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# Generalized thermoelastic waves in anisotropic plates sandwiched between liquid layers

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**Abstract**

The propagation of waves in a homogeneous, transversely isotropic, thermally conducting plate bordered with layers of inviscid liquid or half-space of inviscid liquid on both sides, is investigated in the context of Lord–Shulman (LS), Green–Lindsay [GL] and Green–Nagdhi (GN) theories of thermoelasticity. Secular equations for homogeneous, transversely isotropic plate in closed form and isolated mathematical conditions for symmetric and antisymmetric wave modes in completely separate terms are derived. The results for isotropic materials, coupled and uncoupled theories of thermoelasticity have been obtained as particular cases. It is shown that the purely transverse motion (SH mode), which is not affected by thermal variations, gets decoupled from rest of the motion of wave propagation and occurs along an in-plane axis of symmetry. The special cases, such as short wavelength waves, thin plate waves and leaky Lamb waves of the secular equations are also discussed. The amplitudes of displacement components and temperature change have also been computed and studied. Finally, the numerical solution is carried out for transversely isotropic plate of zinc material bordered with water. The dispersion curves for symmetric and antisymmetric wave modes, attenuation coefficient and amplitudes of displacement and temperature change in case of fundamental symmetric ($S_0$) and skew symmetric ($A_0$) modes along the direction making an angle of 45° with the axis of symmetry are presented in order to illustrate and compare the theoretical results. The theory and numerical computations are found to be in close agreement.


## 1. Introduction

The coupling between thermal and strain fields gives rise to the coupled theory of thermoelasticity. Chadwick and Sneddon [1] discussed in detail the influence of volume and thermal changes coupled with each other in the form of plane harmonic waves. A list of

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Nowacki's papers on the coupled theory of thermoelasticity can be found in his monumental books [2,3]. Kozlov [4] has considered the influence of coupling between temperature and strain fields on the dynamic behavior of rectangular plates. Massalas et al. [5] studied the influence of a constant heat flux on the static and dynamic response of simply supported and clamped circular plate with immovably constrained edge. Kumar [6] studied the coupled thermoelastic waves in plates of thickness '$d$' subjected to axially symmetric hydrostatic tension. Saxena and Dhaliwal [7] studied two-dimensional problems of axisymmetric and plane strain cases in coupled thermoelastic wave propagation in a homogeneous isotropic plate. Chadwick and Seet [8] and Chadwick [9] have also studied the propagation of plane harmonic waves in homogeneous transversely isotropic and anisotropic materials in the context of coupled theory of thermoelasticity, respectively. The classical theory of heat conduction predicts an infinite speed of heat transportation, which contradicts the physical facts. During the last three decades, non-classical theories have been developed to alleviate this paradox. Lord and Shulman [10] incorporated a flux-rate term in Fourier's law of heat conduction in order to formulate a generalized theory that admits finite speed for thermal signals. Green and Lindsay [11] included a temperature rate among the constitutive variables to develop a temperature-rate-dependent thermoelasticity that does not violate the classical Fourier's law of heat conduction when the body under consideration has a center of symmetry; and this theory also predicts a finite speed of heat propagation. According to these theories, heat propagation should be viewed as a wave phenomenon rather than a diffusion phenomenon. A wave-like thermal disturbance referred to as 'second sound' by Chandrasekharaiah [12]. The above theories are also supported by experiments exhibited the actual occurrence of second sound at low temperatures and small intervals of time. Researchers such as Ackerman et al., Guyer and Krumhansal, and Ackerman and Overtone [13–15] experimentally proved for solid helium that thermal waves (second sound) propagating with finite, though quite large, speed also exist, although for most of the solids, the corresponding frequency window namely, range of frequency of thermal excitations in which thermal waves can be detected is extremely limited. The recent relevant theoretical developments on the subjects of finite velocity of heat propagation are due to Green and Nagdhi [16–18], who provide sufficient basic modifications in the constitutive equations that permit treatment of much wider class of heat flow problems. Sharma et al. [19], Sharma [20] and Sharma and Singh [21] studied the propagation of thermoelastic waves in homogeneous isotropic plates subjected to stress-free insulated, stress-free isothermal, rigidly fixed insulated and rigidly fixed isothermal boundary conditions in the context of Conventional–Coupled (CT), Lord–Shulman (LS), Green–Lindsay (GL) and Green–Nagdhi (GN) theories of thermoelasticity. The secular equations for the symmetric and antisymmetric wave modes in the plate have been derived in the compact form and solved numerically. Sharma et al. [22] studied the propagation of thermoelastic waves in homogeneous transversely isotropic plates subjected to stress-free insulated, stress-free isothermal, rigidly fixed insulated and rigidly fixed isothermal boundary conditions.

Recently, resurgent interest in Lamb waves was partially initiated by its application of multisensors [23–25]. The density and viscosity sensing with Lamb waves is based on the principle that the presence of a liquid in contact with a solid plate changes the propagation velocity and the amplitude of the Lamb waves in the plate of free boundaries due to the inertial and viscous effects of the liquid. Schoch [26] first investigated the effect of an inviscid liquid on the propagation of Lamb waves. When a plate of finite thickness is bordered with a half-space homogeneous liquid on both sides, part of Lamb wave energy in the plate is coupled into the liquid as radiation; most of the energy is still in the solid. This type of disturbance is called the leaky Lamb wave. Schoch derived the dispersion relation for leaky Lamb waves for an isotropic plate and an inviscid liquid. Incidentally, the dispersion equations also have an interface wave solution whose velocity is slightly less than the bulk sound velocity in the liquid and most of energy is in the liquid. It is often called the Scholte wave after Scholte [27]. Kurtz and Bolt [28] derived a dispersion equation for bending waves when a plate is in contact with an inviscid fluid based on the acoustic impedance concept. Watkins et al. [29] calculated the attenuation of Lamb waves in the presence of an inviscid liquid using an acoustic impedance method. Wu and Zhu [30] studied the propagation of Lamb waves in a plate bordered with inviscid liquid layers on both sides. The dispersion equations of this case were derived and solved numerically. It was also shown that the acoustic impedance approach is valid only when the plate thickness is much smaller than the wavelength of the transverse wave in the solid. Zhu and Wu [31] derived the dispersion equations of Lamb waves of a plate bordered with viscous liquid layer or half-space viscous liquid on both sides. Sharma and Pathania [32,33] derived the dispersion equations of Lamb waves of a thermoelastic and generalized thermoelastic plate bordered with inviscid liquid layer or half-space inviscid liquid on both sides. Sharma et al. [34] also studied the dispersion equations of a transversely isotropic thermoelastic plate bordered with layers of inviscid liquid on both sides in the context of coupled theory of thermoelasticity.

In the present paper, we have discussed the analysis of Lamb type wave propagation in an infinite homogeneous, transversely isotropic, thermoelastic plate bordered with an inviscid liquid layers or half-spaces on both sides in the context of generalized (LS, GL, GN) theories of thermoelasticity. More general dispersion equations of Lamb type waves are derived and discussed in coupled and uncoupled theories of thermoelasticity. The secular equations for leaky Lamb waves have been deduced. Numerical solution of the dispersion equations for zinc material is also presented.

## 2. Formulation of the problem

We consider an infinite homogeneous, transversely isotropic, thermally conducting elastic plate of thickness $2d$ initially at uniform temperature $T_0$. We take origin of the co-ordinate system $(x, y, z)$ on the middle surface of the plate. The $x - y$ plane is chosen to coincide with the middle surface and the $z$-axis normal to it along the thickness as shown in Fig. 1. The surface $z = \pm d$ is subjected to different boundary conditions.

We take $x - z$ as the plane of incidence and assume that the solutions are explicitly independent of $y$ but implicit dependence is there so that component $v$ of displacement is non-vanishing. The basic governing equations for homogeneous transversely isotropic medium in generalized thermoelasticity, in the absence of body forces and heat sources, are given by

$$c_{66}v_{,xx} + c_{44}v_{,zz} = \rho\ddot{v}, \tag{1}$$

$$c_{11}u_{,xx} + c_{44}u_{,zz} + (c_{13} + c_{44})w_{,xz} - \beta_1(T + t_1\delta_{2k}\dot{T})_{,x} = \rho\ddot{u}, \tag{2}$$

$$(c_{13} + c_{44})u_{,xz} + c_{44}w_{,xx} + c_{33}w_{,zz} - \beta_3(T + t_1\delta_{2k}\dot{T})_{,z} = \rho\ddot{w}, \tag{3}$$

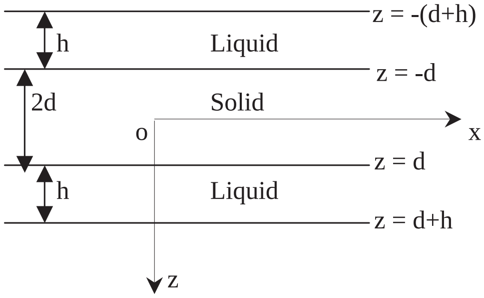



Fig. 1. Geometry of the problem.

$$K_1T_{,xx} + K_3T_{,zz} - \rho C_e(\dot{T} + t_0\ddot{T}) = T_0(\beta_1\dot{u}_{,x} + \beta_3\dot{w}_{,z} + t_0\delta_{1k}(\beta_1\ddot{u}_{,x} + \beta_3\ddot{w}_{,z})), \tag{4}$$

where $\beta_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3, \beta_3 = 2c_{13}\alpha_1 + c_{33}\alpha_3$.

The dot notation is used for time differentiation and comma denotes spatial derivatives. $\rho$ is the density of the solid plate, $C_e$ is the specific heat at constant strain of the solid plate, $\alpha_1, K_1$ are the coefficients of linear thermal expansion and thermal conductivity, in the direction orthogonal to axis of symmetry, $\alpha_3, K_3$ are the corresponding quantities along the axis of symmetry, $c_{ij}$'s are the elastic parameters. Here $\vec{u}(x,z,t) = (u,v,w)$ is the displacement vector, $T(x,z,t)$ is the temperature change, $T_0$ is the reference temperature, $t_0$ and $t_1$ are the thermal relaxation times. $\delta_{ik}$ is Kronecker's delta with $k = 1$ for LS theory and $k = 2$ for Green–Lindsay (GL) theory. Eqs. (1)–(4) in non-dimensional form can be written as

$$c_4v_{,xx} + c_2v_{,zz} = \ddot{v}, \tag{5}$$

$$u_{,xx} + c_2u_{,zz} + c_3w_{,xz} - (T + t_1\delta_{2k}\dot{T})_{,x} = \ddot{u}, \tag{6}$$

$$c_3u_{,xz} + c_2w_{,xx} + c_1w_{,zz} - \bar{\beta}(T + t_1\delta_{2k}\dot{T})_{,z} = \ddot{w}, \tag{7}$$

$$T_{,xx} + \bar{K}T_{,zz} - (\dot{T} + t_0\ddot{T}) = \in(\dot{u}_{,x} + \bar{\beta}\dot{w}_{,z} + t_0\delta_{1k}(\ddot{u}_{,x} + \bar{\beta}\ddot{w}_{,z})), \tag{8}$$

where we have defined the following quantities:

$$u' = \frac{\rho\omega^* v_1 u}{\beta_1 T_0}, \quad v' = \frac{\rho\omega^* v_1 v}{\beta_1 T_0}, \quad w' = \frac{\rho\omega^* v_1 w}{\beta_1 T_0}, \quad x' = \frac{\omega^* x}{v_1}, \quad z' = \frac{\omega^* z}{v_1}, \quad t' = \omega^* t,$$
$$t_1' = \omega^* t_1, \quad t_0' = \omega^* t_0, \quad v_1^2 = \frac{c_{11}}{\rho}, \quad \bar{\beta} = \frac{\beta_3}{\beta_1}, \quad \bar{K} = \frac{K_3}{K_1}, \quad \sigma_{ij}' = \frac{\sigma_{ij}}{\beta_1 T_0}, \quad T' = \frac{T}{T_0}, \quad c_1 = \frac{c_{33}}{c_{11}},$$
$$c_2 = \frac{c_{44}}{c_{11}}, \quad c_3 = \frac{c_{13} + c_{44}}{c_{11}}, \quad c_4 = \frac{c_{66}}{c_{11}}, \quad c_{66} = \frac{c_{11} - c_{23}}{2}, \quad d' = \frac{\omega^* d}{v_1}, \quad h' = \frac{\omega^* h}{v_1}, \quad \xi' = \frac{\xi v_1}{\omega^*},$$
$$c' = \frac{c}{v_1}, \quad \omega' = \frac{\omega}{\omega^*}, \quad \omega^* = \frac{C_e c_{11}}{K_1}, \quad \in = \frac{\beta_1^2 T_0}{\rho C_e c_{11}}. \tag{9}$$

Here $\omega^*$ is the characteristic frequency of the solid plate, $\in$ is the thermomechanical coupling constant and $v_1$ is the longitudinal wave velocity in the medium. The primes have been suppressed for convenience.

