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Letter to the Editor

Higher-order mixed method for time integration in dynamic structural analysis

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1. Introduction

The direct integration schemes commonly used in engineering practice include various difference methods and linear acceleration methods [1]. These schemes are either first or second order accurate, and therefore their accuracy is relatively low especially in the high-frequency range. In addition, most of them have inherent algorithmic damping. Although such damping can help to damp out the spurious high-frequency responses, it may distort the high-frequency responses in cases where they are of relevance. Hence, it is desirable if the numerical dissipation in the high-frequency range is controllable.

To overcome these problems, a number of higher order numerical integration algorithms have been put forward. Based on Hamilton's law of varying action, Riff and Baruch [2] used cubic interpolation functions to construct a time integration scheme, which was found to be fourth order accurate and conditionally stable. Argyris et al. [3] and Gellert [4] used Hermitian shape functions and point collocation method to derive algorithms corresponding to the Padé approximations, and the algorithms were shown to be unconditionally stable and fourth order accurate. Also using cubic shape functions and the collocation method, Golley [5] obtained a third order accurate and conditionally stable algorithm with the second order Gauss quadrature points as the collocation points. Kujawski and Gallagher [6] derived a fourth order accurate unconditionally stable algorithm from a generalized least-squares procedure for undamped systems. Tarnow and Simo [7] demonstrated the use of sub-stepping technique to get fourth order accurate solutions from the second order accurate algorithms by evaluating the solutions at three fractional steps.

The search for numerical integration schemes with arbitrary order of accuracy mainly started in the 1990s. Zhong and Williams [8] proposed a precise time step integration method (PTSIM), which was an exact solution of the homogenous equations of motion. The accuracy of the

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algorithm is restricted only by the simulation accuracy of loading. Fung [9] put forward a complex-time-step method to construct a family of unconditionally stable higher order accurate algorithms with controllable numerical dissipation. However to maintain higher order accuracy, the excitation may need some modifications. In a bid to improve the stability for linear second order differential equations based on the weighted residual method, Fung [10] presented a strategy for determination of the weighting parameters for unconditional stability and arbitrary order of accuracy.

A higher order mixed method (MM) for time integration in dynamic structural analysis is proposed here. It uses a combination of the weighted residual method and the collocation method. The dynamic response within a time interval is interpolated between the dynamic responses at the endpoints using a fifth order polynomial. The optimum selection of various working parameters is discussed. The application of the method to single-degree-of-freedom (s.d.o.f.) systems is discussed. The accuracy of the time integration schemes presented are studied and compared with those of other commonly used schemes.

2. Higher order MM

The equation of motion for a system with only one degree of freedom (d.o.f.) can be written as

$$\ddot{x}(t) + 2\omega\zeta\dot{x}(t) + \omega^2 x(t) = p(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{1}$$

in which ζ is the damping ratio, ω is the undamped circular natural frequency of the system, $p(t)$ is the modal forcing excitation, and x, $\dot{x}$ and $\ddot{x}$ are, respectively, the displacement, velocity and acceleration. Introducing $t = t_k + \tau\Delta t$ where t_k is a typical discrete time for numerical computation and τ is the non-dimensional time, Eq. (1) can be normalized as

$$x^{(2)}(\tau) + 2\zeta\omega\Delta t x^{(1)}(\tau) + \omega^2\Delta t^2 x^{(0)}(\tau) = p(t_k + \tau\Delta t)\Delta t^2, \tag{2}$$

in which the time derivatives of x with respect to τ are defined as

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \frac{1}{\Delta t^\alpha}\frac{\mathrm{d}^\alpha x}{\mathrm{d}\tau^\alpha} = \frac{1}{\Delta t^\alpha}x^{(\alpha)}(\tau) \quad (\alpha = 1, 2) \tag{3}$$

and Δt is the time step. The dynamic response vectors at the start and end points of the time interval $[t_k, t_{k+1}]$ are, respectively, written in terms of the derivatives as

$$\{\delta_{k,0}\}^{\mathrm{T}} = [x_k^{(0)} \quad x_k^{(1)}\Delta t \quad x_k^{(2)}\Delta t^2], \tag{4}$$

$$\{\delta_{k,1}\}^{\mathrm{T}} = [x_{k+1}^{(0)} \quad x_{k+1}^{(1)}\Delta t \quad x_{k+1}^{(2)}\Delta t^2]. \tag{5}$$

The interpolated function of displacement $x(\tau)$ can be expressed as

$$x(\tau) = \{\delta_k\}^{\mathrm{T}}[C_0]\{T\}, \tag{6}$$

where

$$\{\delta_k\}^{\mathrm{T}} = [\{\delta_{k,1}\}^{\mathrm{T}} \quad \{\delta_{k,0}\}^{\mathrm{T}}], \tag{7}$$

$$[C_0] = \begin{bmatrix} 6 & -15 & 10 & 0 & 0 & 0 \\ -3 & 7 & -4 & 0 & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 & 0 & 0 \\ -6 & 15 & -10 & 0 & 0 & 1 \\ -3 & 8 & -6 & 0 & 1 & 0 \\ -0.5 & 1.5 & -1.5 & 0.5 & 0 & 0 \end{bmatrix}, \tag{8}$$

$$\{T\}^{\mathrm{T}} = [\tau^5 \quad \tau^4 \quad \tau^3 \quad \tau^2 \quad \tau \quad 1]. \tag{9}$$

