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A combined transfer matrix and analogue beam method for free vibration analysis of composite beams

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Abstract

In this paper, an exact dynamic field transfer matrix for free vibration analysis of composite beam is presented. The analysis of composite beams is carried out using a combination between the transfer matrix and the analog beam methods (TMABM). A composite beam is composed of an upper slab and a lower beam, connected at the interface by shear transmitting studs. The theory of analogue beam includes the coupling between the bending and torsional modes of deformation, which is usually present in laminated composite beams due to ply orientation. The application of this method is demonstrated by investigation of the free vibration characteristics of a composite beam for which some comparative results are available. The method developed in this paper can mainly be applied in the field of dynamic analysis of composite bridges. Although, the method is complicated, it is more accurate and could prove to be a good tool for design purposes.

1. Introduction

The basic idea of the analogue beam method is to replace the real beam with an analogue beam where all the shear deformation is concentrated in a thin horizontal layer, called the shear layer. When the correct stiffness is assigned to this layer it is possible to get the real beam, and its analogue, to behave the same way in the overall sense, i.e., they have the identical deformation, bending moment, and shear force.

The method of analysis is based on two-kinematical assumptions [1]. In the first one, each sub-beam behaves as a simple Bernouli-Euler beam, i.e., the shear deformation within each beam is

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neglected. All shear deformation is therefore concentrated in the shear layer. In the second, the vertical displacements of the sub-beams are the same, i.e.; the shear layer is transversely rigid.

Most of the studies on composite beams have concentrated on their strength rather than their elastic behaviour [1–6]. Several authors have investigated the free vibration characteristics of composite beams [3,5,6], but only a few have taken into account the effects of shear deformation. Banerjee and Williams [4,5] have developed the dynamic stiffness matrix of composite beams in order to investigate their free vibration characteristics. However, their work on the subject did not account for the effects of shear deformation and rotatory inertia, which can be important for composite beams because they are usually more sensitive to these effects than are their metallic counterparts, due mainly to the low shear moduli of fibrous composites. In their subsequent work they extended the dynamic stiffness method to include the effects of shear deformation. The natural frequencies in this work were calculated using the algorithm of Wittrick and Williams [7].

Lee [8] used the energy method (Rayleigh-Ritz method) to develop his theory to calculate the natural frequencies of thin orthotropic composite shells.

However, the objective of this paper is to investigate the free vibration characteristics of composite beams using a combination of the analogue-beam theory and the transfer matrix method. The analogue beam method is developed first and then used in conjunction with the stiffness matrix to yield natural frequencies in free vibration.

2. The mathematical model

The model used in this analysis is a composite steel-concrete beam [2] which is composed of a concrete slab and a steel beam, connected at their interface by a shear transmitting device such as studs, as shown in Fig. 1.

The purpose of the shear studs is to transmit the horizontal shear force between the slab and the beam. The shear interface is of course not completely rigid but has a force (Q)–displacement (δ) relationship of the type shown in Fig. 2. As the composite beam is loaded, some slip must take place at the beam–slab interface. The composite beam therefore will no longer follow the Navier hypothesis of beam theory, even though the slab and the beam separately can still be expected to

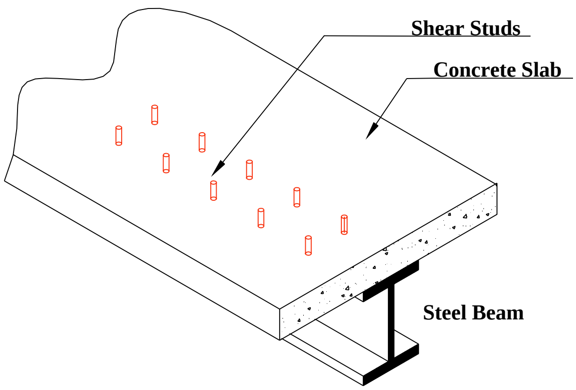

Fig. 1. Composite beam cross-section.

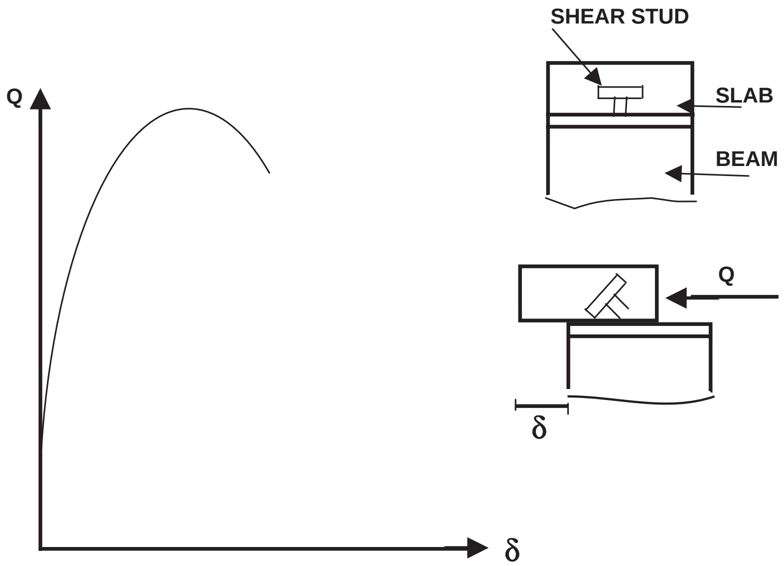

Fig. 2. Typical slip behavior of a shear stud.

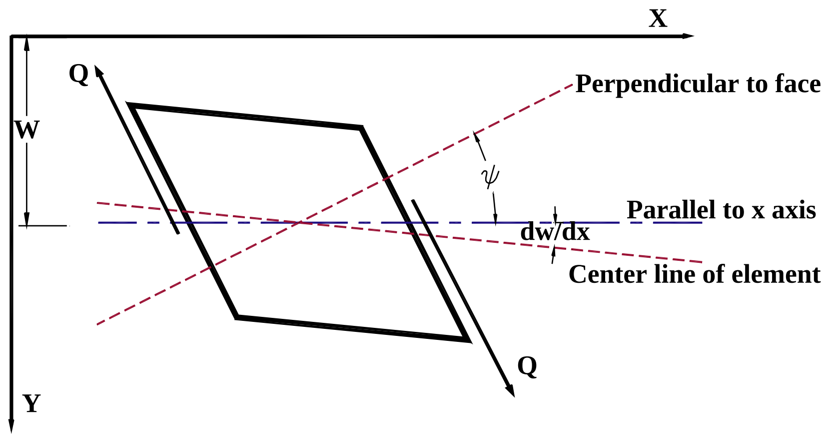

Fig. 3. Effect of shear on beam deflection.

do so. Shear deformation of the steel beam can be important for composite beams, but it is not included in this analysis.