In the liquid boundary layers, we have

$$u_j = \phi_{j,x} + \psi_{j,z}, \quad w_j = \phi_{j,z} - \psi_{j,x}, \quad j = 1, 2, \tag{10}$$

where $\phi_j$ and $\psi_j$, $j = 1, 2$ are respectively, the scalar velocity potential and vector velocity component along the $y$ direction for the top liquid layer ($j = 1$) and for the bottom liquid layer ($j = 2$), $u_j$ and $w_j$ are, respectively, $x$ and $z$ components of the particle velocity in the layers of liquid. The potential functions $\phi_j$ satisfy the non-dimensional basic governing equations

$$\nabla^2 \phi_j - \frac{1}{\delta_L^2} \ddot{\phi}_j = 0, \quad j = 1, 2, \tag{11}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad \delta_L^2 = \frac{c_L^2}{v_1^2}, \quad c_L^2 = \frac{\lambda_L}{\rho_L}.$$

Here $c_L$ is the velocity of sound in the liquid, $\rho_L$ is the density of the liquid and $\lambda_L$ is the bulk modulus.

## 3. Boundary conditions

The boundary conditions at the solid–liquid interfaces $z = \pm d$ to be satisfied are:

(i) The magnitude of the normal component of the stress tensor of the plate should be equal to the pressure of the liquid. This implies that

$$(c_3 - c_2)u_{,x} + c_1 w_{,z} - \bar{\beta}(T + t_1 \delta_{2k} \dot{T}) = \frac{\rho_L \omega^2}{\rho} \phi_j, \quad j = 1, 2.$$

(ii) The tangential component of the stress tensor should be zero implying that $u_{,z} + w_{,x} = 0$

(iii) The normal component of displacement of the solid should be equal to that of the liquid. This leads to $w = \phi_{j,z}, \quad j = 1, 2$.

(iv) The thermal boundary condition is given by $T_{,z} + HT = 0$, where $H$ is Biot's heat transfer coefficient.

## 4. Solution of the problem

We assume solutions of the form

$$\{u, v, w, T, \phi_1, \phi_2\} = \{1, V, W, S, \Phi_1, \Phi_2\} U \mathrm{e}^{\mathrm{i}\xi(x \sin\theta + mz - ct)}, \tag{12}$$

where $c = \omega/\xi$ is the non-dimensional phase velocity, $\omega$ is the frequency and $\xi$ is the wave number. Here $\theta$ is the angle of inclination of wave normal with axis of symmetry $z$-axis; $m$ is still unknown parameter. Here $V, W, S$ and $\Phi_1, \Phi_2$ are, respectively, the amplitude ratios of displacements $v, w$ temperature $T$ and potentials $\phi_1, \phi_2$ to that of displacement $u$. Upon using solutions (12) in Eqs. (5)–(8), we obtain

$$(c_2m^2 + c_4s^2 - c^2)V = 0, \tag{13a}$$

$$c_2m^2 + s^2 - c^2 + c_3msW + sc\tau_1 S = 0, \tag{13b}$$

$$c_3sm + (c_1m^2 + c_2s^2 - c^2)W + \bar{\beta}mc\tau_1 S = 0, \tag{13c}$$

$$\mathrm{i}\in\xi sc^2\tau_0' + \mathrm{i}\in\xi\bar{\beta}c^2m\tau_0'W - (m^2\bar{K} + s^2 - c^2\tau_0)S = 0, \tag{13d}$$

where $\tau_0 = t_0 + \mathrm{i}\omega^{-1}$, $\tau_0' = t_0\delta_{1k} + \mathrm{i}\omega^{-1}$, $\tau_1 = t_1\delta_{2k} + \mathrm{i}\omega^{-1}$.

Eq. (13a) in the above system corresponds to purely transverse wave mode (SH) that decoupled from rest of the motion and is not affected by the thermal variations. The characteristic roots corresponding to Eq. (13a) are given by

$$m_9, m_{10} = \pm\sqrt{\frac{c^2 - c_4s^2}{c_2}}. \tag{14}$$

We ignore it in the following analysis, as this corresponds to SH wave modes.

From the rest of Eqs. (13), which corresponds to the coupled longitudinal, shear vertical (SV) and thermal (T-mode) motion, one can obtain

$$(m^2 - m_1^2)(m^2 - m_3^2)(m^2 - m_5^2) = 0, \tag{15}$$

where $m_k^2$, $k = 1, 3, 5$ are the roots of the equation

$$m^6 + Am^4 + Bm^2 + C = 0. \tag{16}$$

Here the coefficients $A, B, C$ are given by

$$\begin{aligned}
A &= \frac{Ps^2 - Jc^2}{c_1c_2} + \frac{s^2 - c^2\tau_0}{\bar{K}} + \frac{\mathrm{i}\in\omega c^2\bar{\beta}^2\tau_0'\tau_1}{c_1\bar{K}},\\
B &= \frac{(c_2s^2 - c^2)(s^2 - c^2)}{c_1c_2} + \frac{(s^2 - c^2\tau_0)(Ps^2 - Jc^2)}{c_1c_2\bar{K}} + \frac{\mathrm{i}\in\omega c^2\tau_0'\tau_1}{c_1c_2\bar{K}}[(\bar{\beta}^2 - 2c_3\bar{\beta} + c_1)s^2 - c^2\bar{\beta}^2],\\
C &= \frac{c_2s^2 - c^2}{c_1c_2\bar{K}}[(s^2 - c^2\tau_0)(s^2 - c^2) + \mathrm{i}\in s^2\omega c^2\tau_0'\tau_1],
\end{aligned} \tag{17}$$

where $P = c_1 + c_2^2 - c_3^2$, $J = c_1 + c_2$, $s = \sin\theta$.

Eq. (16) being cubic in $m^2$ admits six solutions for $m$, which also have the property $m_2 = -m_1$, $m_4 = -m_3$, $m_6 = -m_5$. For each $m_q$, $q = 1, 2, 3, 4, 5, 6$ the amplitude ratios $W$ and $S$ can be expressed as

$$W_q = \begin{cases} \dfrac{m_qa_q}{s}, & q = 1, 2, 3, 4,\\[2ex] -\dfrac{[c_2m_q^2 + s^2 - c^2 + sc\tau_1 S_q]}{c_3m_qs}, & q = 5, 6, \end{cases} \tag{18}$$

$$S_q = \begin{cases} -\dfrac{[(c_2 + c_3 a_q)m_q^2 + s^2 - c^2]}{sc\tau_1}, & q = 1, 2, 3, 4, \\ \dfrac{[c_1 c_2 m_q^4 + (Ps^2 - Jc^2)m_q^2 + (c_2 s^2 - c^2)(s^2 - c^2)]a_q}{sc\bar{\beta}\tau_1(c_2 m_q^2 + (1 - c_3/\bar{\beta})s^2 - c^2)}, & q = 5, 6, \end{cases} \tag{19}$$

where

$$a_q = \bar{\beta}\frac{c_2 m_q^2 + (1 - c_3/\bar{\beta})s^2 - c^2}{(c_1 - c_3\bar{\beta})m_q^2 + c_2 s^2 - c^2}, \quad q = 1, 2, 3, 4, 5, 6. \tag{20}$$

Combining Eqs. (18) and (19) with stress–strain–temperature relations we rewrite the formal solution for the displacements, temperature, stresses and temperature gradient as

$$(u, w, T) = \sum_{q=1}^{6}(1, W_q, S_q)U_q \exp[\mathrm{i}\xi(xs + m_q z - ct)], \quad -d < z < d, \tag{21}$$

$$(\sigma_{zz}, \sigma_{xz}, w, T_{,z}) = \sum_{q=1}^{6}\mathrm{i}\xi(D_{1q}, D_{2q}, D_{3q}, D_{4q})U_q \exp[\mathrm{i}\xi(xs + m_q z - ct)]. \tag{22}$$

Again using solutions (12) in Eq. (11), we obtain

$$(\Phi_1, \Phi_2) = \sum_{q=7}^{8}\mathrm{i}\xi(D_{1q}, D_{3q})U_q \exp[\mathrm{i}\xi(xs + m_q z - ct)], \quad d < z < d + h \text{ and } -(d + h) < z < -d, \tag{23}$$

where

$$m_7, m_8 = \pm\sqrt{\frac{c^2}{\delta_L^2} - s^2} \tag{24}$$

are the characteristic roots corresponding to Eq. (11).

Here

$$D_{1q} = \begin{cases} (c_3 - c_2)s + c_1 m_q W_q + c\bar{\beta}\tau_1 S_q, & q = 1, 2, \ldots, 6, \\ -\omega^2\rho_L/\mathrm{i}\xi\rho, & q = 7, 8, \end{cases}$$

$$D_{2q} = \begin{cases} c_2(m_q + sW_q), & q = 1, 2, \ldots, 6, \\ 0, & q = 7, 8, \end{cases}$$

$$D_{3q} = \begin{cases} W_q/\mathrm{i}\xi, & q = 1, 2, \ldots, 6, \\ -m_q, & q = 7, 8, \end{cases}$$

$$D_{4q} = \begin{cases} (m_q + H/\mathrm{i}\xi)S_q, & q = 1, 2, \ldots, 6 \\ 0, & q = 7, 8. \end{cases} \tag{25}$$

The main difference between this case and the case of leaky Lamb waves is that the functions $\Phi_1$ and $\Phi_2$ here are chosen in such a way that the acoustical pressure is zero at $z = \pm(d + h)$, in other words $\Phi_1$ and $\Phi_2$ here are of standing wave solutions, for leaky Lamb waves they are of traveling waves.

## 5. Derivation of the secular equations

By invoking stress free and thermal boundary conditions at plate surfaces $z = \pm d$, we obtain a system of eight simultaneous linear equations in amplitudes $U_q$, $q = 1, 23, 4, 5, 6, 7, 8$ as

$$\sum_{q=1}^{8} D_{1q} E_q U_q = 0, \quad \sum_{q=1}^{8} D_{2q} E_q U_q = 0, \quad \sum_{q=1}^{8} D_{3q} E_q U_q = 0, \quad \sum_{q=1}^{8} D_{4q} E_q U_q = 0. \tag{26}$$

where $E_q = \mathrm{e}^{\pm \mathrm{i}\xi m_q d}$, $q = 1, 2, 3, \ldots, 8$.

System of Eqs. (26) have a non-trivial solution if the determinant of the coefficient of $U_q$, $q = 1, 2, 3 \ldots 8$ vanishes, which leads to a characteristic equation for the propagation of modified guided thermoelastic waves in the plate. We refer such waves as plate waves rather than Lamb waves whose properties were originally derived by Lamb in 1917 for isotropic elastic solids. The characteristic equation for the thermoelastic plate waves in this case, after applying lengthy algebraic reductions and manipulations of the determinant along with conditions $\gamma^* \neq 0$ and $\gamma^* \neq (2n-1)(\pi/2)$, $n = 1, 2, 3 \ldots$ leads to the following secular equations:

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} - \frac{D_{13}}{D_{11}} \frac{G_3}{G_1} \left[\frac{T_3}{T_5}\right]^{\pm 1} - \frac{D_{17} T_7}{D_{37} D_{11} G_1 [T_5]^{\pm 1}} [D_{31} G_1 - D_{33} G_3 + D_{35} G_5] = -\frac{D_{15}}{D_{11}} \frac{G_5}{G_1}. \tag{27}$$

Here the superscript $+1$ corresponds to skew symmetric and $-1$ refers to symmetric modes and

$$T_k = \tan \gamma m_k, \quad k = 1, 3, 5, \quad T_7 = \tan \gamma^* m_7, \quad \gamma = \xi d, \quad \gamma^* = \xi h,$$
$$G_1 = D_{23}(D'_{45} + \mathrm{i} D''_{45} T_5^{\pm 1}) - D_{25}(D'_{43} + \mathrm{i} D''_{43} T_3^{\pm 1}),$$
$$G_3 = D_{21}(D'_{45} + \mathrm{i} D''_{45} T_5^{\pm 1}) - D_{25}(D'_{41} + \mathrm{i} D''_{41} T_1^{\pm 1}),$$
$$G_5 = D_{21}(D'_{43} + \mathrm{i} D''_{43} T_3^{\pm 1}) - D_{23}(D'_{41} + \mathrm{i} D''_{41} T_1^{\pm 1}),$$

where $D'_{4q} = m_q S_q$, $D''_{4q} = H S_q / \mathrm{i}\xi$.

