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Journal of Sound and Vibration 277 (2004) 815–824
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Free vibration of a strong non-linear system described with complex functions

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Received 23 April 2003; accepted 10 September 2003

Abstract

In this paper two approximate analytical methods for solving strong non-linear differential equations with complex functions are developed. Besides the adopted elliptic Krylov–Bogolubov procedure, an alternative analytical method based on the variation of the parameters as well as the elliptic Krylov–Bogolubov is introduced. The difference is that the alternative method considers two approximate first order differential equations. The suggested procedures are compared and tested for the system with strong cubic non-linearity and a small non-linearity of Van der Pol type. Comparing the approximate analytical solutions with exact numerical solution it is concluded that the difference is negligible.

1. Introduction

There are a few papers dealing with the problem of analytical approximate solving procedures of the non-linear differential equation with complex function. In most of the papers the non-linearity is small and the known approximate analytical procedures developed for a second order differential equation with small non-linearity are adopted for this special type of two coupled second order differential equations [1–5]. In Refs. [6,7] an extension of the solving procedure is presented considering the systems with strong cubic non-linearity. Based on the solving methods developed for the one-degree-of-freedom Duffing equation, new methods for obtaining the approximate analytical solution for the differential equation with complex function and cubic non-linearity are tested. In Ref. [8] the class of non-linear differential equations with complex function is extended and new types of cubic non-linearity are introduced. In Refs. [9,10] some particular solutions for these types of non-linearity are considered.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.09.032

In this paper the differential equation with the following type of cubic non-linearity

$$a\ddot{z} + b_1 z + b_3 z(z\bar{z}) = \varepsilon Z(z, \dot{z}, cc) \tag{1}$$

and with initial conditions

$$z(0) = z_0, \quad \dot{z}(0) = 0 \tag{2}$$

is considered. z is a complex function, $\bar{z}$ the complex conjugate function, a, b_1 and b_3 are constant coefficients and εZ is a small function dependent on the complex function and complex conjugate function and their time derivatives. This type of non-linearity is of special interest as it describes the non-linear elastic property of the rotor which represents the fundamental working element of most machines. The initial conditions correspond to the case without impact. In Ref. [11] an approximate analytical solution of the pure-cubic complex differential equation (in Eq. (1) it is $b_1 = 0$) with initial conditions (2) is given.

In this paper the extension of the previous results is carried out and an approximate analytical solution is suggested for the general differential equation (1) with initial conditions (2). The exact solution of the strong non-linear generating differential equation is introduced. The trial solution of the differential equation with small non-linearity is assumed to have the form of the generating solution but with time variable parameters. The trial solution satisfies some constraints but does not satisfy the differential equation (1). Two types of procedures are suggested: one, the ordinary elliptic Krylov–Bogolubov method of averaging [12,13] of the exact system of two first order differential equations and the second, forming of two approximate first order differential equations of motion assuming the first time derivative of the trial solution to be the same as for the generating solution. Both solving procedures are applied to the case when the small function is of the Van der Pol type. The exact numeric solution is compared with solutions obtained applying both suggested methods.

2. The elliptic Krylov–Bogolubov method

The elliptic Krylov–Bogolubov method gives the approximate solution of Eq. (1) based on the introduction of time variable parameters in the generating solution of the same differential equation with $\varepsilon = 0$ called the generating differential equation.

For the case when the small non-linearities are neglected, i.e., $\varepsilon = 0$ the generating differential equation is obtained

$$a\ddot{z} + b_1 z + b_3 z(z\bar{z}) = 0. \tag{3}$$

For the initial conditions (2) the closed-form solution of Eq. (1) is

$$z = (A + \mathrm{i}B)\mathrm{cn}(\omega t, m) \equiv (A + \mathrm{i}B)\mathrm{cn}, \tag{4}$$

where cn is the Jacobi elliptic function [14] with parameter ω

$$\omega = \sqrt{\frac{b_1 + b_3(A^2 + B^2)}{a}} \tag{5}$$

and modulus of the Jacobi elliptic function

$$m = \frac{b_3(A^2 + B^2)}{2a\omega^2}. \tag{6}$$

For the initial conditions (2) it is $A = A_0$ and $B = B_0$ as $z_0 = A_0 + \mathrm{i}B_0$ where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. It is obvious that the parameter ω and the modulus of the Jacobi elliptic function m depend on the initial values A_0 and B_0.