To illustrate this method further, the problem of the vibrating beam should be solved taking into consideration the effect of shear deflection. Consider a beam of length (ℓ), with the following properties that are constant over the length: cross-sectional area (A), second moment of area (I), and mass per unit length (μ). The slope ($\mathrm{d}w/\mathrm{d}x$) of the centreline of the beam is affected by both the bending moment and the shear force. The action of the bending moment rotates the face of the cross-section through an angle (ψ), and from there the shearing action turns the centreline to adopt the slope ($\mathrm{d}w/\mathrm{d}x$), the angle of the face of the beam remaining unchanged (Fig. 3).

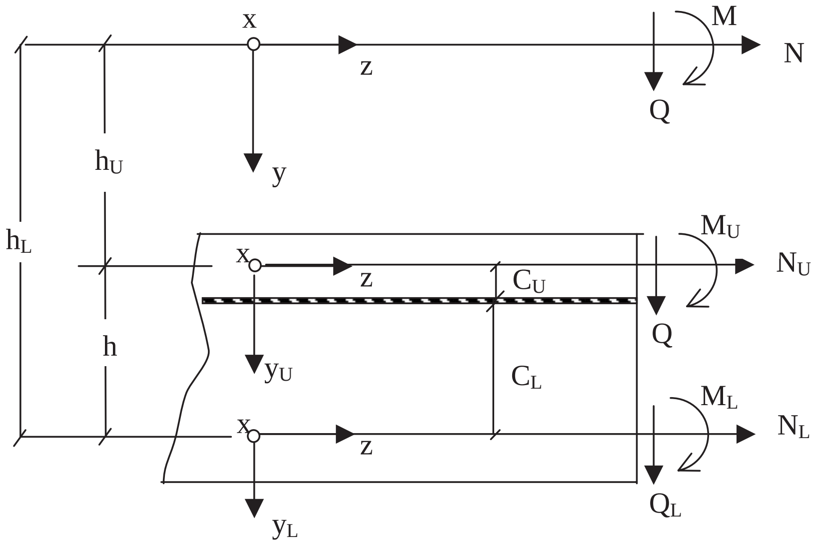

Fig. 4. Global and local internal forces.

The resultant axial forces N_U and N_L and the moments M_U and M_L in the sub-beams are illustrated in Fig. 4. The total bending moment (M) can be decomposed into two components

$$M = M_t + M_c, \tag{1}$$

where M_t can be identified as the bending moment in the beam from what can be called its "truss action", i.e., from the axial force in the sub-beams, while M_c represents the combined bending moment from the individual beam action of the sub-beams acting independently.

$$M_t = (EI)_t \frac{\mathrm{d}\psi}{\mathrm{d}x}, \tag{2}$$

$$M_c = -(EI)_c \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \tag{3}$$

where I_t is the moment of inertia of the beam as a truss, I_c the moment of inertia of the sub-beams acting independently, E the elastic modulus and w the deflection. $(EI)_t$ and $(EI)_c$ represent the bending rigidities for the truss component and for the beam component, respectively.

Eq. (1) can be rewritten as

$$M = (EI)_t \frac{\mathrm{d}\psi}{\mathrm{d}x} - (EI)_c \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}. \tag{4}$$

The shear force is somewhat more complicated. The horizontal shear force in the shear layer, q per unit length, can be expressed as

$$q = kh\left(\psi + \frac{\mathrm{d}w}{\mathrm{d}x}\right), \tag{5}$$

where k is the shear stiffness of the shear layer and h is the distance between the centroids of the sub-beams (the distance between the local z-axis).

The line force (q) acts at the interface between the sub-beams. Moment equilibrium of the two sub-beams element yields

$$Q_U = \frac{dM_U}{dx} + qC_U, \tag{6a}$$

$$Q_L = \frac{dM_L}{dx} + qC_L. \tag{6b}$$

The total shear force is therefore

$$Q = Q_U + Q_L = \frac{dM_U}{dx} + \frac{dM_L}{dx} + qh = \frac{dM_C}{dx} + qh, \tag{7}$$

where $M_C = M_U + M_L$, $h = C_U + C_L$ and also as shown in Fig. 4 that $h = h_L - h_U$.

In a similar way to the total bending moment, the total shear force Q can also be thought of as having two components such as

$$Q = Q_c + Q_t. \tag{8}$$

From Eqs. (3), (5) and (7) then,

$$Q = kh^2\left(\psi + \frac{dw}{dx}\right) - (EI)_c\frac{d^3w}{dx^3}. \tag{9}$$

Then ψ can be expressed as

$$\psi = \frac{1}{kh^2}Q - \frac{1}{kh^2}\frac{dM_C}{dx} - \frac{dw}{dx}. \tag{10}$$

Differentiation of Eq. (10) and substitution in Eq. (4) yield

$$M = \frac{(EI)_t}{kh^2}\frac{dQ}{dx} + \frac{(EI)_t(EI)_c}{kh^2}\frac{d^4w}{dx^4} - EI\frac{d^2w}{dx^2}, \tag{11}$$

where $EI = (EI)_t + (EI)_c$.

If a sinusoidal variation of w with circular frequency ω, is assumed, then

$$w(x,t) = W(x)\sin\omega t,$$

where $W(x)$, is the amplitude of the sinusoidally varying vertical displacement.

The equilibrium considerations give the equations

$$\frac{dQ}{dx} = -\mu\omega^2 W, \quad \frac{dM}{dx} = Q, \tag{12}$$

where ω is the circular frequency of the beam.

Taking the second derivative of Eq. (11) with respect to x and combining it with Eq. (12) gives the equation

$$\frac{d^6W}{dx^6} - C_1\frac{d^4W}{dx^4} - C_2\frac{d^2W}{dx^2} + C_3W = 0, \tag{13}$$

where

$$C_1 = \frac{kh^2 EI\ell^2}{(EI)_c (EI)_t}, \quad C_2 = \frac{\omega^2 \mu \ell^4}{(EI)_c}, \quad C_3 = \frac{\omega^2 \mu k h^2 \ell^6}{(EI)_c (EI)_t}.$$

Since Eq. (13) is an ordinary differential equation with constant coefficients, its solution is of the form

$$W = \bar{C} e^{\lambda x/\ell}, \tag{14}$$

where $\bar{C}$ is constant.

