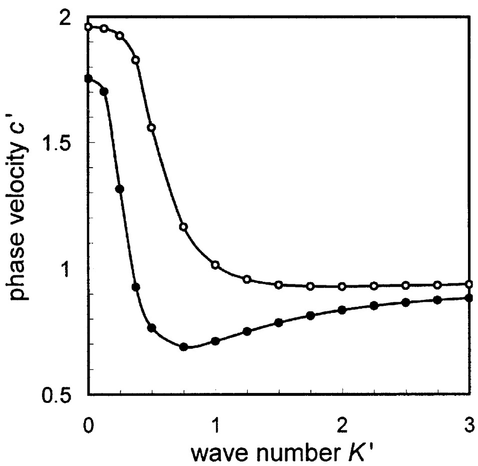

Fig. 2. The first dispersion curves for both three and one layers: —○—, 3 layers; —●—, 1 layer.

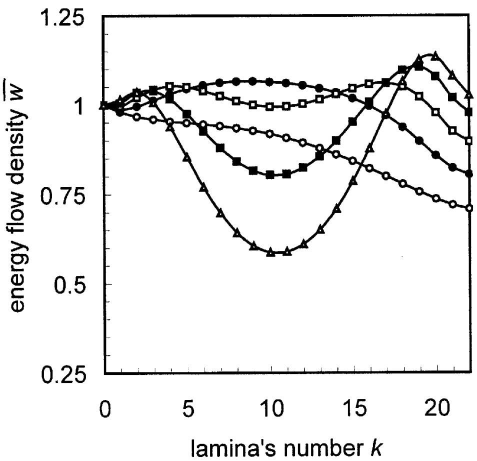

Fig. 3. The means of the non-dimensional energy flow densities for three layers: —○—, $K'=1$; —●—, $K'=2$; —□—, $K'=3$; —■—, $K'=4$; —△—, $K'=5$.

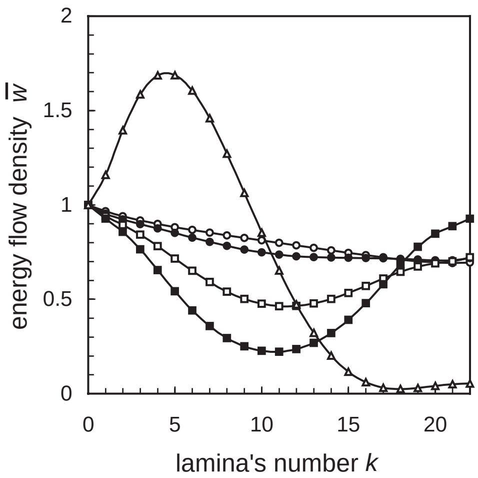

Fig. 4. The means of the non-dimensional energy flow densities for one layer: —○—, $K' = 1$; —●—, $K' = 2$; —□—, $K' = 3$; —■—, $K' = 4$; —△—, $K' = 5$.

The curve for the three-layered cylinder reflects higher phase velocities than the curve for the one-layered cylinder.

The mean values of the energy flow density, normalized to 1 at the inner radius, and versus the lamina's number are shown in Fig. 3 for the three-layered cylinder and in Fig. 4 for the pure transtropic cylinder. The curves are plotted for wave number $K' = 1, 2, \ldots, 5$. The shapes of these curves point at significant differences among them in dependence on the wave number, especially for $K' = 5$. In both, Figs. 3 and 4 it appears that at some wave numbers more energy is transported through the middle of the layered cylinder than along its inner and outer edges, while at other wave numbers the reverse is true. It can be due to some of the waves being predominantly transversal and others longitudinal. These waves can be identified by a detailed analysis of their amplitudes. This matter is an object of the contemporary research.

5. Conclusions

The energy flow of elastic waves propagating along the axis of a thick-walled, three-layered (transtropic–isotropic–transtropic) composite shell was investigated. The inner and outer transtropic GFRP layers of equal thickness surround the middle isotropic rubber layer. The dispersion curves were computed and plotted. It was shown how the isotropic rubber core influences the dispersion curves and the energy flow. The results for some values of the first dispersion curve were compared with those obtained for the pure transtropic cylinder without the rubber core.

The mean value of the energy flow density was determined for the cylinder considered. The non-dimensional phase velocity was computed for some values from a chosen interval of the wave numbers. The mean values of the energy flow densities were presented graphically for both three and one-layered cylinder.

The presented method and the created computational program can be useful in solving some energy flow problems in the propagation of elastic waves along arbitrarily laminated cylinders (bars, pipe ducts, etc.) with various kinds of the anisotropy. By the control of the anisotropy, the optimization of an energy flow distribution in machine parts, buildings, armatures and constructions will be possible.

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