If we let $\rho_L$ approach to zero, Eq. (27) reduces to the dispersion equation

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} - \frac{D_{13}}{D_{11}} \frac{G_3}{G_1} \left[\frac{T_3}{T_5}\right]^{\pm 1} = -\frac{D_{15}}{D_{11}} \frac{G_5}{G_1}. \tag{28}$$

Here $G_1, G_3, G_5$ are now given by

$$G_1 = D_{23} D'_{45} - D_{25} D'_{43}, \quad G_3 = D_{21} D'_{45} - D_{25} D'_{41}, \quad G_5 = D_{21} D'_{43} - D_{23} D'_{41} \tag{29a}$$

for a stress-free thermally insulated $(H \to 0)$plate and

$$G_1 = D_{23} D''_{45} T_5^{\pm 1} - D_{25} D''_{43} T_3^{\pm 1},$$
$$G_3 = D_{21} D''_{45} T_5^{\pm 1} - D_{25} D''_{41} T_1^{\pm 1},$$
$$G_5 = D_{21} D''_{43} T_3^{\pm 1} - D_{23} D''_{41} T_1^{\pm 1} \tag{29b}$$

for stress-free isothermal $(H \to \infty)$plate.

Eq. (28) is the same as obtained and discussed by Sharma et al. [22] in case of homogeneous transversely isotropic thermoelastic plate.

The secular equation (27) is the transcendental equation, which contains complete information about the phase velocity, wave number and attenuation coefficient of the plate waves. In general, wave number and hence the phase velocity of the waves is complex quantity, therefore the waves are attenuated in space. If we write

$$c^{-1} = V_P^{-1} + \mathrm{i}\omega^{-1}Q, \tag{30}$$

where $V_P$ and $Q$ are real, the exponent $\mathrm{e}^{\mathrm{i}\xi(x-ct)}$ in the plane wave solution (12) becomes

$$\frac{\mathrm{i}\omega}{V_P}\{x\sin\theta - V_P t\} - Qx\sin\theta.$$

This shows that $V_P$ is the propagation speed and $Q$ the attenuation coefficient of the waves. Upon using Eq. (30) in Eq. (27) the values of $V_P$ and $Q$ for different modes can be obtained.

For isotropic materials, we have

$$c_1 = 1, \quad c_2 = \delta^2, \quad c_3 = 1-\delta^2, \quad \bar{\beta} = 1, \quad \bar{K} = 1, \quad \alpha^2 = c^2 - 1 \quad \text{and} \quad m_q = \alpha_q.$$

Therefore, Eqs. (18) and (19) reduce to

$$W_q = \begin{cases} \alpha_q, & q = 1,2,3,4, \\ -\alpha_q^{-1}, & q = 5,6, \end{cases}$$

$$S_q = \begin{cases} -(\alpha_q^2 - \alpha^2)/c\tau_1, & q = 1,2,3,4, \\ 0, & q = 5,6. \end{cases} \tag{31}$$

The secular Eq. (27) becomes

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} - \frac{(\alpha_1^2 - \alpha^2)}{(\alpha_3^2 - \alpha^2)}\frac{(\xi\alpha_1 + HT_1^{\pm 1})}{(\xi\alpha_3 + HT_3^{\pm 1})}\left[\frac{T_3}{T_5}\right]^{\pm 1} + \frac{\omega^2\rho_L\alpha_1}{\rho\xi^2\alpha_7\delta^2}\frac{(\alpha_5^2+1)}{(\alpha_5^2-1)^2}\frac{T_7}{[T_5]^{\pm 1}}\left[1 - \frac{\alpha_3(\alpha_1^2-\alpha^2)(\xi\alpha_1 + HT_1^{\pm 1})}{\alpha_1(\alpha_3^2-\alpha^2)(\xi\alpha_3 + HT_3^{\pm 1})}\right]$$
$$= -\frac{4\alpha_1\alpha_5}{(\alpha_5^2-1)^2}\left[1 - \frac{\alpha_3(\alpha_1^2-\alpha^2)(\xi\alpha_1 + HT_1^{\pm 1})}{\alpha_1(\alpha_3^2-\alpha^2)(\xi\alpha_3 + HT_3^{\pm 1})}\right]. \tag{32}$$

In the absence of liquid $\rho_L \to 0$, Eq. (32) reduces to

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} - \frac{(\alpha_1^2 - \alpha^2)}{(\alpha_3^2 - \alpha^2)}\frac{(\xi\alpha_1 + HT_1^{\pm 1})}{(\xi\alpha_3 + HT_3^{\pm 1})}\left[\frac{T_3}{T_5}\right]^{\pm 1} = -\frac{4\alpha_1\alpha_5}{(\alpha_5^2-1)^2}\left[1 - \frac{\alpha_3(\alpha_1^2-\alpha^2)(\xi\alpha_1 + HT_1^{\pm 1})}{\alpha_1(\alpha_3^2-\alpha^2)(\xi\alpha_3 + HT_3^{\pm 1})}\right]. \tag{33}$$

For stress-free thermally insulated plate $H \to 0$, Eq. (33) leads to

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} - \frac{\alpha_1}{\alpha_3}\frac{(\alpha_1^2 - \alpha^2)}{(\alpha_3^2 - \alpha^2)}\left[\frac{T_3}{T_5}\right]^{\pm 1} = -\frac{4\alpha_1\alpha_5}{\left(\alpha_5^2-1\right)^2}\frac{(\alpha_3^2-\alpha_1^2)}{(\alpha_3^2-\alpha^2)}, \tag{34a}$$

and for stress-free thermally insulated plate $H \to \infty$, Eq. (33) leads to

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} - \frac{\alpha_3}{\alpha_1}\frac{(\alpha_1^2 - \alpha^2)}{(\alpha_3^2 - \alpha^2)}\left[\frac{T_3}{T_5}\right]^{\pm 1} = -\frac{(\alpha_5^2-1)^2}{4\alpha_1\alpha_5}\frac{(\alpha_3^2-\alpha_1^2)}{(\alpha_3^2-\alpha^2)}. \tag{34b}$$

Eq. (34) is the same as obtained and discussed by Sharma et al. [19] in case of homogeneous isotropic thermoelastic plate with stress-free thermally insulated plate.

The secular equation in case of coupled thermoelasticity (CT) can be obtained by setting $t_0 = 0 = t_1$ in the above analysis.

## 6. Discussion of the secular equation

### 6.1. Regions of the secular equation

Here depending on whether $m_1, m_3, m_5$ being purely imaginary or complex, the frequency Eq. (27) is correspondingly altered as follows.

#### 6.1.1. Region I

When the roots of characteristic Eq. (16) are of type $m_k^2 = -m_k'^2,\ k = 1, 3, 5$; $m_k = \mathrm{i}m_k'$ is purely imaginary or complex number. This ensures that the superposition of partial waves has the property of "exponential decay". In this case the secular equation is written from Eq. (27) by replacing circular tangent functions of $m_k, k = 1, 3, 5$ with hyperbolic tangent functions of $m_k', k = 1, 3, 5$:

$$\left[\frac{\tan h\,\gamma m_1'}{\tan h\,\gamma m_5'}\right]^{\pm 1} - \frac{D_{13}}{D_{11}}\frac{G_3'}{G_1'}\left[\frac{\tan h\,\gamma m_3'}{\tan h\,\gamma m_5'}\right]^{\pm 1} \pm \frac{iD_{17}T_7}{D_{37}D_{11}G_1'[\tan h\,\gamma m_5']^{\pm 1}} \times [D_{31}G_1' - D_{33}G_3' + D_{35}G_5'] = -\frac{D_{15}}{D_{11}}\frac{G_5'}{G_1'}, \tag{35}$$

where

$$G_1' = D_{23}(D_{45}' - D_{45}''[\tan h\,\gamma m_5']^{\pm 1}) - D_{25}(D_{43}' - D_{43}''[\tan h\,\gamma m_3']^{\pm 1}),$$
$$G_3' = D_{21}(D_{45}' - D_{45}''[\tan h\,\gamma m_5']^{\pm 1}) - D_{25}(D_{41}' - D_{41}''[\tan h\,\gamma m_1']^{\pm 1}),$$
$$G_5' = D_{21}(D_{43}' - D_{43}''[\tan h\,\gamma m_3']^{\pm 1}) - D_{23}(D_{41}' - D_{41}''[\tan h\,\gamma m_1']^{\pm 1}).$$

Here $D_{1q}, D_{2q}, D_{3q}D_{4q}, W_q, S_q, q = 1, 2, \ldots, 8$ can be obtained on replacing $m_k$ by $\mathrm{i}m_k',\ k = 1, 3, 5$ in the corresponding expressions.

#### 6.1.2. Region II

In case two of the roots of Eq. (16) are of the type $m_k^2 = -m_k'^2,\ k = 1, 3$; then the frequency equation can be obtained from Eq. (27) by replacing circular tangent functions of $m_k,\ k = 1, 3$ with hyperbolic tangent functions of $m_k',\ k = 1, 3$:

$$\left[\frac{\tan h\,\gamma m_1'}{\tan \gamma m_5}\right]^{\pm 1} - \frac{D_{13}}{D_{11}}\frac{G_3'}{G_1'}\left[\frac{\tan h\,\gamma m_3'}{\tan \gamma m_5}\right]^{\pm 1} \pm \frac{\mathrm{i}D_{17}T_7}{D_{37}D_{11}G_1'[\tan \gamma m_5]^{\pm 1}} \times [D_{31}G_1' - D_{33}G_3' + D_{35}G_5'] = \pm\frac{\mathrm{i}D_{15}}{D_{11}}\frac{G_5'}{G_1'}, \tag{36}$$

where

$$G_1' = D_{23}(D_{45}' + \mathrm{i}D_{45}''[\tan\gamma m_5]^{\pm 1}) - D_{25}(D_{43}' - D_{43}''[\tan h\,\gamma m_3']^{\pm 1}),$$
$$G_3' = D_{21}(D_{45}' + iD_{45}''[\tan\gamma m_5]^{\pm 1}) - D_{25}(D_{41}' - D_{41}''[\tan h\,\gamma m_1']^{\pm 1}),$$
$$G_5' = D_{21}(D_{43}' - D_{43}''[\tan h\,\gamma m_3']^{\pm 1}) - D_{23}(D_{41}' - D_{41}''[\tan h\,\gamma m_1']^{\pm 1}).$$

Here $D_{1q}, D_{2q}, D_{3q}D_{4q}, W_q, S_q,\ q = 1, 2, \ldots, 8$ are obtained by replacing $m_k$ with $\mathrm{i}m_k',\ k = 1, 3$.

### 6.1.3. Region III

In the general case the roots $m_k^2,\ k = 1, 3, 5$ are complex numbers, and then the frequency equation is given by Eq. (27).