This solution, substituted in Eq. (13), leads to the characteristic equation in λ:

$$\lambda^6 - C_1\lambda^4 - C_2\lambda^2 + C_3 = 0. \tag{15}$$

After extensive algebra, the roots of this equation are $\pm\lambda_1$, $\pm\lambda_2$, and $\pm\lambda_3$, where [3–5]:

$$\lambda_1 = [-2r^{1/3}\cos(\phi/3) + C_1/3]^{1/2},$$
$$\lambda_2 = [-2r^{1/3}\cos[(\phi - 2\pi)/3] + C_1/3]^{1/2},$$
$$\lambda_3 = [-2r^{1/3}\cos[(\phi + 2\pi)/3] + C_1/3]^{1/2},$$

with

$$r = \frac{1}{729} C_1^6 + \frac{1}{81} C_1^4 C_2 + \frac{1}{27} C_1^2 C_2^2 + \frac{1}{27} C_2^3,$$
$$\phi = \cos^{-1}\left[\frac{27C_3 - 9C_1C_2 - 2C_1^3}{2(C_1^2 + 3C_2)^{3/2}}\right],$$

and using the relations

$$e^{\pm\theta} = \cosh\theta \pm \sinh\theta, \quad e^{\pm j\theta} = \cos\theta \pm j\sin\theta.$$

The solution can be written in the form

$$W = \bar{C}_1 \cos\left(\lambda_1 \frac{x}{\ell}\right) + \bar{C}_2 \sin\left(\lambda_1 \frac{x}{\ell}\right) + \bar{C}_3 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \bar{C}_4 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \bar{C}_5 \cosh\left(\lambda_3 \frac{x}{\ell}\right) + \bar{C}_6 \sinh\left(\lambda_3 \frac{x}{\ell}\right).$$

Since the solution for all the variables is of the same form, it is better to start off most conveniently with the solution of Q,

$$Q = A_1 \cos\left(\lambda_1 \frac{x}{\ell}\right) + A_2 \sin\left(\lambda_1 \frac{x}{\ell}\right) + A_3 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + A_4 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + A_5 \cosh\left(\lambda_3 \frac{x}{\ell}\right) + A_6 \sinh\left(\lambda_3 \frac{x}{\ell}\right). \tag{16}$$

From Eq. (12) the deflection will be:

$$W = -\alpha_1 A_1 \sinh\left(\lambda_1 \frac{x}{\ell}\right) + \alpha_1 A_2 \cos\left(\lambda_1 \frac{x}{\ell}\right) + \alpha_2 A_3 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \alpha_2 A_4 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \alpha_3 A_5 \sinh\left(\lambda_3 \frac{x}{\ell}\right) + \alpha_3 A_6 \cosh\left(\lambda_3 \frac{x}{\ell}\right), \tag{17}$$

where

$$\alpha_1 = -\frac{\lambda_1}{\mu\omega^2\ell}, \quad \alpha_2 = -\frac{\lambda_2}{\mu\omega^2\ell}, \quad \alpha_3 = -\frac{\lambda_3}{\mu\omega^2\ell}.$$

The derivative of the deflection is given by

$$W' = -\frac{\alpha_1\lambda_1}{\ell} A_1 \cos\left(\lambda_1 \frac{x}{\ell}\right) - \frac{\alpha_1\lambda_1}{\ell} A_2 \sin\left(\lambda_1 \frac{x}{\ell}\right) + \frac{\alpha_2\lambda_2}{\ell} A_3 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \frac{\alpha_2\lambda_2}{\ell} A_4 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \frac{\alpha_3\lambda_3}{\ell} A_5 \cosh\left(\lambda_3 \frac{x}{\ell}\right) + \frac{\alpha_3\lambda_3}{\ell} A_6 \sinh\left(\lambda_3 \frac{x}{\ell}\right). \tag{18}$$

Using Eq. (17) and substituting in Eq. (9), then Ψ can be expressed as

$$\psi = \beta_1 A_1 \cos\left(\lambda_1 \frac{x}{\ell}\right) + \beta_1 A_2 \sin\left(\lambda_1 \frac{x}{\ell}\right) + \beta_2 A_3 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \beta_2 A_4 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \beta_2 A_5 \cosh\left(\lambda_3 \frac{x}{\ell}\right) + \beta_2 A_6 \sinh\left(\lambda_3 \frac{x}{\ell}\right), \tag{19}$$

where

$$\beta_1 = \left[\frac{1}{kh^2}\left(1 - \frac{(EI)_c\lambda_1^4}{\mu\omega^2\ell^4}\right) - \frac{\lambda_1^2}{\mu\omega^2\ell^2}\right], \quad \beta_2 = \left[\frac{1}{kh^2}\left(1 - \frac{(EI)_c\lambda_2^4}{\mu\omega^2\ell^4}\right) + \frac{\lambda_2^2}{\mu\omega^2\ell^2}\right]$$
$$\beta_3 = \left[\frac{1}{kh^2}\left(1 - \frac{(EI)_c\lambda_3^4}{\mu\omega^2\ell^4}\right) + \frac{\lambda_3^2}{\mu\omega^2\ell^2}\right].$$

Finally, from Eqs. (2) and (3) the expression for M_t and M_c will be

$$M_t = -\gamma_1 A_1 \sin\left(\lambda_1 \frac{x}{\ell}\right) + \gamma_1 A_2 \cos\left(\lambda_1 \frac{x}{\ell}\right) + \gamma_2 A_3 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \gamma_2 A_4 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \gamma_3 A_5 \sinh\left(\lambda_3 \frac{x}{\ell}\right) + \gamma_3 A_6 \cosh\left(\lambda_3 \frac{x}{\ell}\right), \tag{20}$$

$$M_c = \delta_1 A_1 \sin\left(\lambda_1 \frac{x}{\ell}\right) - \delta_1 A_2 \cos\left(\lambda_1 \frac{x}{\ell}\right) + \delta_2 A_3 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \delta_2 A_4 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \delta_3 A_5 \sinh\left(\lambda_3 \frac{x}{\ell}\right) + \delta_3 A_6 \cosh\left(\lambda_3 \frac{x}{\ell}\right), \tag{21}$$

where

$$\gamma_1 = \frac{(EI)_t\beta_1\lambda_1}{\ell}, \quad \gamma_2 = \frac{(EI)_t\beta_2\lambda_2}{\ell}, \quad \gamma_3 = \frac{(EI)_t\beta_3\lambda_3}{\ell},$$

$$\delta_1 = \frac{(EI)_c\lambda_1^3}{\mu\omega^2\ell^3}, \quad \delta_2 = \frac{(EI)_c\lambda_2^3}{\mu\omega^2\ell^3}, \quad \delta_3 = \frac{(EI)_c\lambda_3^3}{\mu\omega^2\ell^3}.$$