The time derivatives of $x(\tau)$ are given by

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}\tau^\alpha} = \{\delta_k\}^{\mathrm{T}}[C_\alpha]\{T\} \quad (\alpha = 1, 2), \tag{10}$$

where the $[C_\alpha]$ matrices are

$$[C_1] = \begin{bmatrix} 0 & 30 & -60 & 30 & 0 & 0 \\ 0 & -15 & 28 & -12 & 0 & 0 \\ 0 & 2.5 & -4 & 1.5 & 0 & 0 \\ 0 & -30 & 60 & -30 & 0 & 0 \\ 0 & -15 & 32 & -18 & 0 & 1 \\ 0 & -2.5 & 6 & -4.5 & 1 & 0 \end{bmatrix}, \tag{11}$$

$$[C_2] = \begin{bmatrix} 0 & 0 & 120 & -180 & 60 & 0 \\ 0 & 0 & -60 & 84 & -24 & 0 \\ 0 & 0 & 10 & -12 & 3 & 0 \\ 0 & 0 & -120 & 180 & -60 & 0 \\ 0 & 0 & -60 & 96 & -36 & 0 \\ 0 & 0 & -10 & 18 & -9 & 1 \end{bmatrix}. \tag{12}$$

Substituting Eq. (10) into Eq. (2), the residual $R(\tau)$ is given by

$$R(\tau) = \{\delta_k\}^{\mathrm{T}}([C_2] + 2\zeta\omega\Delta t[C_1] + \omega^2\Delta t^2[C_0])\{T\} - p(t_k + \tau\Delta t)\Delta t^2. \tag{13}$$

The present MM is a higher order method built upon the weighted residual method and the collocation method. Prescribing that Eq. (2) is satisfied at $\tau = 1$ (i.e., $t_{k+1} = t_k + \Delta t$) and $\tau = \theta$ (i.e., $t_{k+\theta} = t_k + \theta\Delta t$), we have, respectively,

$$x^{(2)}(1) + 2\zeta\omega\Delta t x^{(1)}(1) + \omega^2\Delta t^2 x^{(0)}(1) = p(t_k + \Delta t)\Delta t^2, \tag{14}$$

$$x^{(2)}(\theta) + 2\zeta\omega\Delta t x^{(1)}(\theta) + \omega^2\Delta t^2 x^{(0)}(\theta) = p(t_k + \theta\Delta t)\Delta t^2, \tag{15}$$

where θ is a parameter that controls the stability of the algorithm. In particular, to establish the relationship between the dynamic response vectors $\{\delta_{k,0}\}$ and $\{\delta_{k,1}\}$, the residual $R(\tau)$ defined in

Eq. (13) is minimized using the weighting function $\psi(\tau)$ by

$$\int_0^1 \psi(\tau)R(\tau)\,\mathrm{d}\tau = 0. \tag{16}$$

Let the weighting function $\psi(\tau)$ be chosen as

$$\psi(\tau) = \tau^{\beta} \tag{17}$$

in terms of a parameter β that controls the accuracy of the algorithms. Substituting Eqs. (6), (10) and (17) into Eqs. (14)–(16), the recurrence formula is obtained as

$$[A]\{\delta_{k,1}\} = [B]\{\delta_{k,0}\} + \{P_k\}, \tag{18}$$

where $[A]$ and $[B]$ are 3×3 coefficient matrices, and $\{P_k\}$ is the load vector given by

$$\{P_k\} = \left[p(t_i + \Delta t)\Delta t^2 \quad \Delta t^2 \int_0^1 p(t_i + \tau\Delta t)\tau^{\beta}\,\mathrm{d}\tau \quad p(t_i + \theta\Delta t)\Delta t^2\right]^{\mathrm{T}}. \tag{19}$$

The integrals in Eqs. (16) and (19) are worked out by Gauss quadrature using three Gauss points. The coefficient matrices $[A]$ and $[B]$ are given explicitly as

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad [B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \tag{20}$$