## 6.2. Waves of short wavelength

Some information on the asymptotic behavior is obtainable by letting $\xi \to \infty$. If we take $\xi > \omega/\sqrt{c_2}$, it follows that $c < \sqrt{c_2}$ and the roots of characteristic equation lies in region I in this case. Then we replace $m_1, m_3$ and $m_5$ in the secular equation by $\mathrm{i}m_1', \mathrm{i}m_3'$ and $\mathrm{i}m_5'$. Here for

$$\xi \to \infty,\quad \frac{\tan h(\gamma m_1')}{\tan h(\gamma m_5')} \to 1,\quad \frac{\tan h(\gamma m_3')}{\tan h(\gamma m_5')} \to 1 \quad \text{and} \quad \frac{\tan(\gamma^* m_7)}{\tanh(\gamma m_5')} \to -i$$

so that secular Eq. (27) reduces to

$$D_{11}G_1'' - D_{13}G_3'' + D_{15}G_5'' = \mp\frac{D_{17}}{D_{37}}[D_{31}G_1'' - D_{33}G_3'' + D_{35}G_5''], \tag{37}$$

where

$$G_1'' = D_{23}(D_{45}' - D_{45}'') - D_{25}(D_{43}' - D_{43}''),$$
$$G_3'' = D_{21}(D_{45}' - D_{45}'') - D_{25}(D_{41}' - D_{41}''),$$
$$G_5'' = D_{21}(D_{43}' - D_{43}'') - D_{23}(D_{41}' - D_{41}''),$$

which is the dispersion equation for thermoelastic Rayleigh waves of an infinite half-space solid bordered with an infinite half-space homogeneous liquid. The dispersion equation for thermoelastic Rayleigh waves of an infinite half-space solid bordered with a homogeneous liquid layer of thickness $h$ is given as

$$D_{11}G_1'' - D_{13}G_3'' + D_{15}G_5'' = \mp\frac{\mathrm{i}D_{17}}{D_{37}}[D_{31}G_1'' - D_{33}G_3'' + D_{35}G_5'']\tan\gamma^* m_7. \tag{38}$$

If $\rho_L$ approaches to zero, Eqs. (37) and (38) reduce to

$$D_{11}G_1'' - D_{13}G_3'' + D_{15}G_5'' = 0, \tag{39}$$

which is the same as obtained and discussed by Sharma et al. [22] in case of homogeneous transversely isotropic thermoelastic plate with stress-free thermally insulated plate.

Eq. (39) is merely Rayleigh surface wave equation. The Rayleigh results enter here since for such small wavelengths, the finite thickness plate appears as semi-infinite medium. Hence vibration energy is transmitted along the surface of the plate. It seems that Eq. (38) can be obtained by multiplying the right side of Eq. (37) by a factor of $\mathrm{i}\tan\gamma^* m_7$. This also seems to be true for the case of Lamb waves. The dispersion equations for leaky Lamb waves, i.e., Lamb waves in a transversely isotropic plate bordered with an infinite half-space homogeneous liquid ($h \to \infty$) at both sides are as follows.

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} - \frac{D_{13}}{D_{11}}\frac{G_3}{G_1}\left[\frac{T_3}{T_5}\right]^{\pm 1} + \frac{\mathrm{i}D_{17}}{D_{37}D_{11}G_1[T_5]^{\pm 1}}[D_{31}G_1 - D_{33}G_3 + D_{35}G_5] = -\frac{D_{15}}{D_{11}}\frac{G_5}{G_1}. \tag{40}$$

If we multiply a factor of $\mathrm{i}\tan\gamma^* m_7$ to the terms containing $\rho_L$ in Eq. (40), the dispersion equation (27) of Lamb waves in a transversely isotropic thermoelastic plate bordered with a homogeneous layer of thickness $h$ on both sides can be obtained. Since Lamb waves are special cases of Rayleigh waves ($d$ approaches infinity), this type of analogy seems reasonable.

### 6.3. Thin plate results

The thin plate limits are specified by $\xi d \ll 1$ when the transverse wavelength of the plate is quite large as compared to the thickness of the plate. So for $\xi > \omega/\sqrt{c_2}$, we have $m_k = \mathrm{i}m'_k,\ k = 1, 3, 5$. In this case the secular equation is written from Eq. (27) just by replacing circular tangent functions with hyperbolic tangent functions. On retaining first two terms in the expansion of hyperbolic tangents, the frequency equation (27) reduces to

$$\gamma^3 - L\gamma^2 + M\gamma + N = 0, \tag{41}$$

where

$$L = \frac{P_2}{Q_2 P^*}, \quad M = \frac{P_1}{Q_2 P^*} - \frac{Q_1}{Q_2}\left(1 + \frac{m_7^2 \xi^2 h^2}{3}\right), \quad N = \frac{P_0}{Q_2 P^*} - \frac{Q_0}{Q_2}\left(1 + \frac{m_7^2 \xi^2 h^2}{3}\right), \quad P^* = -\frac{\mathrm{i}hm_7 D_{17}}{dD_{37}},$$

$$P_0 = D_{11}m'_1(D_{23}D'_{45} - D_{25}D'_{43}) - D_{13}m'_3(D_{21}D'_{45} - D_{25}D'_{41}) + D_{15}m'_5(D_{21}D'_{43} - D_{23}D'_{41}),$$

$$\begin{aligned} P_1 = {} & D_{11}m'_1(D_{25}D''_{43}m'_3 - D_{23}D''_{45}m'_5) - D_{13}m'_3(D_{25}D''_{41}m'_1 - D_{21}D''_{45}m'_5) \\ & + D_{15}m'_5(D_{23}D''_{41}m'_1 - D_{21}D''_{43}m'_3), \end{aligned}$$

$$\begin{aligned} Q_2 = {} & \tfrac{1}{3}[D_{31}(D_{25}D''_{43}m'^3_3 - D_{23}D''_{45}m'^3_5) - D_{33}(D_{25}D''_{41}m'^3_1 - D_{21}D''_{45}m'^3_5) \\ & + D_{35}(D_{23}D''_{41}m'^3_1 - D_{21}D''_{43}m'^3_3)], \end{aligned}$$

where $P_2$ and $Q_0$ can be obtained from $P_0$ by replacing $m'_k$ with $m'^3_k/3$ and $D_{1k}m'_k$ with $D_{3k}$, respectively, and $Q_1$ can be obtained from $P_1$ by replacing $D_{1k}m'_k$ with $D_{3k},\ k = 1, 3, 5$.

For isotropic case, frequency Eq. (41) for $H \to 0$, on discarding the terms of higher order than $(c/\delta)^4$ reduces to

$$c = 2\xi d\delta\sqrt{\frac{(1-\delta^2)}{3}}\left\{1 - \frac{\rho_L h}{\rho d}\right\}^{-1/2}, \tag{42}$$

which in the absence of liquid, $\rho_L \to 0$ becomes

$$c = 2\xi d\delta\sqrt{\frac{(1-\delta^2)}{3}}. \tag{43}$$

This result, with linear dependence of $c$ on $\xi$ agrees with that derived from classical plate theory in elastokinetics and of course pertains to the flexure vibration and represents only a single vibrational mode in limited frequency range in the over all frequency spectrum. No effect of thermomechanical coupling has been observed on thin plates in this case. But the presence of liquid on both sides of the plate affects the phase velocity of flexural vibrational mode as a periodic function of liquid layers width.

In region II the antisymmetric case has no roots and the secular equation in symmetric case becomes

$$D_{11}G_1'''m_3' - D_{13}G_3'''m_1' - \mathrm{i}\xi^2 m_1'm_3'm_7 dh\frac{D_{17}}{D_{37}}[D_{31}G_1''' - D_{33}G_3''' + D_{35}G_5'''] = -\frac{\mathrm{i}m_1'm_3'D_{15}G_5'''}{m_5}, \quad (44)$$

where

$$G_1''' = D_{23}(D_{45}' + \mathrm{i}D_{45}''/\gamma m_5) - D_{25}(D_{43}' - D_{43}''/\gamma m_3'),$$
$$G_3''' = D_{21}(D_{45}' + \mathrm{i}D_{45}''/\gamma m_5) - D_{25}(D_{41}' - D_{41}''/\gamma m_1'),$$
$$G_5''' = D_{21}(D_{43}' - D_{43}''/\gamma m_3') - D_{23}(D_{41}' - D_{41}''/\gamma m_1').$$

Eq. (44) for isotropic case reduces to

$$\alpha'^2 - \alpha_1^2 - \alpha_3^2 + \frac{4\xi^2\alpha_1^2\alpha_3^2}{(\xi^2 - m_5^2)^2} = -\frac{\rho_L\omega^2(\xi^2 + m_5^2)\alpha_1^2\alpha_3^2 d\tan\gamma^* m_7}{\gamma\rho\delta^2(\xi^2 - m_5^2)^2}, \quad (45)$$

which for CT theory implies that

$$c = 2\delta\sqrt{1 - \delta^2/(1+\epsilon)}\left[E(1 \pm \sqrt{1 - F/E^2})/2\right]^{1/2}, \quad (46a)$$

where $E = 1 + 1/4\delta^2\mathrm{i}\omega^{-1}(1 + \epsilon - \delta^2)$ and $F = (1-\delta^2)(1+\epsilon)/\delta^2\mathrm{i}\omega^{-1}(1+\epsilon-\delta^2)^2$.

For LS theory, we have

$$c = 2\delta\sqrt{1 - \delta^2/(1 + \mathrm{i}\omega^{-1}\epsilon/\tau_0)}\left[E_1(1 \pm \sqrt{1 - F_1/E_1^2})/2\right]^{1/2}, \quad (46b)$$

where

$$E_1 = 1 + 1/4\delta^2\tau_0(1 + (\mathrm{i}\omega^{-1}\epsilon/\tau_0) - \delta^2)$$

and

$$F_1 = (1-\delta^2)(1 + \epsilon\mathrm{i}\omega^{-1}/\tau_0)/\delta^2\tau_0(1 + (\epsilon\mathrm{i}\omega^{-1}/\tau_0) - \delta^2)^2.$$

In case of GL theory, we have

$$c = 2\delta\sqrt{1 - \delta^2\mathrm{i}\omega^{-1}/\tau_0(1 + \tau_1\epsilon/\tau_0)}\left\lfloor E_2(1 \pm \sqrt{1 - F_2/E_2^2})/2\right\rfloor^{1/2}, \quad (46c)$$

where

$$E_2 = 1 + 1/4\delta^2(\tau_0(1 + \tau_1\epsilon/\tau_0) - \delta^2\mathrm{i}\omega^{-1})$$

and

$$F_2 = (1-\delta^2)\tau_0(1 + \epsilon\tau_1/\tau_0)/\delta^2(\tau_0(1 + \epsilon\tau_1/\tau_0) - \delta^2\mathrm{i}\omega^{-1})^2.$$

Thus, the phase velocity is approximately given by $c = 2\delta\sqrt{1-\delta^2/(1+\in)}$, which is the thin plate or plane stress analogue of the bar velocity of longitudinal rod theory in coupled thermoelasticity. In general, here the wave mode depends upon the thermoelastic coupling parameter and thermal relaxation times whose phase velocities are given by Eq. (46) in case of generalized thermoelasticity. The phase velocity in this case also depends upon the thickness of the plate in addition to its periodic dependence on the thickness of the liquid layers. Thus in case of thin plates $\xi d \ll 1$, fundamental symmetric ($S_0$) mode becomes dispersionless, the phase velocity is equal to the group velocity and equal to $2\delta\sqrt{1-\delta^2/(1+\in)}$ approximately, thermal relaxation time being small. The fundamental skew symmetric ($A_0$) mode meanwhile, becomes the flexural or bending wave of the plate with its phase velocity approximately equal to $2\xi d\delta[(1-\delta^2)/3]^{1/2}$ and its group velocity equals to $4\xi d\delta[(1-\delta^2)/3]^{1/2}$. For a thin plate, $A_0$ mode is essentially a transverse mode, i.e., the $z$-component of the displacement dominates. On the contrary, for $S_0$ mode of a thin plate the $x$-component of the displacement dominates. For ideal liquid (no viscosity) energy of Lamb waves can only be coupled into the liquid through the $z$-component of displacement at the plate surface. This explains why $A_0$ mode of a thin plate is the choice for biosensing applications. The above thin plate analysis reduces to that of Wu and Zhu [30] in elastokinetics, i.e., in case of uncoupled thermoelasticity (UCT) when we set $\in = 0,\ t_0 = 0 = t_1$.

