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Letter to the Editor

Comparison of two Lindstedt–Poincaré-type perturbation methods

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1. Introduction

The standard Lindstedt–Poincaré method is one of the important perturbation techniques widely used in the study of non-linear oscillations [1–3]. But the method is only for solving problems with small parameters. It is the small parameter that restricts the applications of the standard Lindstedt–Poincaré method. To overcome the limitations, some modified Lindstedt–Poincaré techniques have been proposed in recent years. For example, He [4] proposed a modified perturbation method (which will be called Method 1), and Hu [5] pointed out that there exists an "innovative" classical perturbation method (Method 2) which is valid for large parameters.

In this work, the Duffing equation will be treated using the two Lindstedt–Poincaré-type perturbation methods mentioned above. A comparison of these two methods will be presented.

The Duffing equation [1–3] is

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where overdots denote differentiation with respect to time t and ε is a positive parameter. The solution of Eq. (1) is assumed in the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots. \tag{2}$$

The fundamental frequency ω^2 is given by

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \tag{3}$$

where the constants ω_i can be identified by means of no secular terms. Introducing the substitution $\tau = \omega t$, $\mathrm{d}/\mathrm{d}t = \omega \mathrm{d}/\mathrm{d}\tau$ into Eq. (1), we obtain

$$\omega^2 x'' + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad x'(0) = 0, \tag{4}$$

where primes designate differentiation with respect to τ.

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2. Approximate solution by Method 1

Substituting Eqs. (2) and (3) into Eq. (4) gives

$$(\omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \cdots) + \omega_0^2(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^3 = 0. \tag{5}$$

This equation is satisfied by setting the coefficients of the powers of ε equal to zero, resulting in

$$x_0'' + x_0 = 0, \tag{6a}$$

$$x_1'' + x_1 = -\frac{\omega_1}{\omega_0^2}x_0'' - \frac{x_0^3}{\omega_0^2}, \tag{6b}$$

$$x_2'' + x_2 = -\frac{\omega_2}{\omega_0^2}x_0'' - \frac{\omega_1}{\omega_0^2}x_1'' - \frac{3}{\omega_0^2}x_0^2 x_1. \tag{6c}$$

Solving Eq. (6) and taking into account the initial conditions given in Eq. (4) gives

$$x_0 = A\cos\omega t, \tag{7a}$$

$$\omega_1 = \tfrac{3}{4}A^2, \quad x_1 = \frac{A^3}{32\,\omega_0^2}(\cos 3\omega t - \cos\omega t), \tag{7b}$$

$$\omega_2 = -\frac{3A^4}{128\omega_0^2}, \quad x_2 = \frac{A^5}{1024\omega_0^4}(23\cos\omega t - 24\cos 3\omega t + \cos 5\omega t). \tag{7c}$$

Then, the first approximate solution to Eq. (1) is

$$x_1^a = A\cos\omega t + \frac{\varepsilon A^3}{32\omega_0^2}(\cos 3\omega t - \cos\omega t) \tag{8}$$

with

$$\omega = \omega_1^a = \sqrt{\omega_0^2 + \tfrac{3}{4}\varepsilon A^2}. \tag{9}$$

The second approximate solution becomes

$$x_2^a = A\cos\omega t + \frac{\varepsilon A^3}{32\omega_0^2}(\cos 3\omega t - \cos\omega t) + \frac{\varepsilon^2 A^5}{1024\omega_0^4}(23\cos\omega t - 24\cos 3\omega t + \cos 5\omega t) \tag{10}$$

with

$$\omega = \omega_2^a = \sqrt{\omega_0^2 + \frac{3}{4}\varepsilon A^2 - \frac{3\varepsilon^2 A^4}{128\omega_0^2}}. \tag{11}$$

3. Approximate solution by Method 2

Eq. (3) can be rewritten as

$$\omega_0^2 = \omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots. \tag{12}$$

Substituting this equation and Eq. (2) into Eq. (4) gives

$$\omega^2(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \cdots) + (\omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^3 = 0. \tag{13}$$

From this equation, we have

$$x_0'' + x_0 = 0, \tag{14a}$$

$$x_1'' + x_1 = \frac{\omega_1}{\omega^2}x_0 - \frac{x_0^3}{\omega^2}, \tag{14b}$$

$$x_2'' + x_2 = \frac{\omega_2}{\omega^2}x_0 + \frac{\omega_1}{\omega^2}x_1 - \frac{3}{\omega^2}x_0^2 x_1. \tag{14c}$$