Eqs. (17)–(21) can be expressed in matrix form:

$$
\begin{bmatrix} W \\ W' \\ \psi \\ M_t \\ M_c \\ Q \end{bmatrix}
=
\begin{bmatrix}
-\alpha_1 \sin\left(\frac{\lambda_1 x}{\ell}\right) & \alpha_1 \cos\left(\frac{\lambda_1 x}{\ell}\right) & \alpha_2 \sinh\left(\frac{\lambda_2 x}{\ell}\right) & \alpha_2 \cosh\left(\frac{\lambda_2 x}{\ell}\right) & \alpha_3 \sinh\left(\frac{\lambda_3 x}{\ell}\right) & \alpha_3 \cosh\left(\frac{\lambda_3 x}{\ell}\right) \\
-\frac{\lambda_1\alpha_1}{\ell} \cos\left(\frac{\lambda_1 x}{\ell}\right) & -\frac{\lambda_1\alpha_1}{\ell} \sin\left(\frac{\lambda_1 x}{\ell}\right) & \frac{\lambda_2\alpha_2}{\ell} \cosh\left(\frac{\lambda_2 x}{\ell}\right) & \frac{\lambda_2\alpha_2}{\ell} \sinh\left(\frac{\lambda_2 x}{\ell}\right) & \frac{\lambda_3\alpha_3}{\ell} \cosh\left(\frac{\lambda_3 x}{\ell}\right) & \frac{\lambda_3\alpha_3}{\ell} \sinh\left(\frac{\lambda_3 x}{\ell}\right) \\
\beta_1 \cos\left(\frac{\lambda_1 x}{\ell}\right) & \beta_1 \sin\left(\frac{\lambda_1 x}{\ell}\right) & \beta_2 \cosh\left(\frac{\lambda_2 x}{\ell}\right) & \beta_2 \sinh\left(\frac{\lambda_2 x}{\ell}\right) & \beta_3 \cosh\left(\frac{\lambda_3 x}{\ell}\right) & \beta_3 \sinh\left(\frac{\lambda_3 x}{\ell}\right) \\
-\gamma_1 \sin\left(\frac{\lambda_1 x}{\ell}\right) & \gamma_1 \cos\left(\frac{\lambda_1 x}{\ell}\right) & \gamma_2 \sinh\left(\frac{\lambda_2 x}{\ell}\right) & \gamma_2 \cosh\left(\frac{\lambda_2 x}{\ell}\right) & \gamma_3 \sinh\left(\frac{\lambda_3 x}{\ell}\right) & \gamma_3 \cosh\left(\frac{\lambda_3 x}{\ell}\right) \\
\delta_1 \sin\left(\frac{\lambda_1 x}{\ell}\right) & -\delta_1 \cos\left(\frac{\lambda_1 x}{\ell}\right) & \delta_2 \sinh\left(\frac{\lambda_2 x}{\ell}\right) & \delta_2 \cosh\left(\frac{\lambda_2 x}{\ell}\right) & \delta_3 \sinh\left(\frac{\lambda_3 x}{\ell}\right) & \delta_3 \cosh\left(\frac{\lambda_3 x}{\ell}\right) \\
\cos\left(\frac{\lambda_1 x}{\ell}\right) & \sin\left(\frac{\lambda_1 x}{\ell}\right) & \cosh\left(\frac{\lambda_2 x}{\ell}\right) & \sinh\left(\frac{\lambda_2 x}{\ell}\right) & \cosh\left(\frac{\lambda_3 x}{\ell}\right) & \sinh\left(\frac{\lambda_3 x}{\ell}\right)
\end{bmatrix}
\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix}.
$$

Or in another form

$$\mathbf{z}(x) = \mathbf{B}(x)\mathbf{a}. \tag{22}$$

At the point ($x = 0$), $\mathbf{z}(x) = \mathbf{z}_{i-1}$, and the matrix equation (22) becomes

$$\begin{bmatrix} W \\ W' \\ \psi \\ M_t \\ M_c \\ Q \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 \\ -\dfrac{\lambda_1\alpha_1}{\ell} & 0 & \dfrac{\lambda_2\alpha_2}{\ell} & 0 & \dfrac{\lambda_3\alpha_3}{\ell} & 0 \\ \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ 0 & -\delta_1 & 0 & \delta_2 & 0 & \delta_3 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix},$$

or

$$\mathbf{z}_{i-1} = \mathbf{B}(0)\mathbf{a}. \tag{23}$$

Therefore, solving for the column vector $\mathbf{a}$, leads to

$$\mathbf{a} = \mathbf{B}^{-1}(0)\mathbf{z}_{i-1}. \tag{24}$$

Substituting Eq. (24) into Eq. (22) yields

$$\mathbf{z}(x) = \mathbf{B}(x)\mathbf{B}^{-1}(0)\mathbf{z}_{i-1}. \tag{25}$$

At the point $x = \ell$, $\mathbf{z}(x) = \mathbf{z}_i$, so that Eq. (25) becomes

$$\mathbf{z}_i = \mathbf{B}(\ell)\mathbf{B}^{-1}(0)\mathbf{z}_{i-1} = U_i\mathbf{z}_{i-1}. \tag{26}$$

Hence the transfer matrix is

$$U_i = \mathbf{B}(\ell)\mathbf{B}^{-1}(0). \tag{27}$$

In this case the inversion of $\mathbf{B}(0)$ is found to be

$$B^{-1}(0) = \begin{bmatrix} 0 & a_{12} & a_{13} & 0 & 0 & a_{16} \\ a_{21} & 0 & 0 & a_{24} & a_{25} & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 & a_{36} \\ a_{41} & 0 & 0 & a_{44} & a_{45} & 0 \\ 0 & a_{52} & a_{53} & 0 & 0 & a_{56} \\ a_{61} & 0 & 0 & a_{64} & a_{65} & 0 \end{bmatrix}. \tag{28}$$