## 7. Uncoupled thermoelasticity

In case of UCT, the thermomechanical-coupling constant vanishes, i.e., $\in = 0$, Eq. (16) reduces to

$$m^4 + \bar{B}m^2 + \bar{C} = 0 \quad \text{and} \quad m^2 = -\frac{(s^2 - c^2\tau_0)}{\bar{K}}, \tag{47}$$

where

$$\bar{B} = \frac{Ps^2 - Jc^2}{c_1c_2}, \quad \bar{C} = \frac{(c_2s^2 - c^2)(s^2 - c^2)}{c_1c_2}.$$

Consequently the secular equation (27) in this case reduces to

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} + \frac{D'_{17}}{D'_{37}D'_{25}}\frac{T_7}{[T_5]^{\pm 1}}[D'_{21}D'_{35} - D'_{25}D'_{31}] = \frac{D'_{15}D'_{21}}{D'_{11}D'_{25}}, \tag{48}$$

where

$$D'_{1q} = \begin{cases} (c_3 - c_2)s + c_1 m_q W_q, & q = 1,5, \\ -\omega^2\rho_L/\mathrm{i}\xi\rho, & q = 7, \end{cases}$$

$$D'_{2q} = \begin{cases} c_2(m_q + sW_q), & q = 1,5, \\ 0, & q = 7, \end{cases}$$

$$D'_{3q} = \begin{cases} W_q/i\xi, & q = 1,5, \\ -m_q, & q = 7, \end{cases}$$

$$W_1 = -\frac{c_3 m_1 s}{c_1 m_1^2 + c_2 s^2 - c^2}, \quad W_5 = -\frac{c_2 m_5^2 + s^2 - c^2}{c_3 m_5 s}, \quad S_1 = 0 = S_5,$$

$$m_1^2 + m_5^2 = -\frac{Ps^2 - Jc^2}{c_1 c_2}, \quad m_1^2 m_5^2 = \frac{(c_2 s^2 - c^2)(s^2 - c^2)}{c_1 c_2}. \tag{49}$$

In case of UCT, Eq. (32) for isotropic plate also reduces to

$$\left[\frac{T_1}{T_5}\right]^{\pm 1} + \frac{\omega^2 \rho_L \alpha_1 (\alpha_5^2 + 1) T_7}{\rho \xi^2 \alpha_7 \delta^2 (\alpha_5^2 - 1)^2 [T_5]^{\pm 1}} = -\frac{4\alpha_1 \alpha_5}{(\alpha_5^2 - 1)^2}. \tag{50}$$

In the absence of liquid $\rho_L \to 0$ Eq. (50) becomes

$$\frac{T_1}{T_5} = \left[-\frac{4\alpha_1 \alpha_5}{(\alpha_5^2 - 1)^2}\right]^{\mp 1}, \tag{51}$$

where $\alpha_1^2 = c^2 - 1$, $\alpha_5^2 = c^2/\delta^2 - 1$. The transcendental equation (51) is the Rayleigh–Lamb equation and was discussed by Graff [35] and Achenbach [36] in case of homogeneous isotropic elastic plates with stress-free boundaries.

## 8. Displacement and temperature amplitudes

Using Eq. (21), the amplitude $w_{asy}$, $z$-component of displacement; the amplitude $u_{sy}$, $x$-component of displacement and the amplitude $T_{sy}$ of temperature change may be calculated as

$$w_{asy} = \mathrm{i}\xi(c_1' s_3 s_5 V_1 W_1 + s_1 c_3' s_5 V_3 W_3 + s_1 s_3 c_5' V_5 W_5) B_1 \mathrm{e}^{\mathrm{i}\xi(xs - ct)} / s_3 s_5 V_1, \tag{52}$$

$$u_{sy} = \mathrm{i}\xi(c_1' c_3 c_5 R_1 + c_1 c_3' c_5 R_3 + c_1 c_3 c_5' R_5) A_1 \mathrm{e}^{\mathrm{i}\xi(xs - ct)} / c_3 c_5 R_1, \tag{53}$$

$$T_{sy} = \mathrm{i}\xi(c_1' c_3 c_5 S_1 R_1 + c_1 c_3' c_5 S_3 R_3 + c_1 c_3 c_5' S_5 R_5) A_1 \mathrm{e}^{\mathrm{i}\xi(xs - ct)} / c_3 c_5 R_1, \tag{54}$$

where $R_1 = D_{23} T_3 D_{15} - D_{25} T_5 D_{13} - D_{17} T_3 T_5 T_7 (D_{23} D_{35} - D_{25} D_{33}) / D_{37}$.

$$R_3 = D_{25} T_5 D_{11} - D_{21} T_1 D_{15} - D_{17} T_1 T_5 T_7 (D_{25} D_{31} - D_{21} D_{35}) / D_{37},$$

$$R_5 = D_{21} T_1 D_{13} - D_{23} T_3 D_{11} - D_{17} T_1 T_3 T_7 (D_{21} D_{33} - D_{23} D_{31}) / D_{37},$$

$$V_1 = D_{25} T_5^{-1} D_{13} - D_{23} T_3^{-1} D_{15} + D_{17} T_3^{-1} T_5^{-1} T_7 (D_{25} D_{33} - D_{23} D_{35}) / D_{37},$$

$$V_3 = D_{21} T_1^{-1} D_{15} - D_{25} T_5^{-1} D_{11} + D_{17} T_1^{-1} T_5^{-1} T_7 (D_{21} D_{35} - D_{25} D_{31}) / D_{37},$$

$$V_5 = D_{23} T_3^{-1} D_{11} - D_{21} T_1^{-1} D_{13} + D_{17} T_1^{-1} T_3^{-1} T_7 (D_{23} D_{31} - D_{12} D_{33}) / D_{37}.$$

Here $c_k = \cos\gamma m_k,\ s_k = \sin\gamma m_k,\ c'_k = \cos\gamma' m_k,\ s'_k = \sin\gamma' m_k,\ k = 1,3,5,\ c_7 = \cos\gamma^* m_7,\ s_7 = \sin\gamma^* m_7$ and $\gamma = \xi d,\ \gamma' = \xi z,\ \gamma^* = \xi h$. The expressions for $-w_{sy}, u_{asy}$ and $T_{asy}$ can be obtained from Eqs. (52) to (54), respectively, by interchanging $s_k$ and $c_k$, $s'_k$ and $c'_k$, $R_k$ and $V_k$, $k = 1,3,5$ and by changing $A_1$ with $B_1$ and vice versa.

## 9. Thermoelasticity without energy dissipation

The basic governing equations of motion and a heat conduction equation in the absence of body forces and heat sources for a plate in the context of thermoelasticity without energy dissipation, are given by [18,37]

$$c_{11}u_{,xx} + c_{44}u_{,zz} + (c_{13} + c_{44})w_{,xz} - \beta_1^* T_{,x} = \rho\ddot{u}, \tag{55}$$

$$(c_{13} + c_{44})u_{,xz} + c_{44}w_{,xx} + c_{33}w_{,zz} - \beta_3^* T_{,z} = \rho\ddot{w}, \tag{56}$$

$$K_1^* T_{,xx} + K_3^* T_{,zz} - \rho C_e \ddot{T} = T_0(\beta_1^* \ddot{u}_{,x} + \beta_3^* \ddot{w}_{,z}), \tag{57}$$

which in non-dimensional form can be written as

$$u_{,xx} + c_2 u_{,zz} + c_3 w_{,xz} - T_{,x} = \ddot{u}, \tag{58}$$

$$c_3 u_{,xz} + c_2 w_{,xx} + c_1 w_{,zz} - \bar{\beta}^* T_{,z} = \ddot{w}, \tag{59}$$

$$T_{,xx} + \bar{K}^* T_{,zz} - \omega^{*'} \ddot{T} = \in^* \omega^{*'} (\ddot{u}_{,x} + \bar{\beta}^* \ddot{w}_{,z}), \tag{60}$$

where

$$u' = \frac{\rho\omega^{*'} v_1 u}{\beta_1^* T_0}, \quad w' = \frac{\rho\omega^{*'} v_1 w}{\beta_1^* T_0}, \quad \bar{\beta}^* = \frac{\beta_3^*}{\beta_1^*}, \quad \bar{K}^* = \frac{K_3^*}{K_1^*}, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\beta_1^* T_0}, \quad \in^* = \frac{\beta_1^* T_0}{\rho C_e c_{11}},$$
$$\omega^{*'} = \frac{c_P^2}{c_T^2}, \quad c_P^2 = \frac{c_{11}}{\rho v_p^2} = \frac{v_1^2}{v_p^2}, \quad c_T^2 = \frac{K_1^*}{\rho C_e v_p^2}, \quad c_S^2 = \frac{c_{44}}{\rho v_p^2} = \frac{v_2^2}{v_p^2}, \quad v_1^2 = \frac{c_{11}}{\rho}, \quad v_2^2 = \frac{c_{44}}{\rho}. \tag{61}$$

The rest of quantities are given by expressions similar to Eq. (9) except $\omega^*$ is replaced by $\omega^{*'}$ where symbols and notation are same as defined earlier, $v_p$ is the standard speed and $\beta_1^*, \beta_3^*$ are the coefficients of volume expansion. Here $K_1^*, K_3^*$ are not the coefficients of thermal conductivity but material constant characteristics of the theory.

$$(c_2 m^2 + s^2 - c^2 + c_3 m s W)\xi^2 + \mathrm{i}\xi s S = 0, \tag{62}$$

$$(c_3 s m + (c_1 m^2 + c_2 s^2 - c^2)W)\xi^2 + \mathrm{i}\xi\bar{\beta}^* m S = 0, \tag{63}$$

$$\mathrm{i}\in^* \xi s c^2 \omega^{*'} + \mathrm{i}\in^* \xi\bar{\beta}^* c^2 m \omega^{*'} W + (m^2 \bar{K}^* + s^2 - c^2 \omega^{*'})S = 0. \tag{64}$$

Eqs. (62)–(64) are the same as Eq. (13). Hence, its solution is given by Eq. (12). Eqs. (15) and (16) are also obtained similarly, but here the coefficients $A, B, C$ are given by

$$\begin{aligned}
A &= \frac{Ps^2 - Jc^2}{c_1 c_2} + \frac{s^2 - c^2\omega^{*'}}{\bar{K}^*} + \frac{\in^*\omega^{*'}c^2\bar{\beta}^{*2}}{c_1\bar{K}^*},\\
B &= \frac{(c_2 s^2 - c^2)(s^2 - c^2)}{c_1 c_2} + \frac{(s^2 - c^2\omega^{*'})(Ps^2 - Jc^2)}{c_1 c_2 \bar{K}^*} + \frac{\in^*\omega^{*'}c^2}{c_1 c_2\bar{K}^*}[(\bar{\beta}^{*2} - 2c_3\bar{\beta}^* + c_1)s^2 - c^2\bar{\beta}^{*2}],\\
C &= \frac{c_2 s^2 - c^2}{c_1 c_2 \bar{K}^*}[(s^2 - c^2\omega^{*'})(s^2 - c^2) + \in^*\omega^{*'}s^2c^2].
\end{aligned} \tag{65}$$

For uncoupled theory ($\in^* = 0$), Eq. (16) again reduces to Eq. (47).

The amplitude ratios $W$ and $S$ can be expressed as

$$W_q = \begin{cases} \dfrac{m_q a_q}{s}, & q = 1,2,3,4, \\ -\dfrac{c_2 m_q^2 + s^2 - c^2 + i\xi^{-1} s S_q}{c_3 m_q s}, & q = 5,6, \end{cases} \tag{66}$$

$$S_q = \begin{cases} \dfrac{\mathrm{i}\xi[(c_2 - c_3 a_q)m_q^2 + s^2 - c^2]}{s}, & q = 1,2,3,4, \\ \dfrac{\mathrm{i}\xi[c_1 c_2 m_q^4 + (Ps^2 - Jc^2)m_q^2 + (c_2 s^2 - c^2)(s^2 - c^2)]a_q}{s\bar{\beta}^*(c_2 m_q^2 + (1 - c_3/\bar{\beta}^*)s^2 - c^2)}, & q = 5,6, \end{cases} \tag{67}$$

where

$$a_q = \bar{\beta}^* \frac{c_2 m_q^2 + (1 - c_3/\bar{\beta}^*)s^2 - c^2}{(c_1 - c_3\bar{\beta}^*)m_q^2 + c_2 s^2 - c^2}, \quad q = 1,2,3,4,5,6$$

and

$$D_{1q} = (c_3 - c_2)s + c_1 m_q W_q - \bar{\beta}^* S_q, \quad q = 1,2,\ldots,6.$$

The rest of the analysis can be made in the similar manner as was done in the case of the other theories of thermoelasticity in the previous section.