Solving Eq. (14) and taking into account the initial conditions given in Eq. (4), we obtain [5]

$$x_0 = A\cos\omega t, \tag{15a}$$

$$\omega_1 = \tfrac{3}{4}A^2, \quad x_1 = \frac{A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t), \tag{15b}$$

$$\omega_2 = -\frac{3A^4}{128\omega^2}, \quad x_2 = \frac{A^5}{1024\omega^4}(\cos 5\omega t - \cos\omega t). \tag{15c}$$

Therefore, the first approximate solution to Eq. (1) is

$$x_1^b = A\cos\omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t) \tag{16}$$

with

$$\omega = \omega_1^b = \sqrt{\omega_0^2 + \tfrac{3}{4}\varepsilon A^2}. \tag{17}$$

The second approximate solution becomes [5]

$$x_2^b = A\cos\omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t) + \frac{\varepsilon^2 A^5}{1024\omega^4}(\cos 5\omega t - \cos\omega t), \tag{18}$$

where

$$\omega = \omega_2^b = \frac{1}{4}\sqrt{8\omega_0^2 + 6\varepsilon A^2 + \sqrt{64\omega_0^4 + 96\omega_0^2\varepsilon A^2 + 30\varepsilon^2 A^4}}. \tag{19}$$

4. Comparisons

4.1. Comparison of the fundamental frequencies

The first approximate solution $\omega_1^b = \omega_1^a$. But the second approximate solution ω_2^a is invalid for the large values of εA^2 (if $\varepsilon A^2 \geqslant (16 + \frac{8}{3}\sqrt{42})\omega_0^2 = 33.282\omega_0^2$, $\omega_0^2 + \frac{3}{4}\varepsilon A^2 - \frac{3}{128}\varepsilon^2 A^4/\omega_0^2 \leqslant 0$). ω_2^b can give excellent approximate frequencies for both small and large values of εA^2 [5].

4.2. Comparison of the time-dependent oscillatory displacement curves with the exact solutions

Since ω_2^a is not valid when $\varepsilon A^2 \geqslant 33.282\omega_0^2$, we will not consider the second approximate solution $x_2^a(t)$ in the following. In the work to follow, we let $\omega_0^2 = 1$. The exact periodic solution $x_e(t)$

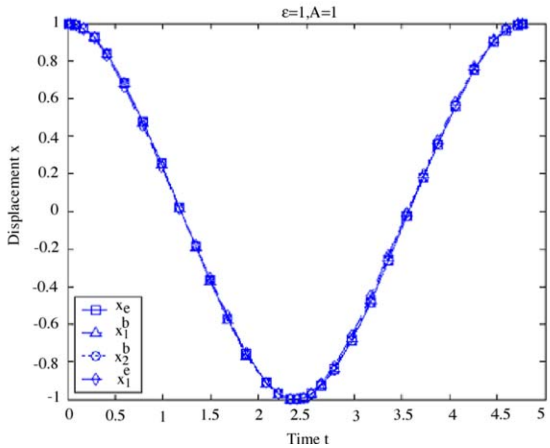

Fig. 1. Comparison of the approximate solutions with the exact solution for $\varepsilon = 1$, $A = 1$.

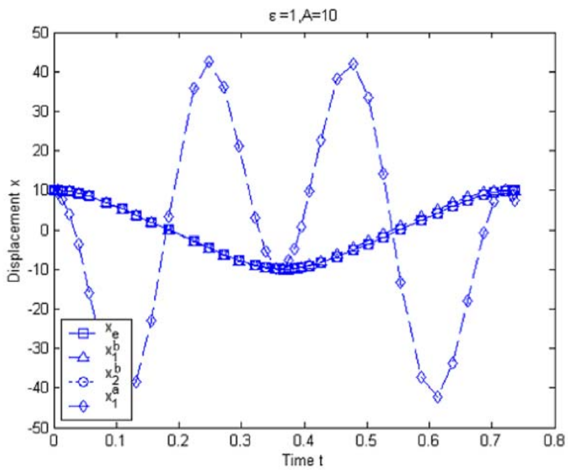

Fig. 2. Comparison of the approximate solutions with the exact solution for $\varepsilon = 1$, $A = 10$.

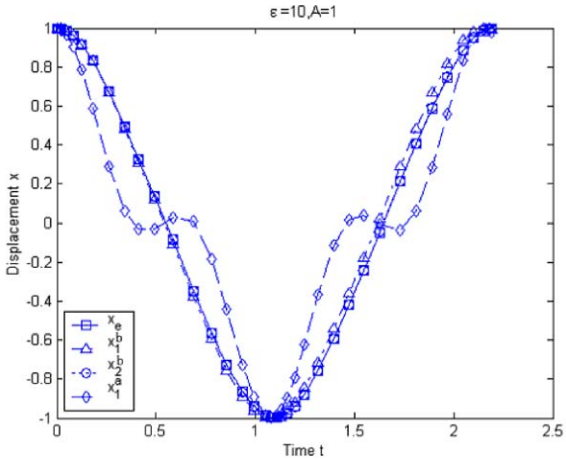

Fig. 3. Comparison of the approximate solutions with the exact solution for $\varepsilon = 10$, $A = 1$.