At the point $x = \ell$ the matrix $\mathbf{B}(\ell)$ can be written as

$$B(\ell) = \begin{bmatrix} -\alpha_1 \sin\lambda_1 & \alpha_1 \cos\lambda_1 & \alpha_2 \sinh\lambda_2 & \alpha_2 \cosh\lambda_2 & \alpha_3 \sinh\lambda_3 & \alpha_3 \cosh\lambda_3 \\ -\frac{\lambda_1\alpha_1}{\ell}\cos\lambda_1 & -\frac{\lambda_1\alpha_1}{\ell}\sin\lambda_1 & \frac{\lambda_2\alpha_2}{\ell}\cosh\lambda_2 & \frac{\lambda_2\alpha_2}{\ell}\sinh\lambda_2 & \frac{\lambda_3\alpha_3}{\ell}\cosh\lambda_3 & \frac{\lambda_3\alpha_3}{\ell}\sinh\lambda_3 \\ \beta_1 \cos\lambda_1 & \beta_1 \sin\lambda_1 & \beta_2 \cosh\lambda_2 & \beta_2 \sinh\lambda_2 & \beta_3 \cosh\lambda_3 & \beta_3 \sinh\lambda_3 \\ -\gamma_1 \sin\lambda_1 & \gamma_1 \cos\lambda_1 & \gamma_2 \sinh\lambda_2 & \gamma_2 \cosh\lambda_2 & \gamma_3 \sinh\lambda_3 & \gamma_3 \cosh\lambda_3 \\ \delta_1 \sin\lambda_1 & -\delta_1 \cos\lambda_1 & \delta_2 \sinh\lambda_2 & \delta_2 \cosh\lambda_2 & \delta_3 \sinh\lambda_3 & \delta_3 \cosh\lambda_3 \\ \cos\lambda_1 & \sin\lambda_1 & \cosh\lambda_2 & \sinh\lambda_2 & \cosh\lambda_3 & \sinh\lambda_3 \end{bmatrix}. \tag{29}$$

The final matrix operation $\mathbf{B}(\ell)\mathbf{B}^{-1}(0)$ then produce the transfer matrix, so that

$$\begin{bmatrix} W \\ W' \\ \psi \\ M_t \\ M_c \\ Q \end{bmatrix}_i = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{bmatrix} \begin{bmatrix} W \\ W' \\ \psi \\ M_t \\ M_c \\ Q \end{bmatrix}_{i-1}, \tag{30}$$

or

$$\mathbf{Z}_i = \mathbf{F}\mathbf{Z}_{i-1}. \tag{31}$$

The coefficients of the field transfer matrix "**F**" are illustrated in Appendix A.

3. Transfer matrix scheme

The actual beam is divided into N elements, as shown in Fig. 5, the field matrix **F** for each element is determined as a function of ω_N^2 by using Eq. (31). The relation between the state vector $\mathbf{Z}_N$ at support N and the state vector $\mathbf{Z}_O$ at support O, using transfer matrix method is

$$\mathbf{Z}_N = \mathbf{T}\mathbf{Z}_O, \tag{32}$$

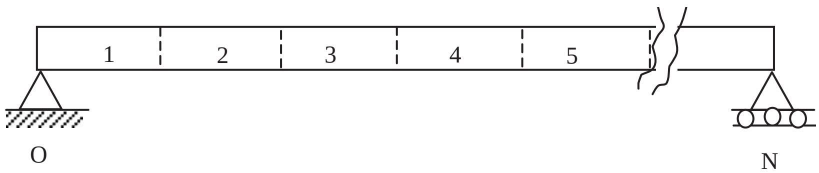

Fig. 5. Schematic diagram of the actual beam.

where

$$\mathbf{T} = \prod_{i=N-1}^{1} \mathbf{F}_i$$

which is called over-all transfer matrix. The coefficients of this matrix (T_{11} to T_{66}) all being function of the circular frequencies ω_n. Expanding Eq. (32) gives six equations, by applying the boundary conditions to these equations the frequency determinant can be easily obtained [9].

3.1. Boundary conditions for analog beam

For the case of simply supported beam, the moments and displacements at both ends are zero, or in view of Eq. (1) by

$$M = M_t + M_c = 0 \tag{33}$$

and

$$W = 0. \tag{34}$$

This can be realized in two different ways [1], either by making the individual moment in each sub-beam zero, i.e.:

$$M_t = M_c = 0. \tag{35}$$

Or, by making the total moment M equal to zero, i.e:

$$M_t = -M_c. \tag{36}$$

The first case, Eq. (35), is called a simple support without shear restraint, Fig. 6(a). This type of support would occur in a simply supported beam without any special restraint at the end. The boundary conditions for this case are:

$$W = 0, \quad M_t = 0, \quad M_c = 0.$$

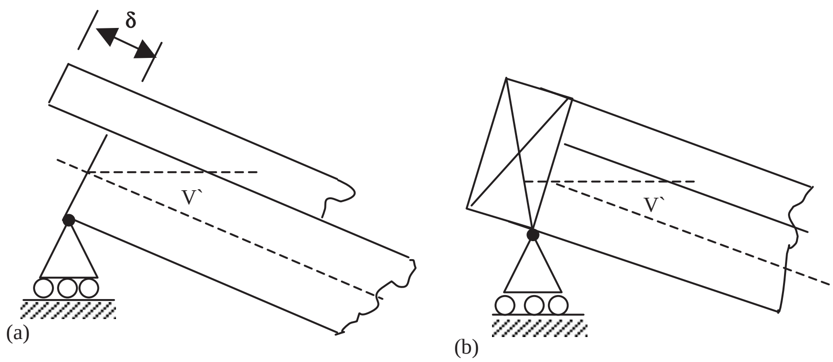

Fig. 6. (a) No shear restraints. (b) Shear restraints.

The second case, Eq. (35), is called simple supported with restraint, Fig. 6(b). The boundary conditions for this case are:

$$W = 0, \quad \psi = -W', \quad M_t = \left(\frac{I_c}{I_t}\right) M_c.$$

4. Application of the model

The above method can now be used to compute the natural frequencies of a simple supported beam with uniformly distributed mass. It is convenient to introduce the following non-dimensional parameters,

$$\eta = \frac{EI_b}{EI_t}, \quad \xi = \frac{EI_t}{kh^2\ell^2}, \quad \varepsilon = \frac{\omega_{TM}}{\omega_{CL}},$$

where η represents the relative importance of bending stiffness EI_b of the slab and beam acting independently and the bending stiffness EI_t caused by the truss action. For typical composite beam η varies from about 0.2 to 1.4 [2]. ξ represents the relative importance of the truss bending stiffness EI_t and shear stiffness k of the shear layer. A very large variation in ξ is possible. ξ equal to zero corresponds to complete interaction (completely rigid shear studs $k = \infty$), and ξ equal to infinity corresponds to zero interaction (no shear studs $k = 0$). While ε represents the relative importance of the natural frequencies calculated by transfer matrix method (ω_{TM}) and the natural frequencies calculated by the classical method (ω_{CL}).