Based on the generating solution (4) the trial solution of Eq. (1) is formed:

$$z = (A(t) + \mathrm{i}B(t))\mathrm{cn}(\psi, m) \equiv (A + \mathrm{i}B)\mathrm{cn}, \tag{7}$$

where $A(t)$ and $B(t)$ are time dependent and the corresponding functions ω (5) and m (6) which depend on A and B are also time dependent. The argument of the elliptic function is $\psi(t) = \int_0^t \omega(s)\,\mathrm{d}s$. The task of finding the solution $z(t)$ is transformed into finding the functions $A(t)$ and $B(t)$ so that expression (7) satisfies Eq. (1). Due to the Krylov–Bogolubov procedure the function $z(t)$ has to satisfy the constraint that the first time derivative of the trial solution must have the same form as for the generating solution (4)

$$\dot{z} = (A + \mathrm{i}B)\omega\mathrm{cn}_{\psi}, \tag{8}$$

where $(_{\psi}) = \partial/\partial\psi$ is the derivative with respect to the argument ψ. It is satisfied for

$$(\dot{A} + \mathrm{i}\dot{B})\mathrm{cn} + (A + \mathrm{i}B)\dot{m}\mathrm{cn}_m = 0, \tag{9}$$

where $(_m) = \partial/\partial m$ is the derivative with respect to the modulus of Jacobi elliptic function and $\dot{m} = (\partial m/\partial A)\dot{A} + (\partial m/\partial B)\dot{B}$. The time derivative of Eq. (8) is

$$\ddot{z} = (\dot{A} + \mathrm{i}\dot{B})\omega\mathrm{cn}_{\psi} + (A + \mathrm{i}B)\dot{\omega}\mathrm{cn}_{\psi} + (A + \mathrm{i}B)\omega^2\mathrm{cn}_{\psi\psi} + (A + \mathrm{i}B)\omega\dot{m}\mathrm{cn}_{\psi m}, \tag{10}$$

where $\dot{\omega} = (\partial/\partial A)\dot{A} + (\partial/\partial B)\dot{B}$, and the derivatives with respect to the argument and to the modulus of Jacobi elliptic function, are $\mathrm{cn}_{\psi\psi} = \partial\mathrm{cn}_{\psi}/\partial\psi$ and $\mathrm{cn}_{\psi m} = \partial\mathrm{cn}_{\psi}/\partial m$, where $\mathrm{cn}_{\psi m} = -(\mathrm{sn}_m\,\mathrm{dn} + \mathrm{sn}\,\mathrm{dn}_m)$. Substituting relations (7) and (10) into Eq. (1) and separating the real and imaginary terms, a system of two first order differential equations is obtained:

$$\begin{aligned} &a(\dot{A}\omega\mathrm{cn}_{\psi} + A\dot{\omega}\mathrm{cn}_{\psi} + A\omega\dot{m}\mathrm{cn}_{\psi}m) = \mathrm{Re}(\varepsilon Z),\\ &a(\dot{B}\omega\mathrm{cn}_{\psi} + B\dot{\omega}\mathrm{cn}_{\psi} + B\omega\dot{m}\mathrm{cn}_{\psi m}) = \mathrm{Im}(\varepsilon Z). \end{aligned} \tag{11}$$

According to Eqs. (9) and (11) and using the form of the time derivatives of functions (5) and (6) with time variable functions, the following system of two first order coupled differential equations is obtained:

$$\begin{aligned} &a\dot{A}\left(\omega\mathrm{cn}_{\psi}\mathrm{cn}_m - \mathrm{cn}\,\mathrm{cn}_{\psi}\frac{a\omega^3}{b_1} - \omega\mathrm{cn}\,\mathrm{cn}_{\psi m}\right) = \mathrm{Re}(\varepsilon Z)\mathrm{cn}_m,\\ &a\dot{B}\left(\omega\mathrm{cn}_{\psi}\mathrm{cn}_m - \mathrm{cn}\,\mathrm{cn}_{\psi}\frac{a\omega^3}{b_1} - \omega\mathrm{cn}\,\mathrm{cn}_{\psi m}\right) = \mathrm{Im}(\varepsilon Z)\mathrm{cn}_m, \end{aligned} \tag{12}$$