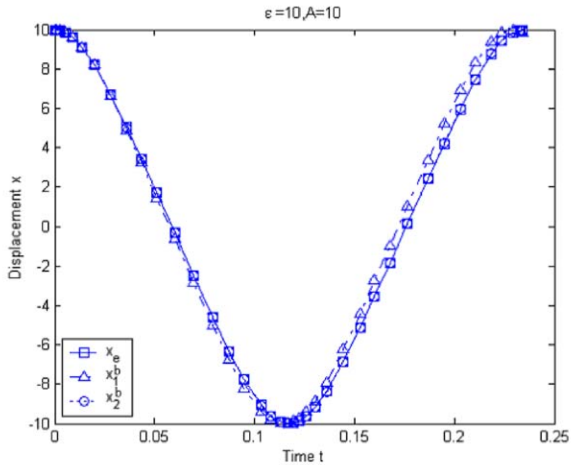

Fig. 4. Comparison of the approximate solutions with the exact solution for $\varepsilon = 10$, $A = 10$.

obtained by integrating Eq. (1), the approximate analytical periodic solutions $x_1^a(t)$, $x_1^b(t)$ and $x_2^b(t)$ computed by Eqs. (8), (16) and (18), respectively, are plotted in Figs. 1–3. Fig. 1 shows that $x_1^a(t)$, $x_1^b(t)$ and $x_2^b(t)$ are close to $x_e(t)$ for $\varepsilon = 1$, $A = 1$. But for $\varepsilon = 1$, $A = 10$ and $\varepsilon = 10$, $A = 1$, Figs. 2 and 3 show that $x_1^a(t)$ is not acceptable. For large values of εA^2, $x_e(t)$, $x_1^b(t)$ and $x_2^b(t)$ are pictured in Figs. 4–9. Fig. 9 indicates that even when $\varepsilon = 1000$ and $A = 1000$, $x_1^b(t)$ and $x_2^b(t)$ can give good approximations, and $x_2^b(t)$ is more accurate than $x_1^b(t)$.

5. Concluding remarks

(1) For Method 1, the second approximate solution is not better than the first approximate solution, and Method 1 is invalid for large parameters. Comparing Eqs. (6) and (7) with the

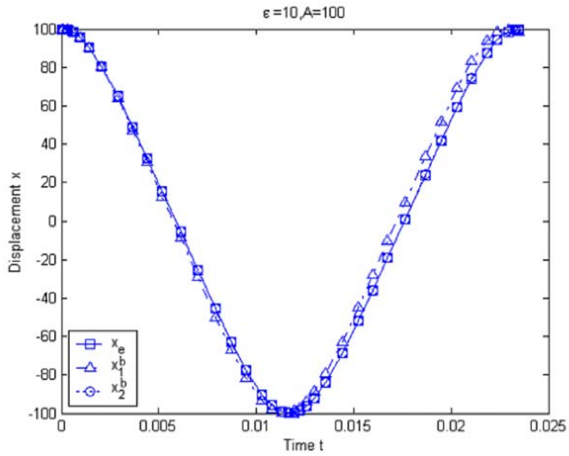

Fig. 5. Comparison of the approximate solutions with the exact solution for $\varepsilon = 10$, $A = 100$.

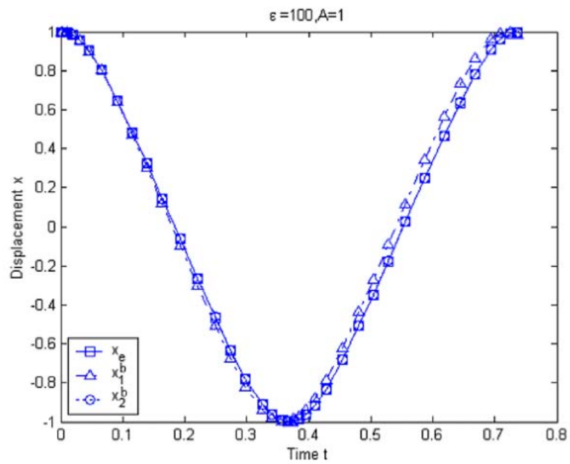

Fig. 6. Comparison of the approximate solutions with the exact solution for $\varepsilon = 100$, $A = 1$.