By applying simple supported boundary conditions at each end of the classical beam, the classical natural frequencies can be expressed as [9,10]

$$\omega_{CL} = \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\mu}}. \tag{37}$$

From the above equation ω_{CL} varies between $\omega_{CL\max}$ and $\omega_{CL\min}$ due to the value of EI, where $\omega_{CL\max}$ corresponds to complete interaction between the beam and slab ($EI = EI_t + EI_c$) and $\omega_{CL\min}$ corresponds to zero interaction ($EI = EI_c$).

It is very important to note that, in the classical beam it is not possible to distinguish between boundary condition with and without shear restraints.

For comparison, natural frequencies can be calculated by applying the TMABM and Eq. (37) using the data listed below:

Bending stiffness for the beam component $(EI)_c = 4 \times 10^6\,\mathrm{mt}^2$;

Length of the beam $(\ell) = 10\,\mathrm{m}$;

Mass per unit length = 1000 kg/m;

Distance between centroids of the sub-beams = 0.3 m.

The first four natural frequencies of the beam have been calculated for the case of no shear restraints ($k = 0$) and $\eta = 1$ and are listed in Table 1. The natural frequencies calculated using TMABM agreed completely with those obtained using Eq. (37).

Table 1
Comparison of natural frequencies calculated using TMABM and Refs. [9,10]. For case of no shear restraints ($k = 0$)

Frequancy	Natural frequency (Hz)	
	TMABM	Refs. [9,10]
ω_1	6.24	6.24
ω_2	24.96	24.96
ω_3	56.18	56.17
ω_4	99.52	99.87

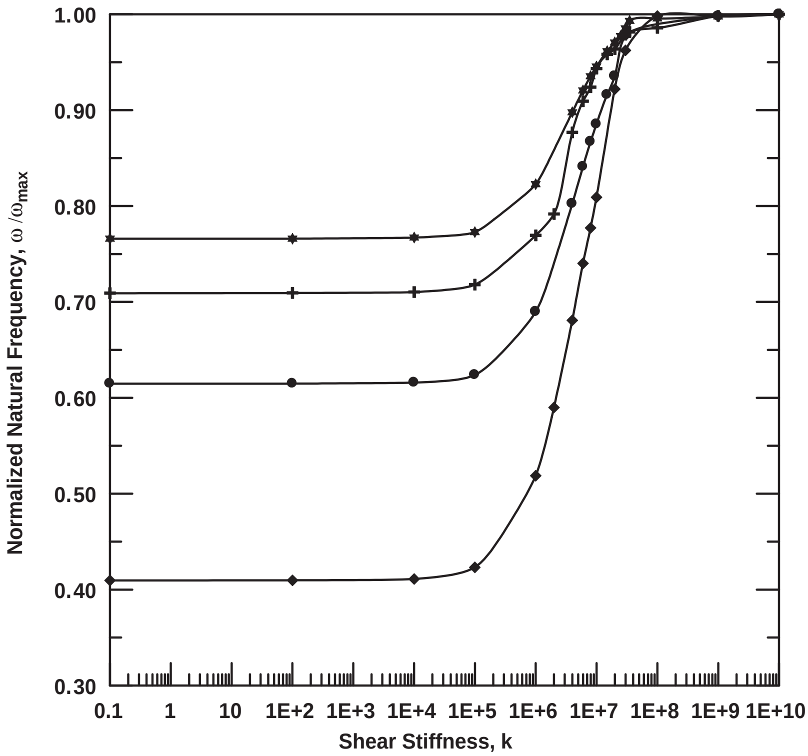

Fig. 7. Variation of normalized natural frequency with shear stiffness (k) for mode 1, ω_1, [—♦— $h = 0.2$, —●— $h = 0.6$, —+— $h = 1.0$, —★— $h = 1.4$].

4.1. Case of no shear restraints at both ends

Figs. 7–9 represent the normalized natural frequencies ($\omega_1, \omega_2, \omega_3$) for the case of no shear restraints at both ends of the beam. The frequencies are normalized with respect to the maximum natural frequency obtained by using Eq. (37).

Different values of "η" have been considered to represent the relative importance of bending stiffness. From the results represented in these figures, it is clear that for sections with complete

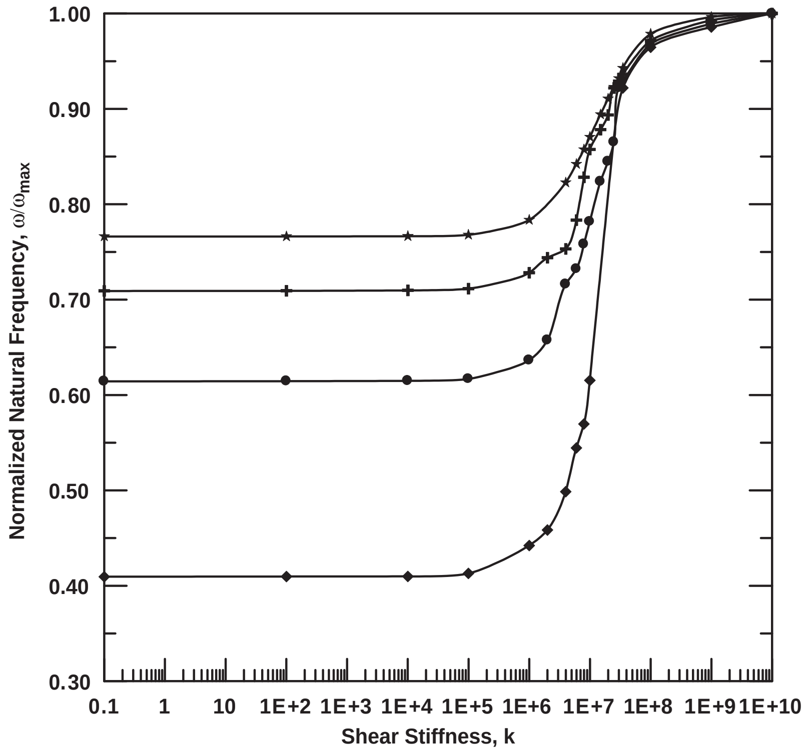

Fig. 8. Variation of normalized natural frequency with Shear stiffness (k) for mode 2, ω_2, [—♦— $h = 0.2$, —●— $h = 0.6$, —+— $h = 1.0$, —★— $h = 1.4$].

interaction between the beam and the slab ($k > 10^7$ N/m^2), the behaviour of the composite beam is identical to that of the equivalent classical beam. However, when sliding occurs ($k < 10^6$ N/m^2), the composite beam presented a lower natural frequency than that obtained by the equivalent classical beam.