## 10. Numerical results and discussion

With the view of illustrating theoretical results obtained in the preceding sections, we now present some numerical results. The materials chosen for this purpose is single crystal of zinc, the physical data for which is given below:

$$\rho = 7.14 \times 10^3\ \mathrm{kg\,m^{-3}}, \quad c_{11} = 1.628 \times 10^{11}\ \mathrm{N\,m^{-2}}, \quad c_{12} = 0.362 \times 10^{11}\ \mathrm{N\,m^{-2}},$$

$c_{13} = 0.508 \times 10^{11}\,\mathrm{N\,m^{-2}}, \quad c_{33} = 0.627 \times 10^{11}\,\mathrm{N\,m^{-2}}, \quad c_{44} = 0.385 \times 10^{11}\,\mathrm{N\,m^{-2}}, \quad T_0 = 296\,\mathrm{K},$

$$\beta_1 = 5.75 \times 10^6\,\mathrm{N\,m^{-2}\,deg^{-1}}, \quad \beta_3 = 5.07 \times 10^6\,\mathrm{N\,m^{-2}\,deg^{-1}}, \quad C_e = 3.9 \times 10^2\,\mathrm{J\,kg\,m^{-1}\,deg},$$

$K_1 = 1.24 \times 10^2\,\mathrm{W\,m^{-1}\,deg^{-1}}, \; K_3 = 1.24 \times 10^2\,\mathrm{W\,m^{-1}\,deg^{-1}}, \; \omega^* = 5.01 \times 10^{11}\,\mathrm{s^{-1}}, \; \in = 0.0221.$

The liquid taken for the purpose of numerical calculations is water, the velocity of sound in which is given by $c_L = 1.5 \times 10^3\,\mathrm{m/s}$. For the purpose of numerical calculations, we take $d = 1$, $h = 1$ and the angle of inclination of the wave with axis of symmetry have been considered $\pi/4$ for numerical calculations.

In the case of GN theory $K_1^*, K_3^*$ are not the coefficients of thermal conductivity but material constant characteristics of the theory and their values have been selected at random. Thus, the numerical results are applicable for zinc crystal like material in this theory. We assume hypothetical values of material parameters [37] $c_P = 1.0$, $c_T = 0.5$ so that $v_p^2 = (c_{11}/\rho)$, $K_1^* = (C_e c_{11}/4)$ and $K_3^* = (C_e c_{33}/4)$. For this choice of parameters, the longitudinal thermoelastic wave happens to be faster than the thermal wave. In this theory the thermal wave speed $c_T$ depends upon $K_1^*, K_3^*$ instead of thermal relaxation times as in the LS and GL theories of thermoelasticity. Here the thermoelastic-coupling constant has been selected as $\in^* = \in = 0.0221$, and values of thermal relaxation times have been estimated from Eq. (2.5) of [12] and taken as $t_0 = 0.75 \times 10^{-13}$ and $t_1 = 0.5 \times 10^{-13}\,\mathrm{s}$ here.

The complex roots of characteristic equation (16) have been computed with the help of reduced Cardano's method, which are then used in various relevant relations. The secular equation (27) is solved for the phase velocity by using iteration method. The sequence of iteration is made to converge after sampling it over about 100 sample values in order to achieve the desired level of accuracy (four decimal places here). The phase velocity profile of first few symmetric and skew symmetric modes has been computed and corresponding dispersion curves for Rayleigh–Lamb type modes are represented graphically in Fig. 2 for wave normal inclination $\theta = 45^\circ$ with the axis of symmetry.

From Fig. 2(a) it is observed that the non-dimensional phase velocity of lowest (acoustic) skew symmetric mode ($A_0$) is observed to increase from zero value at vanishing wave number to become closer to Rayleigh wave velocity at higher wave numbers in direction of wave propagation. The velocity of the acoustic symmetric mode ($S_0$) become dispersionless, i.e., remains constant with variation in wave number and gets significantly reduced and affected in the presence of liquid that remains closer to the velocity of thermoelastic Rayleigh waves in a solid half-space bordered with a liquid layers on both sides with increasing wave number in the considered direction of wave propagation. The phase velocity of higher (optical) modes of wave propagation, symmetric and skew symmetric; attain quite large values at vanishing wave number, which sharply slashes down to become steady and asymptotic to the reduced Rayleigh wave velocity with increasing wave number. It is observed that in the context of various theories of thermoelasticity (CT, LS and GL), the various modes of propagation have higher velocity in coupled thermoelasticity (CT) followed by generalized thermoelasticity LS and GL. The velocity of higher modes is observed to develop at a rate, which is approximately $n$-time, the magnitude of the velocity of first mode ($n = 1$). It is noticed that due to the damping effect of the liquid on both sides of the plate, the phase velocity of fundamental mode

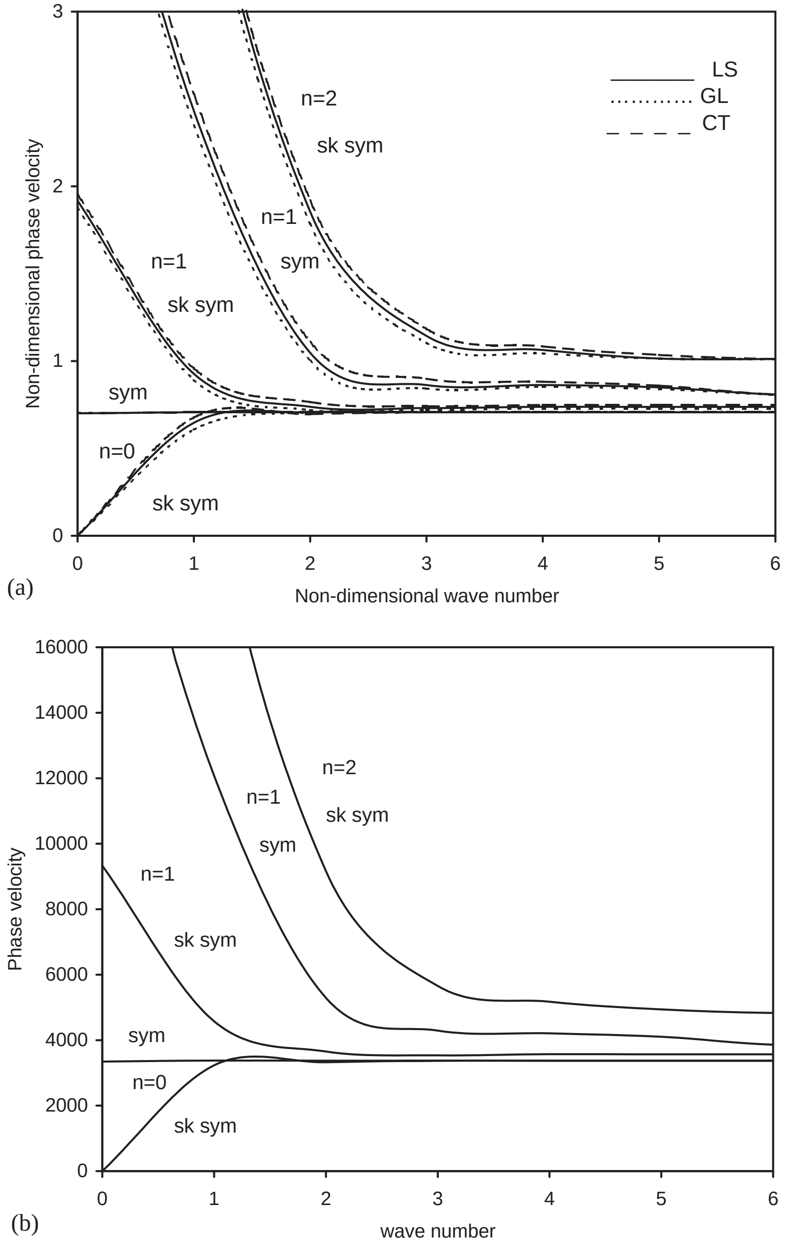



Fig. 2. (a) Phase velocity profile of modified Lamb wave modes of vibrations of a plate bordered with liquid layers with wave number for CT, LS and GL theories. (b) Dimensional phase velocity profile of modified Lamb wave modes of vibrations of a plate bordered with liquid layers with wave number in case of CT theory.

decreases in case of symmetric ($S_0$) modes from a value more than unity in the direction of propagation and increases in case of skew symmetric ($A_0$) modes from zero at vanishing wave numbers to attain certain value in considered direction of propagation which is reduced

Rayleigh wave velocity. It is also observed that the phase velocity of the Lamb type plate waves falls significantly at vanishing wave number in the presence of liquid layers. The dispersion curves become more smoothen in this case because of the shock absorption nature of the liquid. The acoustic skew symmetric ($A_0$) mode is observed to be most affected and sensitive. These numerically computed results are found to be quite in agreement with the corresponding results and their trends are also similar to those of Sharma and Pathania [32,33] and Sharma et al. [19,22,34]. At higher wave numbers the phase velocities of various optical modes become asymptotically closer to the shear wave speed in contrast to that of acoustical modes which tends to the Rayleigh wave speed. Because a finite thickness plate appears to be half-space in such situations and the vibration energy is mainly transmitted through the surface (interface) of the plate. The free surfaces admit a Rayleigh-type surface wave with complex wave number and hence phase velocity. Consequently, the surface wave propagates with attenuation due to the radiation of energy into the medium. This radiated energy will be reflected back to the center of the plate by the lower and upper surfaces. Thus the attenuated surface wave on the free surface is enhanced by this reflected energy to form a propagation wave. In fact, the multiple reflections between the upper and lower surfaces of the plate form caustics at one of the free surface and a strong stress concentration arises due to which wave field becomes unbounded in the limit $d \to \infty$. The unbounded displacement field is characterized by the singularities of circular tangent functions. It is also observed that as the thickness of the plate increases, the phase velocity decreases. This can be explained by the fact that as the thickness of the plate increases, the coupling effect of various interacting fields also increases resulting in lower phase velocity. It can also be observed that the Rayleigh wave velocity is reached at lower wave number as the thickness increases, because the transportation of energy mainly takes place in the neighborhood of the free surfaces of the plate in this case. The dimensional phase velocity profile with wave number in case of coupled theory of thermoelasticity (CT) is also presented in Fig. 2(b) for demonstration and clarification purpose. The phase velocities of symmetric and skew symmetric modes of wave propagation in the context of GN and UCT theories of thermoelasticity have been computed for various values of wave number from the corresponding dispersion relation for stress-free thermoelastic plate bordered with layers of inviscid liquid on both sides. The dispersion curves for Rayleigh–Lamb type modes are presented in Figs. 3(a) and (b) for symmetric and skew symmetric modes, respectively. From the figures it is observed the phase velocity of lowest (acoustic) and first mode of GN theory have same trend of variation as in case of CT, LS and GL theories. But for second and higher modes of wave propagation in the specified direction, the phase velocity reverses its trend in contrast to other counterpart theories of thermoelasticity. The effect of non-dissipation thermal energy is quite visible from the graphs. From the comparison of these two figures, it is observed that symmetric case has slightly higher magnitudes than the skew symmetric case but the trend is same in both the cases. The variation of attenuation coefficient with wave number for symmetric and skew symmetric acoustic modes is represented in Fig. 4. The attenuation coefficient for acoustic and first mode has almost negligible variation with wave number in the considered direction of wave propagation in various theories of thermoelasticity. For higher mode of wave propagation, the magnitude of attenuation coefficient increases monotonically to attain a maximum value for CT, LS and GL theories and then slashes down sharply to zero with the increase in wave number except for third mode. For third mode of wave propagation the magnitude of attenuation coefficient for GL theory shoots up to 7.55217 in region $1 \leqslant \xi d \leqslant 4$ at $\xi d = 3$ and then slashes