where $\varepsilon Z \equiv \varepsilon Z((A + \mathrm{i}B)\mathrm{cn}, (A + \mathrm{i}B)\omega\mathrm{cn}_{\psi}, cc)$. The differential equations (12) represent the transformed version of the differential equation (1) and the solutions $A(t)$ and $B(t)$ form the exact solution (7) of the differential equation (1). Unfortunately, it is impossible to find the closed form solution for system (12). It is at this point the usual averaging procedure is introduced. The period of averaging corresponds to the period of the Jacobi elliptic functions. That is, Eq. (12) is transformed to the averaged system

$$a\dot{A}\omega\langle \mathrm{cn}_{\psi}\mathrm{cn}_m - \mathrm{cn}\,\mathrm{cn}_{\psi m}\rangle = \langle \mathrm{Re}(\varepsilon Z)\mathrm{cn}_m\rangle,$$
$$a\dot{B}\omega\langle \mathrm{cn}_{\psi}\mathrm{cn}_m - \mathrm{cn}\,\mathrm{cn}_{\psi m}\rangle = \langle \mathrm{Im}(\varepsilon Z)\mathrm{cn}_m\rangle, \tag{13}$$

where $\langle\,\cdot\,\rangle = (1/(4K))\int_0^{4K}(\cdot)\,\mathrm{d}\psi$, where K is the total elliptic integral of the first kind [15]. Solving the system of differential equations (13) the time variable functions $A(t)$ and $B(t)$ are obtained. Substituting the solution $A(t)$ and $B(t)$ into relations (5) and (6) the expressions for ω and m are denoted.

The suggested procedure is valid for all types of small function, but it has some disadvantages. The averaging procedure for the differential equations (13) is not an easy task as is discussed in the paper of Coppola and Rand [13]. Besides, in general, for the case when the small function on the right-hand side of Eq. (1) is a non-symmetrical one (corresponding to its real and imaginary part) the averaged system of differential equations (13) is a non-linear coupled system of differential equations from which an analytical solution cannot possibly be obtained and the discussion based only on the numerical experiment.

3. The approximate first order differential equations

To avoid the disadvantages of the elliptic Krylov–Bogolubov method a new approximate procedure for solving the differential equation (1) is introduced. It is based on forming a system of two approximate first order differential equations. In this procedure the trial solution is assumed in the same form (7) as in the previous method and has to satisfy Eq. (1). The first time derivative of solution (7) is assumed in the form (8) and the other terms are neglected. Relating to this assumption the first order differential equations obtained by substitution of Eq. (7) and the second time derivative of Eq. (8) into Eq. (1) are approximate ones. Namely, they are not forced to satisfy relation (9).

For simplicity, introduce solution (7) of Eq. (1) in the polar form

$$z = C(t)\exp(\mathrm{i}\alpha(t))\mathrm{cn}(\psi, m) \equiv C\exp(\mathrm{i}\alpha)\mathrm{cn}, \tag{14}$$

where $C(t)$ and $\alpha(t)$ are time-dependent functions, $\psi(t) = \int_0^t \omega(C)\mathrm{d}t$ is the argument of the elliptic function cn and the function ω and the modulus m of the Jacobi elliptic function are, respectively,

$$\omega = \sqrt{\frac{b_1 + b_3 C(t)^2}{a}}, \quad m = \frac{b_3 C(t)^2}{2a\omega^2}. \tag{15}$$

In relation (17) the functions ω and m are time dependent as is $C \equiv C(t)$. The task of finding the solution $z(t)$ is transformed into finding the functions $C(t)$ and $\alpha(t)$ so that expression (14) satisfies Eq. (1). Substituting Eq. (14) and its second time derivative into Eq. (1) and separating the real and imaginary terms the system of two first order differential equations is obtained:

$$a\dot{C}(\omega \mathrm{cn}_{\psi} + C\omega' \mathrm{cn}_{\psi} + C\omega m' \mathrm{cn}_{\psi m}) = \mathrm{Re}(\varepsilon Z\exp(-\mathrm{i}\alpha)),$$
$$Ca\dot{\alpha}\omega \mathrm{cn}_{\psi} = \mathrm{Im}(\varepsilon Z\exp(-\mathrm{i}\alpha)), \tag{16}$$

where $Z \equiv Z(C\exp(\mathrm{i}\alpha)\mathrm{cn}, C\omega\exp(\mathrm{i}\alpha)\mathrm{cn}_{\psi}, cc)$, $(_{\psi}) = \partial/\partial\psi$ is the derivative with respect to the argument ψ, $(_m) = \partial/\partial m$ is the derivative with respect to the modulus of Jacobi elliptic function m

and $(') = \mathrm{d}/\mathrm{d}C$. Integrating the differential equations (16) for the initial conditions $C(0) = C_0$ and $\alpha(0) = \alpha_0$ the functions $C(t)$ and $\alpha(t)$ are obtained. The expressions for ω and m are obtained by substituting the solution $C(t)$ into relations (15).