5. Conclusions

Explicit expressions for field transfer matrix of a uniform elastic composite beam have been derived. A combined transfer matrix-analog beam method (TMABM) is presented and applied to a simple supported beam with uniformly distributed mass to investigate the natural frequencies. The method can be applied to both cases of the no shear restraint and with shear restraint at both ends. The results obtained from the present method were verified with that of the classical method and a good agreement was achieved. Also, the effect of changing the values of significant parameters η and K on the natural frequencies of the elastic composite beams has been studied. The results obtained from this study indicate that the natural frequency strongly depends on these values and that it is very important to consider these variables especially for the case when

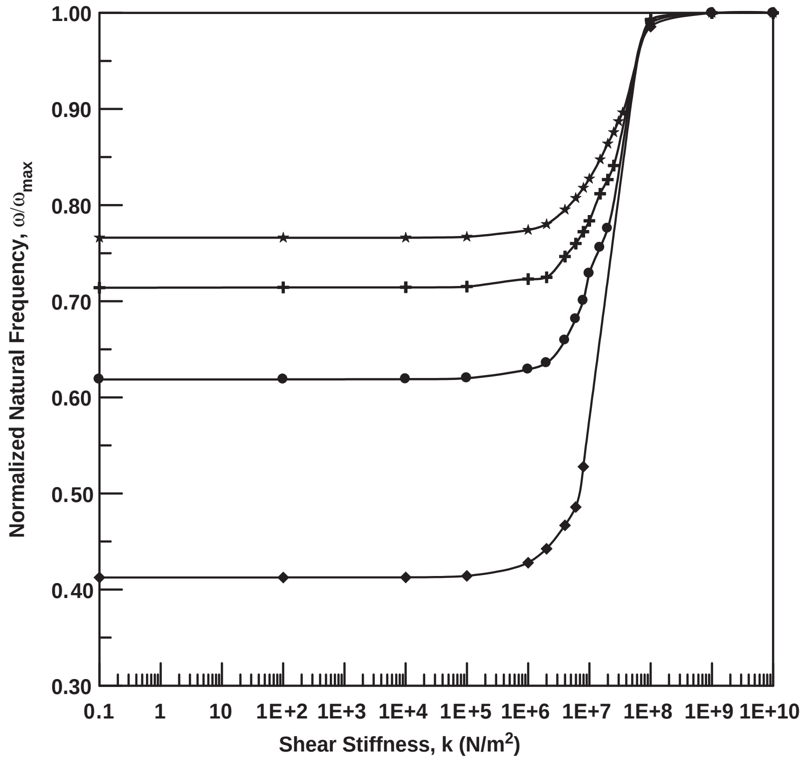

Fig. 9. Variation of normalized natural frequency with Shear stiffness (k) for mode 3, ω_3, [—◆— $h = 0.2$, —●— $h = 0.6$, —+— $h = 1.0$, —★— $h = 1.4$].

studying the bridges and structures that are exposed to dynamic loading. Although, the method is more complicated, it is more accurate and could prove to be a good tool for design purposes. The developed model can be applied to calculate the natural frequencies of multi-span composite bridges with various intermediate conditions beside the effects of the intermediate conditions such as rigid supports.

Appendix A. The coefficient of the field transfer matrix "F"

$$T_{11} = a_{21}\alpha_1 \cos\lambda_1 + a_{41}\alpha_2 \cosh\lambda_2 + a_{61}\alpha_3 \cosh\lambda_3,$$
$$T_{12} = -a_{12}\alpha_1 \sin\lambda_1 + a_{32}\alpha_2 \sinh\lambda_2 + a_{52}\alpha_3 \sinh\lambda_3,$$
$$T_{13} = -a_{13}\alpha_1 \sin\lambda_1 + a_{33}\alpha_2 \sinh\lambda_2 + a_{53}\alpha_3 \sinh\lambda_3,$$
$$T_{14} = a_{24}\alpha_1 \cos\lambda_1 + a_{44}\alpha_2 \cosh\lambda_2 + a_{64}\alpha_3 \cosh\lambda_3,$$
$$T_{15} = a_{25}\alpha_1 \cos\lambda_1 + a_{45}\alpha_2 \cosh\lambda_2 + a_{65}\alpha_3 \cosh\lambda_3,$$
$$T_{16} = -a_{16}\alpha_1 \sin\lambda_1 + a_{36}\alpha_2 \sinh\lambda_2 + a_{56}\alpha_3 \sinh\lambda_3,$$

$$T_{21} = -a_{21}\frac{\lambda_1\alpha_1}{\ell}\sin\lambda_1 + a_{41}\frac{\lambda_2\alpha_2}{\ell}\sinh\lambda_2 + a_{61}\frac{\lambda_3\alpha_3}{\ell}\sinh\lambda_3,$$
$$T_{22} = -a_{12}\frac{\lambda_1\alpha_1}{\ell}\cos\lambda_1 + a_{32}\frac{\lambda_2\alpha_2}{\ell}\cosh\lambda_2 + a_{52}\frac{\lambda_3\alpha_3}{\ell}\cosh\lambda_3$$
$$T_{23} = -a_{13}\frac{\lambda_1\alpha_1}{\ell}\cos\lambda_1 + a_{33}\frac{\lambda_2\alpha_2}{\ell}\cosh\lambda_2 + a_{53}\frac{\lambda_3\alpha_3}{\ell}\cosh\lambda_{33},$$
$$T_{24} = -a_{24}\frac{\lambda_1\alpha_1}{\ell}\sin\lambda_1 + a_{44}\frac{\lambda_2\alpha_2}{\ell}\sinh\lambda_2 + a_{64}\frac{\lambda_3\alpha_3}{\ell}\sinh\lambda_3,$$
$$T_{25} = -a_{25}\frac{\lambda_1\alpha_1}{\ell}\sin\lambda_1 + a_{45}\frac{\lambda_2\alpha_2}{\ell}\sinh\lambda_2 + a_{65}\frac{\lambda_3\alpha_3}{\ell}\sinh\lambda_3,$$
$$T_{26} = -a_{16}\frac{\lambda_1\alpha_1}{\ell}\cos\lambda_1 + a_{36}\frac{\lambda_2\alpha_2}{\ell}\cosh\lambda_2 + a_{56}\frac{\lambda_3\alpha_3}{\ell}\cosh\lambda_3,$$