corresponding results obtained by the standard Lindstedt–Poincaré method (for example, Eqs. (2.58)–(2.60), and Eqs. (2.63) and (2.64) in Ref. [1]), it follows that Method 1 is identical to the standard Lindstedt–Poincaré method in nature. It is interesting to note that there are x_0'' and x_1'' on the right-hand sides of Eqs. (6), whereas there are only x_0 and x_1 on the right-hand sides of Eqs. (14). Obviously, if x_0 and x_1 are approximate solutions, then the accuracy of x_0'' and x_1'' is worse than that of x_0 and x_1. Perhaps this difference gives one of the reasons Method 1 (or the standard Lindstedt–Poincaré method) is invalid for larger parameters.

(2) Hu [1] pointed out that maybe using the expansion

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{20}$$

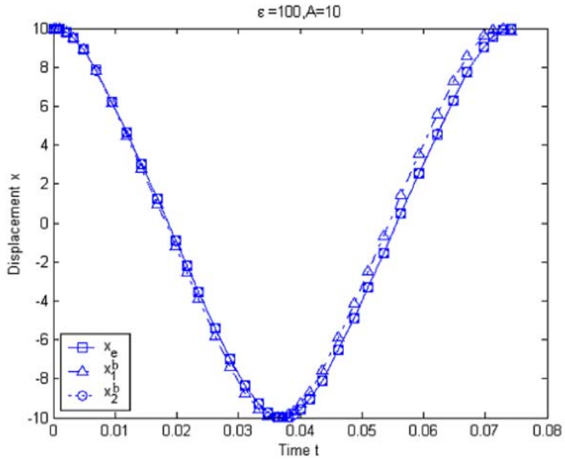

Fig. 7. Comparison of the approximate solutions with the exact solution for $\varepsilon = 100$, $A = 10$.

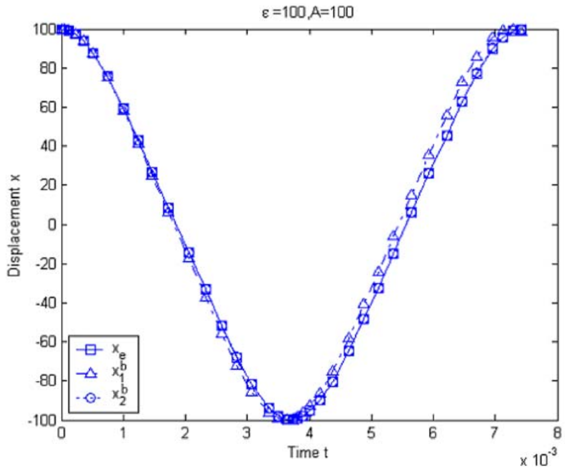

Fig. 8. Comparison of the approximate solutions with the exact solution for $\varepsilon = 100$, $A = 100$.

instead of expansion (3) is the reason why the standard classical perturbation method is not able to provide accurate results when the parameter is large. But in this paper, we see that although both Methods 1 and 2 use the same expansion (3), only Method 2 works for strongly nonlinear systems. Method 1 substitutes expansion (3) for ω^2 in Eq. (4), whereas Method 2 substitutes expansion (3) (or expansion (12)) for ω_0^2 in Eq. (4), which is the most important difference between the two methods! It should also be noted that Method 2 works even when the linear part of restoring force is zero [6].

It therefore appears that the work presented here contributes to explaining why the ‘‘innovative’’ classical perturbation method (Method 2) works for large parameters.

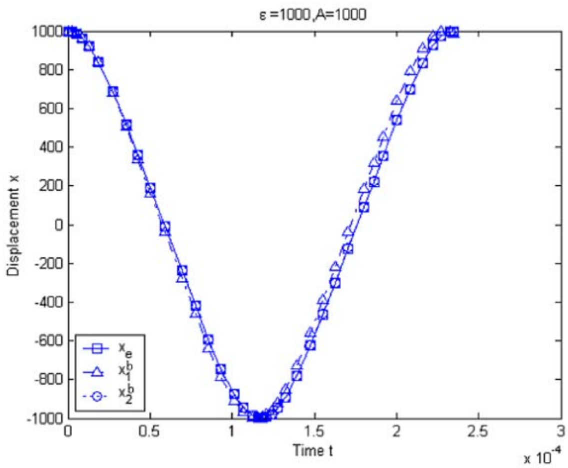

Fig. 9. Comparison of the approximate solutions with the exact solution for $\varepsilon = 1000$, $A = 1000$.

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