This type of differential equation (16) is very convenient for the case when the right-hand side member of the first equation (16) vanishes.

4. Strong cubic non-linear Van der Pol differential equation with complex function

The differential equation of the Van der Pol type with complex function is

$$\ddot{z} + z + z(z\bar{z}) = \varepsilon g(1 \pm pz\bar{z})\mathrm{i}\dot{z}, \tag{17}$$

where g and p are constants and $\varepsilon \ll 1$ is a small parameter.

Let the differential equation (17) be transformed into the system of two first order differential equations according to elliptic Krylov–Bogolubov procedure. For the small non-linear function $\varepsilon Z = \varepsilon g(1 + z\bar{z})\mathrm{i}\dot{z}$ the real and the imaginary parts are

$$\begin{aligned} \mathrm{Re}(\varepsilon Z) &= -\varepsilon g(1 \pm p(A^2 + B^2)\mathrm{cn}^2)B\omega\mathrm{cn}_\psi, \\ \mathrm{Im}(\varepsilon Z) &= \varepsilon g(1 \pm p(A^2 + B^2)\mathrm{cn}^2)A\omega\mathrm{cn}_\psi. \end{aligned} \tag{18}$$

The averaged differential equations (13) are

$$\begin{aligned} \dot{A}\langle \mathrm{cn}_\psi \mathrm{cn}_m - \mathrm{cn}\,\mathrm{cn}_{\psi m} \rangle &= -\varepsilon g B \langle (1 \pm p(A^2 + B^2)\mathrm{cn}^2)\mathrm{cn}_\psi \mathrm{cn}_m \rangle, \\ \dot{B}\langle \mathrm{cn}_\psi \mathrm{cn}_m - \mathrm{cn}\,\mathrm{cn}_{\psi m} \rangle &= \varepsilon g A \langle (1 \pm p(A^2 + B^2)\mathrm{cn}^2)\mathrm{cn}_\psi \mathrm{cn}_m \rangle. \end{aligned} \tag{19}$$

Analyzing relations (19) it is obvious that dividing the equations gives $A\dot{A} = -B\dot{B}$. It means that for the initial conditions $A(0) = A_0$ and $B(0) = B_0$ it is $A^2 + B^2 = C_0^2 = const.$ According to relations (5) and (6) it can be concluded that $\omega = \sqrt{1 + C_0^2} = const.$, $m = C_0^2/2\omega^2 = const.$ Integrating the differential equations (19) for initial conditions (2) and using the aforementioned conclusion the solution $A(t)$ and $B(t)$ is obtained.

According to Eq. (16) the approximate first order differential equations which correspond to the differential equation (17) are

$$\begin{aligned} &a\dot{C}(\omega\mathrm{cn}_\psi + C\omega'\mathrm{cn}_\psi + C\omega m'\mathrm{cn}_{\psi m}) = 0, \\ &\dot{\alpha} = \varepsilon g(1 \pm pC^2\mathrm{cn}^2). \end{aligned} \tag{20}$$

For the initial condition $C(0) = C_0$ the solution of the first equation $(20)_1$ is $C = C_0 = const.$ The second equation $(20)_2$ transforms to

$$\dot{\alpha} = \varepsilon g(1 \pm pC_0^2\mathrm{cn}^2(\omega_0 t, m_0)), \tag{21}$$

where $\omega_0 = \sqrt{1 + C_0^2} = const.$ and $m_0 = C_0^2/2(1 + C_0^2) = const.$ Integrating Eq. (21) for the initial condition $\alpha(0) = \alpha_0$ it is

$$\alpha = \alpha_0 + \varepsilon g\left\{\left[1 \mp \frac{pC_0^2}{m_0}(1 - m_0)\right]t \pm \frac{pC_0^2}{m_0\omega_0}E(\omega_0 t, m_0)\right\}, \tag{22}$$

where $E(\omega_0 t, m_0)$ is the incomplete elliptic integral of the second kind [16]. Finally, the approximate solution of Eq. (17) is

$$z = C_0 \exp(\mathrm{i}\alpha_0)\mathrm{cn}(\omega_0 t, m_0) \times \exp\left(\varepsilon g \mathrm{i}\left\{\left[1 \mp \frac{pC_0^2}{m_0}(1-m_0)\right]t \pm \frac{pC_0^2}{m_0\omega_0}E(\omega_0 t, m_0)\right\}\right). \tag{23}$$