$$T_{31} = a_{21}\beta_1\sin\lambda_1 + a_{41}\beta_2\sinh\lambda_2 + a_{61}\beta_3\sinh\lambda_3,$$
$$T_{32} = a_{12}\beta_1\cos\lambda_1 + a_{32}\beta_2\cosh\lambda_2 + a_{52}\beta_3\cosh\lambda_3,$$
$$T_{33} = a_{13}\beta_1\cos\lambda_1 + a_{33}\beta_2\cosh\lambda_2 + a_{53}\beta_3\cosh\lambda_3,$$
$$T_{34} = a_{24}\beta_1\sin\lambda_1 + a_{44}\beta_2\sinh\lambda_2 + a_{64}\beta_3\sinh\lambda_3,$$
$$T_{35} = a_{25}\beta_1\sin\lambda_1 + a_{45}\beta_2\sinh\lambda_2 + a_{65}\beta_3\sinh\lambda_3,$$
$$T_{36} = a_{16}\beta_1\cos\lambda_1 + a_{36}\beta_2\cosh\lambda_2 + a_{56}\beta_3\cosh\lambda_3,$$

$$T_{41} = a_{21}\gamma_1\cos\lambda_1 + a_{41}\gamma_2\cosh\lambda_2 + a_{61}\gamma_3\cosh\lambda_3,$$
$$T_{42} = -a_{12}\gamma_1\sin\lambda_1 + a_{32}\gamma_2\sinh\lambda_2 + a_{52}\gamma_3\sinh\lambda_3,$$
$$T_{43} = -a_{13}\gamma_1\sin\lambda_1 + a_{33}\gamma_2\sinh\lambda_2 + a_{53}\gamma_3\sinh\lambda_3,$$
$$T_{44} = a_{24}\gamma_1\cos\lambda_1 + a_{44}\gamma_2\cosh\lambda_2 + a_{64}\gamma_3\cosh\lambda_3,$$
$$T_{45} = a_{25}\gamma_1\cos\lambda_1 + a_{45}\gamma_2\cosh\lambda_2 + a_{65}\gamma_3\cosh\lambda_3,$$
$$T_{46} = -a_{16}\gamma_1\sin\lambda_1 + a_{36}\gamma_2\sinh\lambda_2 + a_{56}\gamma_3\sinh\lambda_3,$$

$$T_{51} = -a_{21}\delta_1\cos\lambda_1 + a_{41}\delta_2\cosh\lambda_2 + a_{61}\delta_3\cosh\lambda_3,$$
$$T_{52} = a_{12}\delta_1\sin\lambda_1 + a_{32}\delta_2\sinh\lambda_2 + a_{52}\delta_3\sinh\lambda_3,$$
$$T_{53} = a_{13}\delta_1\sin\lambda_1 + a_{33}\delta_2\sinh\lambda_2 + a_{53}\delta_3\sinh\lambda_3,$$
$$T_{54} = -a_{24}\delta_1\cos\lambda_1 + a_{44}\delta_2\cosh\lambda_2 + a_{64}\delta_3\cosh\lambda_3,$$
$$T_{55} = -a_{25}\delta_1\cos\lambda_1 + a_{45}\delta_2\cosh\lambda_2 + a_{65}\delta_3\cosh\lambda_3,$$
$$T_{56} = a_{16}\delta_1\sin\lambda_1 + a_{36}\delta_2\sinh\lambda_2 + a_{56}\delta_3\sinh\lambda_3,$$

$$T_{61} = a_{21}\sin\lambda_1 + a_{41}\sinh\lambda_2 + a_{61}\sinh\lambda_3,\quad T_{62} = a_{12}\cos\lambda_1 + a_{32}\cosh\lambda_2 + a_{52}\cosh\lambda_3,$$
$$T_{63} = a_{13}\cos\lambda_1 + a_{33}\cosh\lambda_2 + a_{53}\cosh\lambda_3,\quad T_{64} = a_{24}\sin\lambda_1 + a_{44}\sinh\lambda_2 + a_{64}\sinh\lambda_3,$$
$$T_{65} = a_{25}\sin\lambda_1 + a_{45}\sinh\lambda_2 + a_{65}\sinh\lambda_3,\quad T_{66} = a_{16}\cos\lambda_1 + a_{36}\cosh\lambda_2 + a_{56}\cosh\lambda_3,$$

where the following abbreviations have been introduced:

$$a_{12} = (\beta_3 - \beta_2)/\Delta_1, \quad a_{13} = (\lambda_2\alpha_2 - \lambda_3\alpha_3)/\ell\Delta_1, \quad a_{16} = (\lambda_3\alpha_3\beta_2 - \lambda_2\alpha_2\beta_3)/\ell\Delta_1,$$
$$a_{21} = (\gamma_2\delta_3 - \gamma_3\delta_2)/\Delta_2, \quad a_{24} = (\delta_2\alpha_3 - \delta_3\alpha_2)/\Delta_2, \quad a_{25} = (-\alpha_3\gamma_2 + \alpha_2\gamma_3)/\Delta_2,$$
$$a_{32} = (\beta_1 - \beta_3)/\Delta_1, \quad a_{33} = (\lambda_3\alpha_3 + \lambda_1\alpha_1)/\ell\Delta_1, \quad a_{36} = (-\lambda_3\alpha_3\beta_1 - \lambda_1\alpha_1\beta_3)/\ell\Delta_1.$$

$$a_{41} = (-\gamma_1\delta_3 - \gamma_3\delta_1)/\Delta_2, \quad a_{44} = (\delta_1\alpha_3 + \delta_3\alpha_1)/\Delta_2, \quad a_{45} = (-\alpha_1\gamma_3 + \alpha_3\gamma_1)/\Delta_2,$$
$$a_{52} = (\beta_2 - \beta_1)/\Delta_1, \quad a_{53} = (-\lambda_2\alpha_2 - \lambda_1\alpha_1)/\ell\Delta_1, \quad a_{56} = (\lambda_1\alpha_1\beta_2 + \lambda_2\alpha_2\beta_1)/\ell\Delta_1,$$
$$a_{61} = (\gamma_1\delta_2 + \gamma_2\delta_1)/\Delta_2, \quad a_{64} = (-\delta_1\alpha_2 - \delta_2\alpha_1)/\Delta_2, \quad a_{65} = (\alpha_1\gamma_2 - \alpha_2\gamma_1)/\Delta_2.$$

and

$$\Delta_1 = \frac{\beta_1}{\ell}(\lambda_2\alpha_2 - \lambda_3\alpha_3) + \frac{\beta_2}{\ell}(\lambda_1\alpha_1 + \lambda_3\alpha_3) + \frac{\beta_3}{\ell}(-\lambda_2\alpha_2 - \lambda_1\alpha_1),$$
$$\Delta_2 = \alpha_1(-\gamma_3\delta_2 + \gamma_2\delta_3) + \alpha_2(-\gamma_3\delta_1 - \gamma_1\delta_3) + \alpha_3(\gamma_1\delta_2 + \gamma_2\delta_1).$$

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