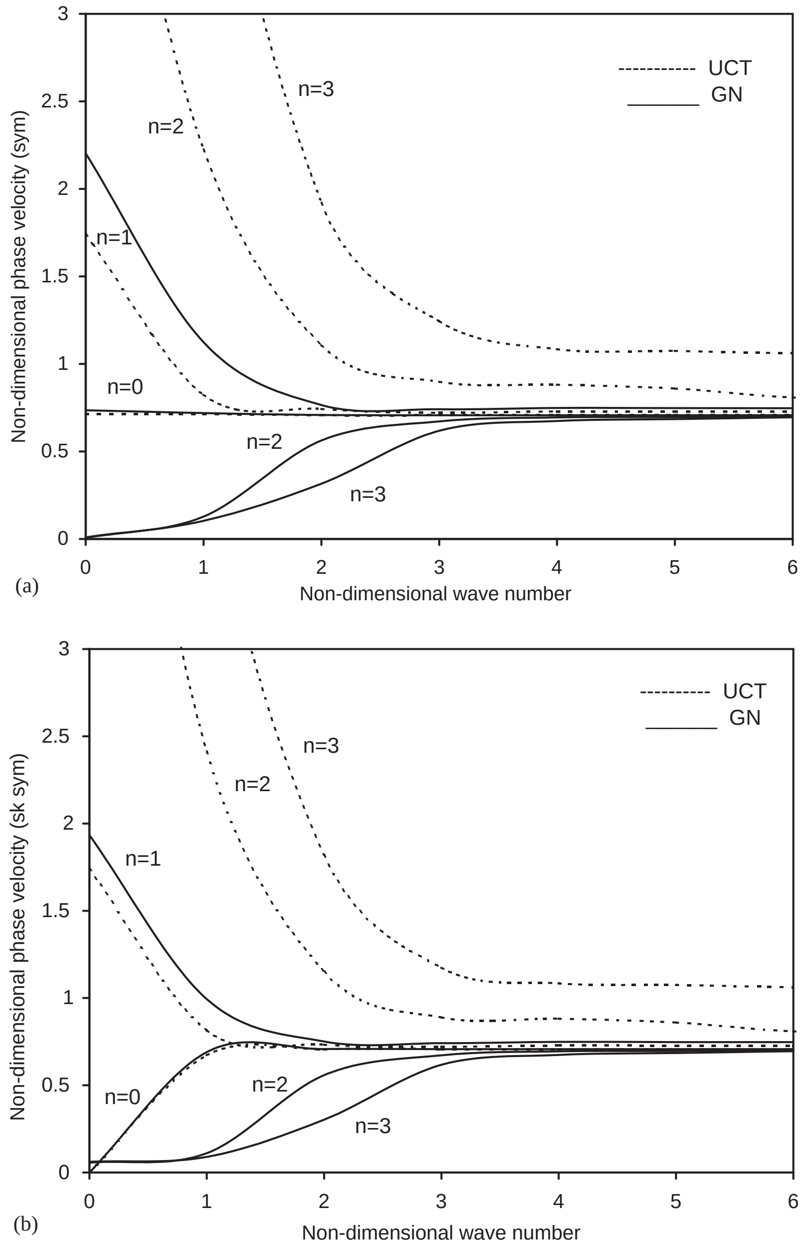



Fig. 3. (a) Phase velocity profile of modified Lamb wave modes of a plate bordered with liquid layers with wave number for GN and UCT theories in symmetric case. (b) Phase velocity profile of modified Lamb wave modes of a plate bordered with liquid layers with wave number for GN and UCT theories in skew symmetric case.

down sharply to zero with the increase in wave number. Also for this mode at vanishing wave number the magnitude of attenuation coefficient is quite higher for CT, LS and GL theories as compared to the other modes.

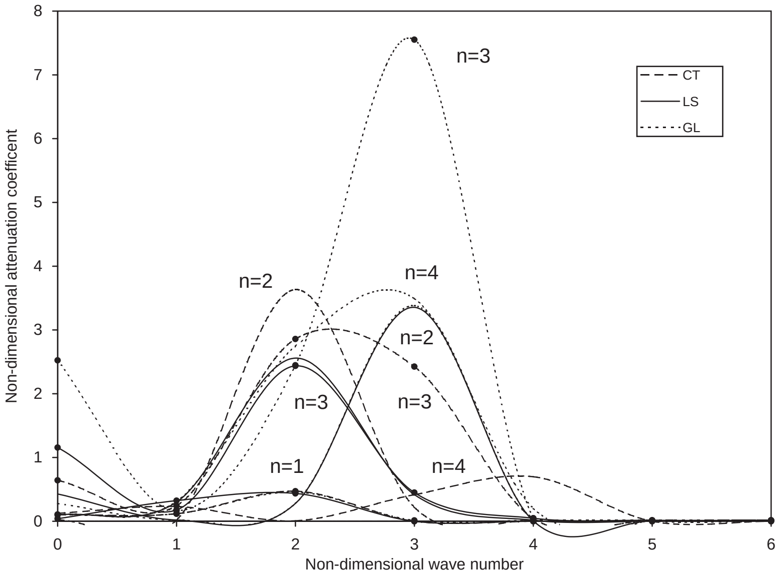



Fig. 4. Variation of attenuation coefficient with wave number for CT, LS and GL thermoelasticity.

Fig. 5(a) contains plots of non-dimensional amplitude of the $z$-component of displacement ($w$) in the direction of wave propagation for three different values of liquid layer thickness in case of generalized thermoelastic plate of acoustic skew symmetric ($A_0$) mode. Three curves namely broken line, solid and dotted correspond to coupled thermoelasticity (CT), LS and GL theories of thermoelasticity for liquid layer thickness $h = 0.25$, 0.5 and 0.75. The displacement amplitude in this case is observed to be maximum at center of plate and minimum at the surfaces. Fig. 5(b) shows that the amplitude of the non-dimensional $x$-component of the skew symmetric displacement $u$ in generalized thermoelastic plate. In this case displacement amplitude is observed to be quite large at plate surfaces and zero at center of the plate. As for as symmetric acoustic ($S_0$) mode is concerned, the amplitude of the $x$-component of displacement ($u$) looks like amplitude of $z$-component of displacement ($w$) for $A_0$ mode as shown by Fig. 5(a). The amplitude of $z$-component of displacement ($w$) looks like the $x$-component ($u$) of $A_0$ mode as shown in Fig. 5(b). The comparison of Figs. 5(a) and (b) reveals that skew symmetric $x$-component of displacement ($u$) is dominant as compared to the symmetric one in this case. Figs. 6(a) and (b) contains the plots of the magnitude of non-dimensional symmetric and skew symmetric temperature change ($T$) in a generalized thermoelastic plate in the considered direction of wave propagation. Three curves namely broken line, solid and dotted correspond to CT, LS and GL theories of thermoelasticity for liquid layer thickness $h = 0.25$, 0.5 and 0.75. These curves correspond to first branch (acoustic) of symmetric ($S_0$) and skew symmetric ($A_0$) mode. It has been noticed that for symmetric temperature mode curves change their behavior from convex upward to convex downward as compared to displacement. It is observed that the variation of the symmetric and skew symmetric temperature change ($T$), respectively, looks like the amplitude of symmetric and skew symmetric $x$-component of strain.

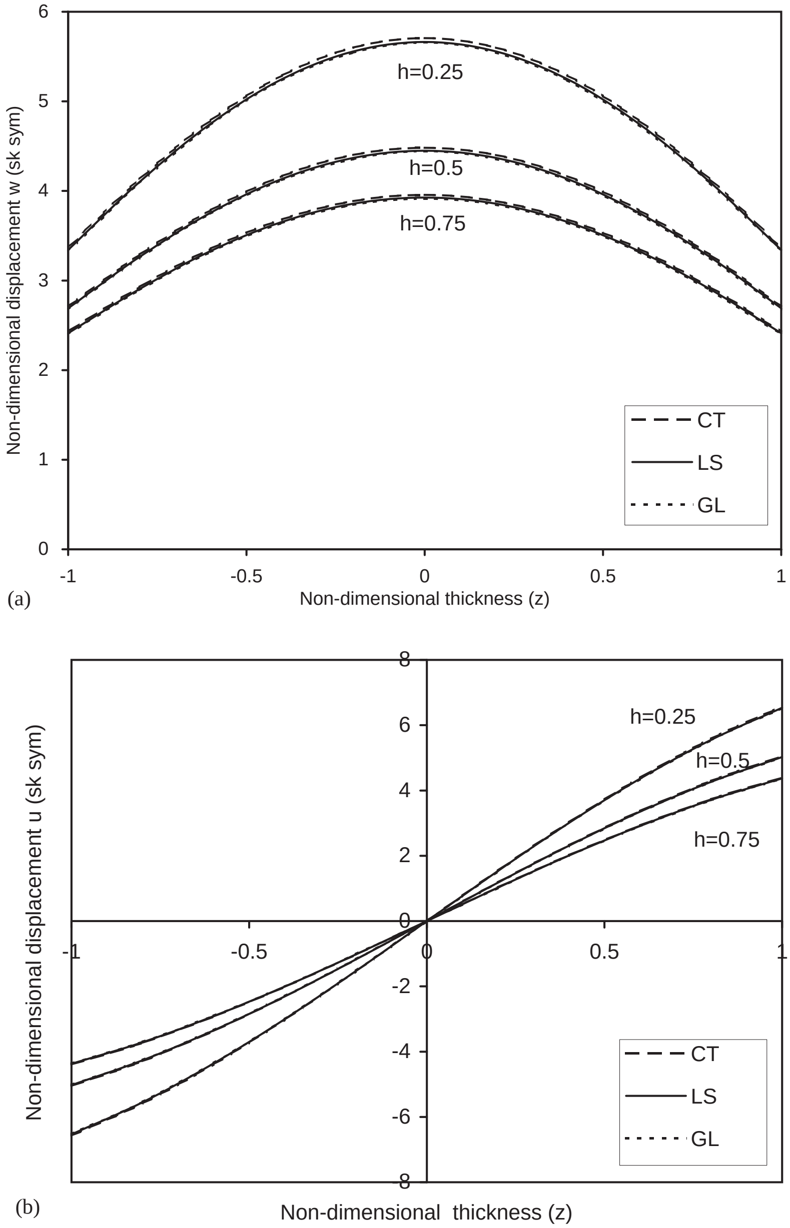



Fig. 5. (a) Variation of amplitude of vertical displacement $w_{asy}$ with thickness of plate for CT, LS and GL thermoelasticity. (b) Variation of amplitude of horizontal displacement $u_{\mathrm{asy}}$ with thickness of plate for CT, LS and GL thermoelasticity.

It is noticed that the thermal relaxations time has negligibly small effect on the displacement and temperature change as compared to the dispersion curves. The magnitude of non-dimensional skew symmetric temperature change has same value as the liquid layer