This solution can be discussed at length. Using the polar form of the solution it can be concluded that the term $C_0 \exp(\mathrm{i}\alpha_0)\mathrm{cn}(\omega_0 t, m_0)$ defines the modulus of the complex function. This value does not depend on the small value εg. The modulus is a periodically time variable function. The period of variation is $T = 4K(m_0)/\omega_0$, where $K(m_0)$ is a complete elliptic integral of the first kind [15]. The argument of the complex function is

$$\beta = t \mp \frac{pC_0^2}{m_0}\left[(1-m_0)t \mp \frac{E(\omega_0 t, m_0)}{\omega_0}\right]. \tag{24}$$

The argument varies due to small function εg. As it is a small value the argument varies slowly in time. The variation is approximately linear in time. Namely, expanding the function $E(\omega_0 t, m_0)$ in series [16] and assuming the first two terms the function β is approximately $\beta \approx t(1 \pm pC_0^2)$. Three separate cases appear: (a) $p > 0$, (b) $p < 0$, (c) $p = 0$. For $p = 0$, when the small function is linear, it is $\beta = t$ and the approximate analytic solution of Eq. (17) is

$$z = C_0 \exp(\mathrm{i}\alpha_0)\mathrm{cn}(\omega_0 t, m_0)\exp(\varepsilon g \mathrm{i} t). \tag{25}$$

For $p > 0$ the angle $\beta \approx t(1 + pC_0^2)$ increases during the time and observing in x–y plane the argument increases in a positive direction.

For $p < 0$ it is $\beta \approx t(1 - pC_0^2)$. The variation of angle depends on the value of pC_0^2. For $pC_0^2 = 1$ it is $\beta \approx 0$ and the argument is constant. For $pC_0^2 < 1$ it is $\beta > 0$ and for $pC_0^2 > 1$ it is $\beta < 0$. The direction of increasing of the argument of the complex function depends on the sign of β. For $\beta > 0$ it is in one direction (positive) and for $\beta < 0$ it is in the opposite direction (negative direction).

This discussion is of special interest for rotor dynamics. Namely, relation (23) corresponds to the motion of the rotor centre during vibrations described with differential equation (17). The modulus of complex function corresponds to the radial position of the rotor centre and the argument to the angle position of the centre of the rotor.

4.1. Example

Consider a numerical example. The parameter value is $p = 1$ and initial conditions are

$$z_0 = 0.5(1 + \mathrm{i}), \quad \dot{z}_0 = 0. \tag{26}$$

Separating the real and the imaginary terms in Eq. (26) using the complex function $z = x + \mathrm{i}y$ where x and y are time-dependent functions and i is the imaginary unit, a system of two second order differential equations is obtained:

$$\begin{aligned} \ddot{x} + x + x(x^2 + y^2) &= -0.01\dot{y}(1 + x^2 + y^2), \\ \ddot{y} + y + y(x^2 + y^2) &= 0.01\dot{x}(1 + x^2 + y^2). \end{aligned} \tag{27}$$

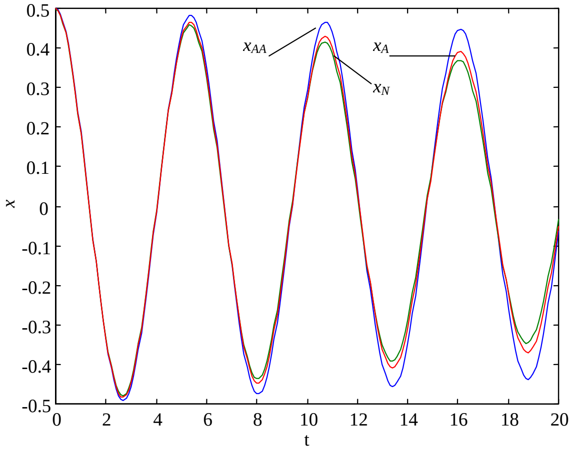

Fig. 1. $x - t$ diagrams obtained: numerical (x_N), applying the elliptic Krylov–Bogolubov method (x_A) and applying the approximate equations (x_{AA}).

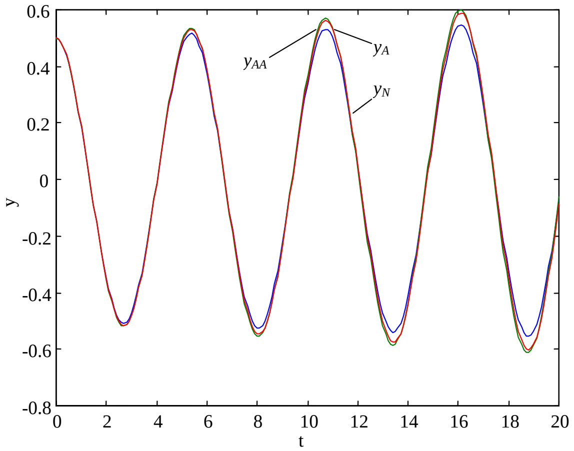

Fig. 2. $y - t$ diagrams obtained: numerical (y_N), applying the elliptic Krylov–Bogolubov method (y_A) and applying the approximate equations (y_{AA}).

The system of coupled differential equations (27) is solved numerically applying the Runge–Kutta procedure. In Figs. 1 and 2 the numeric solution x_N and y_N is plotted.

In Figs. 1 and 2, also, the approximate analytic solution x_A and y_A of the system of differential equations

$$\begin{aligned} \dot{A}\langle \mathrm{cn}_\psi \mathrm{cn}_m - \mathrm{cn}\,\mathrm{cn}_{\psi m} \rangle &= -0.01B\left\langle \left(1+\tfrac{1}{2}\mathrm{cn}^2\right)\mathrm{cn}_\psi \mathrm{cn}_m \right\rangle, \\ \dot{B}\langle \mathrm{cn}_\psi \mathrm{cn}_m - \mathrm{cn}\,\mathrm{cn}_{\psi m} \rangle &= 0.01A\left\langle \left(1+\tfrac{1}{2}\mathrm{cn}^2\right)\mathrm{cn}_\psi \mathrm{cn}_m \right\rangle, \end{aligned} \tag{28}$$

obtained applying the elliptic Krylov–Bogolubov method is shown.

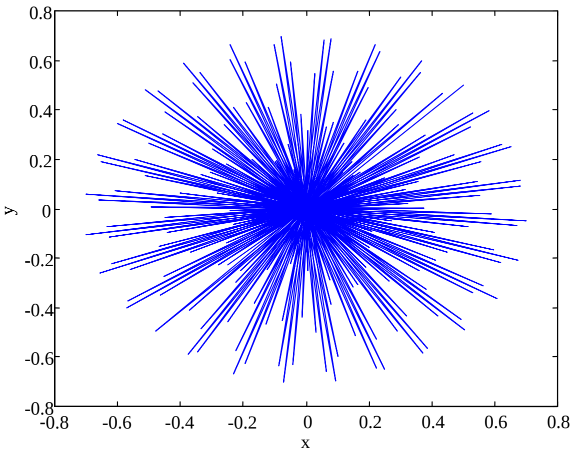

Fig. 3. The solution in x–y plane.

The approximate solution (23) x_{AA} and y_{AA} for the corresponding initial conditions (2) in the polar form $C_0 = \sqrt{2}/2$ and $\alpha_0 = \pi/4$ is

$$x_{AA} = \frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4} - 0.0015t + 0.0245E\left(\sqrt{\frac{3}{2}}t, \frac{1}{6}\right)\right)\,\mathrm{cn}\left(\sqrt{\frac{3}{2}}t, \frac{1}{6}\right),$$

$$y_{AA} = \frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{4} - 0.0015t + 0.0245E\left(\sqrt{\frac{3}{2}}t, \frac{1}{6}\right)\right)\mathrm{cn}\left(\sqrt{\frac{3}{2}}t, \frac{1}{6}\right). \tag{29}$$

Comparing the exact numeric solution x_N and y_N in Figs. 1 and 2 with the approximate solution x_A and y_A obtained by the averaging elliptic Krylov–Bogolubov method and the approximate solution x_{AA} and y_{AA}, it can be concluded that both groups of approximate solutions are on the top of the exact numeric solution. The elliptic Krylov–Bogolubov method gives better results but requires a very complex calculating procedure. The second method gives satisfactory results and due to its simplicity is more convenient to be applied in engineering practice.