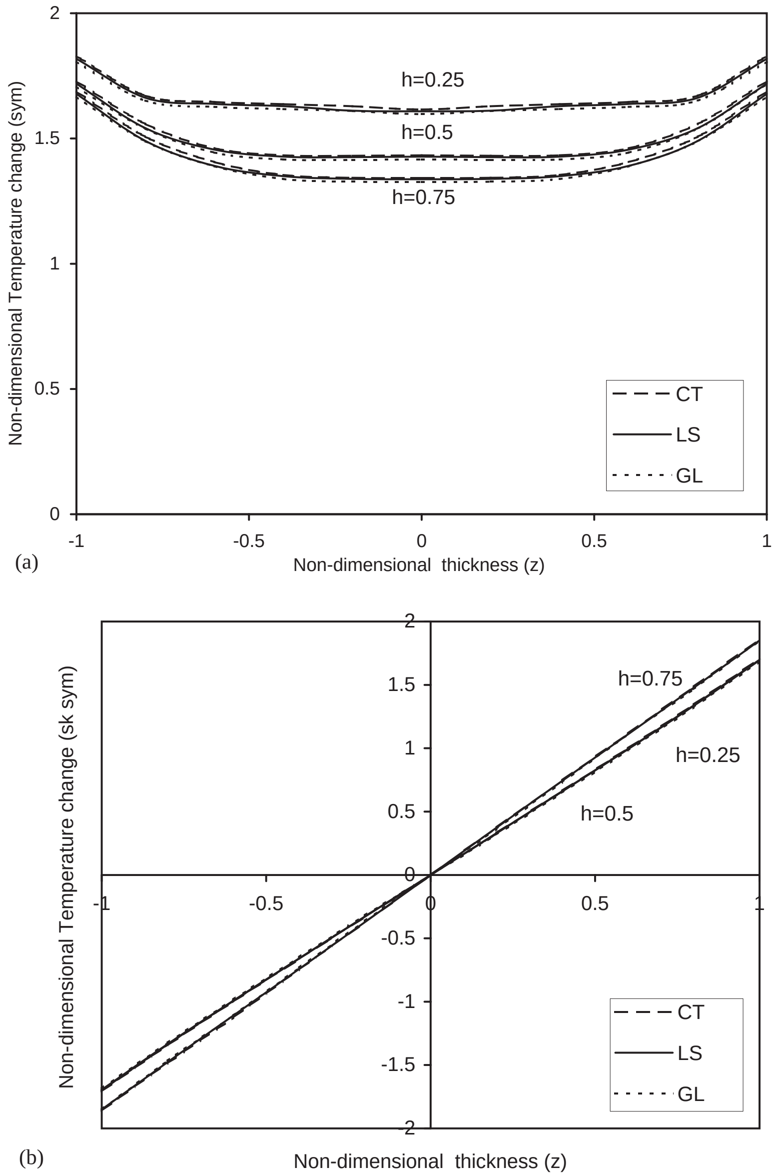
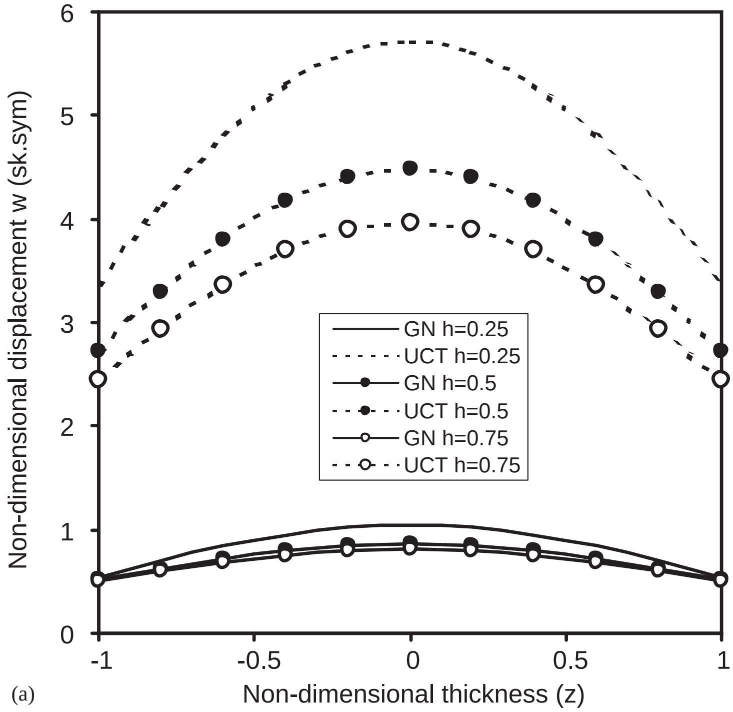
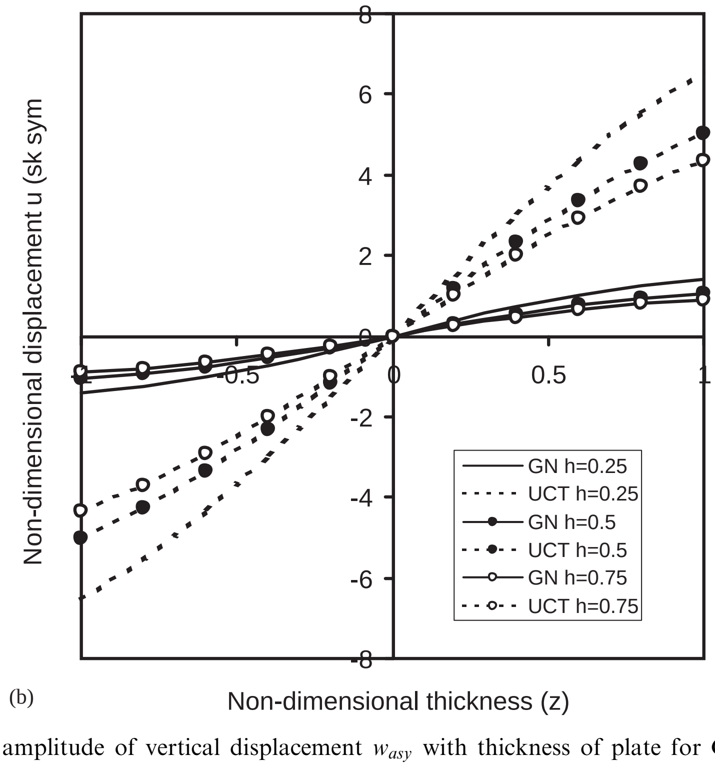



Fig. 6. (a) Variation of amplitude of symmetric temperature change with thickness of plate for CT, LS and GL thermoelasticity. (b) Variation of amplitude of skew symmetric temperature change with thickness of plate for CT, LS and GL thermoelasticity.

thickness decreases. It is also noticed that the displacements $u$ and $w$ has significantly higher magnitude in both symmetric and skew symmetric modes of wave propagation as compared to temperature change in the present situation. It is observed from the generated numerical data that the non-dimensional displacements and temperature change have slightly




(a)




(b)

Fig. 7. (a) Variation of amplitude of vertical displacement $w_{asy}$ with thickness of plate for GN and UCT theories. (b) Variation of amplitude of horizontal displacement $u_{asy}$ with thickness of plate for GN and UCT theories.

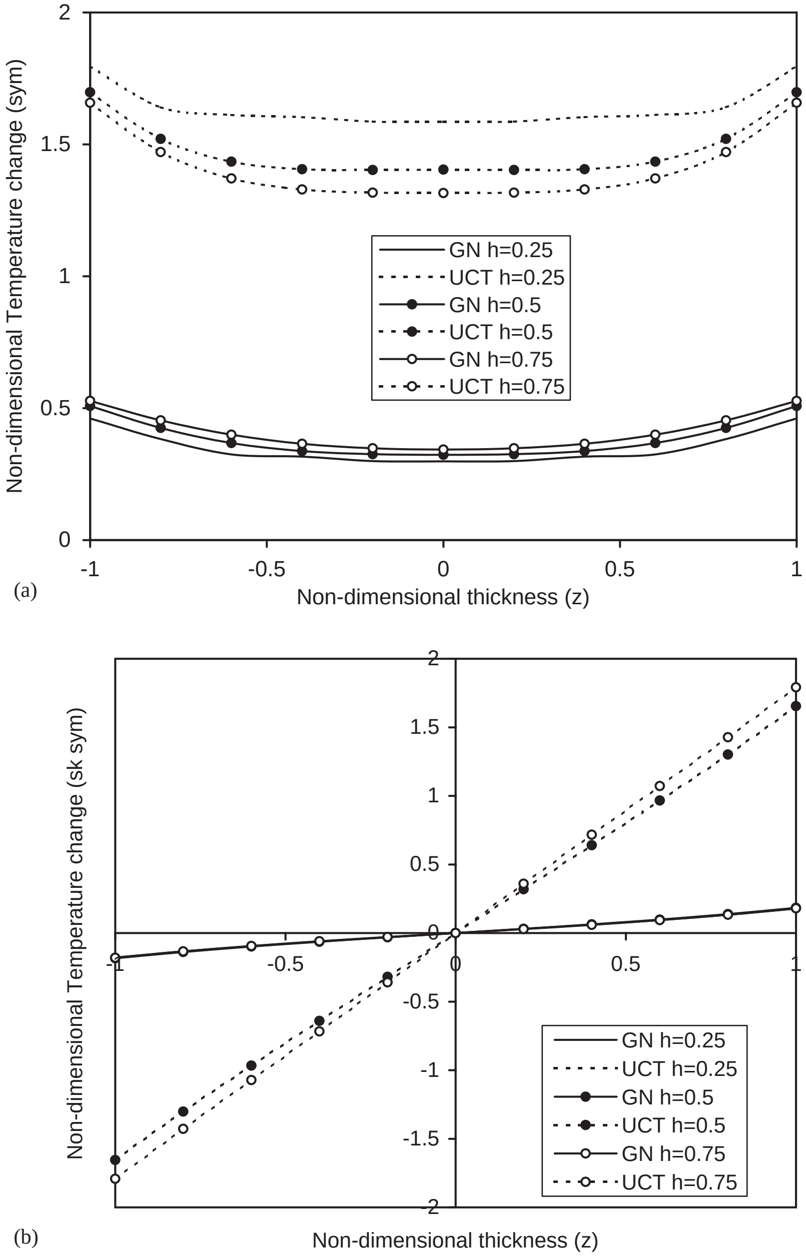



Fig. 8. (a) Variation of amplitude of symmetric temperature change with thickness of plate for GN and UCT theories. (b) Variation of amplitude of skew symmetric temperature change with thickness of plate for GN and UCT theories.

higher magnitudes in coupled thermoelasticity (CT) followed by LS and GL theories of thermoelasticity.

Figs. 7 and 8 represents the amplitudes of displacement and temperature change for symmetric and skew symmetric modes in the presence of liquid layers thickness $h = 0.25$, 0.5 and 0.75 in case

GN and UCT theories of thermoelasticity. The comparison of curves for UCT and GN theory of thermoelasticity in various figures reveals that the magnitude of different considered functions is quiet low in case of latter as compared to that of former one due to thermal variations. From these figures it is observed that the displacement and temperature amplitude have same trend of variation as in case of CT, LS and GL theories.

## 11. Conclusions

The propagation of waves in a homogeneous, transversely isotropic, thermally conducting plate bordered with layers of inviscid liquid or half-space of inviscid liquid on both sides, is investigated in the context of generalized theories of thermoelasticity. The results for isotropic materials, coupled and uncoupled theories of thermoelasticity have been obtained as particular cases. It is shown that the purely transverse motion (SH mode), which is not affected by thermal variations, gets decoupled from rest of the motion of wave propagation and occurs along an in-plane axis of symmetry. The special cases, such as short wavelength waves and thin plate waves, of the secular equations are also discussed. The attenuation coefficients and amplitudes of displacement components and temperature change have also been computed and studied. For the waves of short wavelength it is observed that the dispersion equation of Lamb waves in an transversely isotropic thermoelastic plate bordered with a homogeneous layer of thickness $h$ on both sides can be obtained by multiplying a factor of $\mathrm{i}\tan\gamma^* m_7$ to terms containing $\rho_L$ in the dispersion equation for leaky Lamb waves, i.e., Lamb waves in an transversely isotropic plate bordered with an infinite half-space homogeneous liquid ($h \to \infty$) at both sides. The periodic tangent function of liquid thickness reflects the periodic nature of the influence due to the presence of the liquid layers of varying thickness, which is also confirmed by our numerical calculations. This analysis includes thermal effects and hence is more general as compared to that of Wu and Zhu [30]. In case of isotropic materials the present analysis reduces to that of Sharma and Pathania [33]. Various regions of the secular equation are also explored. In region I, the secular equation for the skew symmetric case pertains to flexural vibrations and represents only a single vibration mode in a limited frequency range in the overall frequency spectrum. This is independent of thermal effects too. For the secular equation in the symmetric case in region II, the phase velocity expression reduces to the thin plate or plate stress analogue of the bar velocity of longitudinal rod theory, which also depends upon the thermal variations and relaxation time and, in the case of uncoupled theory, reduces to the result obtained by Graff [35].

It is observed that the phase velocity of the acoustic symmetric mode ($S_0$) become dispersionless and gets significantly reduced and affected in the presence of liquid. For increasing wave number this remains closer to the velocity of thermoelastic leaky Lamb waves in a solid half-space bordered with a liquid layers on both sides because in this case the energy transmission takes place mainly along the surface (interface) of the plate. The acoustic skew symmetric mode ($A_0$) has zero velocity at vanishing wave number, which increases to become closer to the velocity of thermoelastic leaky Lamb wave with increasing wave number but also gets modified due to the presence of liquid. The phase velocities of optical modes of propagation attain large values at vanishing wave number, which slashes down to become steady and asymptotic to the reduced Rayleigh wave velocity with increasing wave number. It is noticed that as the inclination of the wave normal with the axis of symmetry progresses, the behavior of attenuation coefficient becomes dispersionless which is also the case with increasing thickness of plate. The hypothetical results in case of GN theory here observed to resemble closely to that of CT, LS and GL theories of thermoelasticity for initial two modes namely acoustic and first modes of wave propagation. The heat is noticed to propagate with a finite, though quite large, speed in the instant case except the fact that the magnitude of phase velocity and amplitudes of different considered functions get suppressed in the presence of liquid.

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