In Fig. 3 the solution of Eq. (27) is plotted in $x - y$ plane. The figure proves the statement of the character of the solution for $p > 0$ as is discussed in the previous section.

5. Conclusions

1. Comparing the exact numeric solution and the approximate solution obtained applying the elliptic Krylov–Bogolubov method for the second order differential equation with complex function, it can be concluded that the difference for small initial conditions and not too long time is negligible. The disadvantage of the method is its complexity and serious averaging calculation.
2. The alternative analytic method is suitable for application only for the case when the non-linearity is small and the time is short. It is preferable to the elliptic Krylov–Bogolubov because

its simplicity. It is particularly convenient for the systems where the right-hand side member of the first equation (16) vanishes. In spite of the estimation error the solution is usually very convenient for qualitative analysis of the dynamic properties of some systems. It is recommended for dynamical analysis of the rotors.

3. In the polar form solution of the Van der Pol differential equation the polar function C is constant and also the frequency ω and the modulus m of the Jacobi elliptic function are constant values. These values depend on the initial value C_0. The modulus of the complex function z does not depend on the small value εg. The modulus of the complex function is a periodical time variable function. The argument of the complex function depends on the small parameter εg and on the parameter p, too. Depending on the value of parameter p and the initial values α_0 and C_0 the argument of complex function may be constant or approximately linearly time dependent.

Acknowledgements

The present study has been supported by Ministry of Science, Technologies and Development, Republic of Serbia (Project No. 1874).

References

[1] L. Cveticanin, Approximate analytical solutions to a class of non-linear equations with complex functions, *Journal of Sound and Vibration* 157 (1992) 289–302.
[2] L. Cveticanin, An asymptotic solution for weak nonlinear vibrations of the rotor, *Mechanism and Machine Theory* 28 (1993) 495–505.
[3] L. Cveticanin, Resonant vibrations of nonlinear rotors, *Mechanism and Machine Theory* 30 (4) (1995) 581–588.
[4] G.M. Mahmoud, Approximate solutions of a class of complex nonlinear dynamical systems, *Physica A* 253 (1998) 211–222.
[5] G.M. Mahmoud, A.H.A. Shaban, On periodic solutions of parametrically excited complex non-linear dynamical systems, *Physica A* 278 (2000) 390–404.
[6] L. Cveticanin, An approximate solution for a system of two coupled differential equations, *Journal of Sound and Vibration* 152 (1992) 375–380.
[7] L. Cveticanin, Analytic methods for solving strongly non-linear differential equations, *Journal of Sound and Vibration* 214 (1998) 325–338.
[8] R. Manasevich, J. Mawhin, F. Zanolin, Periodic solutions of some complex-valued Lienard and Rayleigh equations, *Nonlinear Analysis* 36 (1999) 997–1014.
[9] L. Cveticanin, Some particular solutions which describe the motion of the rotor, *Journal of Sound and Vibration* 212 (1998) 173–178.
[10] L. Cveticanin, Analytic approach for the solution of the complex-valued strong non-linear differential equation of Duffing type, *Physica A* 297 (2001) 348–360.
[11] L. Cveticanin, Analytic solutions of the system of two coupled pure cubic non-linear oscillators equations, *Journal of Sound and Vibration* 245 (2001) 571–580.
[12] S.B. Yuste, J.D. Bejarano, Improvement of a Krylov–Bogolubov method that uses Jacobi elliptic functions, *Journal of Sound and Vibration* 139 (1990) 151–163.
[13] V.T.Coppola, R.H. Rand, Averaging using elliptic functions: approximation of limit cycles, *Acta Mechanica* 81 (1990) 125–142.

[14] P.F. Byrd, M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer, Berlin, 1954.
[15] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Nauka, Moscow, 1979 (in Russian).
[16] I.S. Gradshtein, I.M. Rjizhik, *Tablicji integralov, summ, rjadov i proizvedenij*, Nauka, Moscow, 1971 (in Russian).