where

$$\begin{aligned}
&a_{11} = \varpi^2, \qquad a_{12} = 2\zeta\varpi, \qquad a_{13} = 1,\\
&a_{21} = \varpi^2(6\beta_6 - 15\beta_5 + 10\beta_4) + 2\zeta\varpi(30\beta_5 - 60\beta_4 + 30\beta_3) + 120\beta_4 - 180\beta_3 + 60\beta_2,\\
&a_{22} = \varpi^2(-3\beta_6 + 7\beta_5 - 4\beta_4) + 2\zeta\varpi(-15\beta_5 + 28\beta_4 - 12\beta_3) - 60\beta_4 + 84\beta_3 - 24\beta_2,\\
&a_{23} = \varpi^2(0.5\beta_6 - \beta_5 + 0.5\beta_4) + 2\zeta\varpi(2.5\beta_5 - 4\beta_4 + 1.5\beta_3) + 10\beta_4 - 12\beta_3 + 3\beta_2,\\
&a_{31} = \varpi^2(6\theta^5 - 15\theta^4 + 10\theta^3) + 2\zeta\varpi(30\theta^4 - 60\theta^3 + 30\theta^2) + 120\theta^3 - 180\theta^2 + 60\theta,\\
&a_{32} = \varpi^2(-3\theta^5 + 7\theta^4 - 4\theta^3) + 2\zeta\varpi(-15\theta^4 + 28\theta^3 - 12\theta^2) - 60\theta^3 + 84\theta^2 - 24\theta,\\
&a_{33} = \varpi^2(0.5\theta^5 - \theta^4 + 0.5\theta^3) + 2\zeta\varpi(2.5\theta^4 - 4\theta^3 + 1.5\theta^2) + 10\theta^3 - 12\theta^2 + 3\theta,\\
&b_{11} = 0.0, \qquad b_{12} = 0.0, \qquad b_{13} = 0.0,\\
&b_{21} = \varpi^2(6\beta_6 - 15\beta_5 + 10\beta_4 - \beta_1) + 2\zeta\varpi(30\beta_5 - 60\beta_4 + 30\beta_3) + 120\beta_4 - 180\beta_3 + 60\beta_2,\\
&b_{22} = \varpi^2(3\beta_6 - 8\beta_5 + 6\beta_4 - \beta_2) + 2\zeta\varpi(15\beta_5 - 32\beta_4 + 18\beta_3 - \beta_1) + 60\beta_4 - 96\beta_3 + 36\beta_2,\\
&b_{23} = \varpi^2(0.5\beta_6 - 1.5\beta_5 + 1.5\beta_4 - 0.5\beta_3) + 2\zeta\varpi(2.5\beta_5 - 6\beta_4 + 4.5\beta_3 - \beta_2)\\
&\qquad + 10\beta_4 - 18\beta_3 + 9\beta_2 - \beta_1,\\
&b_{31} = \varpi^2(6\theta^5 - 15\theta^4 + 10\theta^3 - 1) + 2\zeta\varpi(30\theta^4 - 60\theta^3 + 30\theta^2) + 120\theta^3 - 180\theta^2 + 60\theta,\\
&b_{32} = \varpi^2(3\theta^5 - 8\theta^4 + 6\theta^3 - \theta) + 2\zeta\varpi(15\theta^4 - 32\theta^3 + 18\theta^2 - 1) + 60\theta^3 - 96\theta^2 + 36\theta,
\end{aligned}$$

$$
\begin{aligned}
b_{33} &= \varpi^2(0.5\theta^5 - 1.5\theta^4 + 1.5\theta^3 - 0.5\theta^2) + 2\zeta\varpi(2.5\theta^4 - 6\theta^3 + 4.5\theta^2 - \theta) \\
&\quad + 10\theta^3 - 18\theta^2 + 9\theta - 1, \\
\beta_1 &= 1/(\beta+1), \quad \beta_2 = 1/(\beta+2), \quad \beta_3 = 1/(\beta+3), \\
\beta_4 &= 1/(\beta+4), \quad \beta_5 = 1/(\beta+5), \quad \beta_6 = 1/(\beta+6).
\end{aligned}
$$

The amplification matrix $[T_e]$ for the MM can be derived from Eq. (18) as

$$[T_e] = [A]^{-1}[B]. \tag{21}$$

The amplification matrix $[T_e]$ and therefore its spectral radius ρ both depend on the parameters θ and β, the damping ratio ζ and the ratio $\Delta t/T$ where T is the undamped natural period. As $\Delta t/T$ tends to infinity, the spectral radius $\rho(\Delta t/T, \zeta, \theta, \beta)$ is independent of the damping ratio ζ and can be rewritten as a function of θ and β only, i.e., $\rho(\infty, \zeta, \theta, \beta) = \bar{\rho}(\theta, \beta)$. The variation of the spectral radius $\bar{\rho}(\theta, \beta)$ against the parameters θ and β is shown in Fig. 1. The choices of parameters θ and β for unconditional stability of this algorithm are

$$
\begin{aligned}
&1.01 \leqslant \theta \leqslant 1.26 \quad \text{for } \beta = 0, \\
&0.98 \leqslant \theta \leqslant 1.35 \quad \text{for } \beta = 1, \\
&0.96 \leqslant \theta \leqslant 1.43 \quad \text{for } \beta = 2, \\
&0.93 \leqslant \theta \leqslant 1.50 \quad \text{for } \beta = 3.
\end{aligned} \tag{22}
$$

Fig. 2 shows the variation of the spectral radius $\rho(\Delta t/T, \zeta, \theta, \beta)$ against the ratio $\Delta t/T$ for the damping ratio $\zeta = 0.01$ and several combinations of parameters θ and β for unconditional stability according to Eq. (22). According to Wood [11], the spectral radius of amplitude against $\Delta t/T$ should stay close to unit level as long as possible and decrease to about 0.5 or 0.8 as $\Delta t/T$

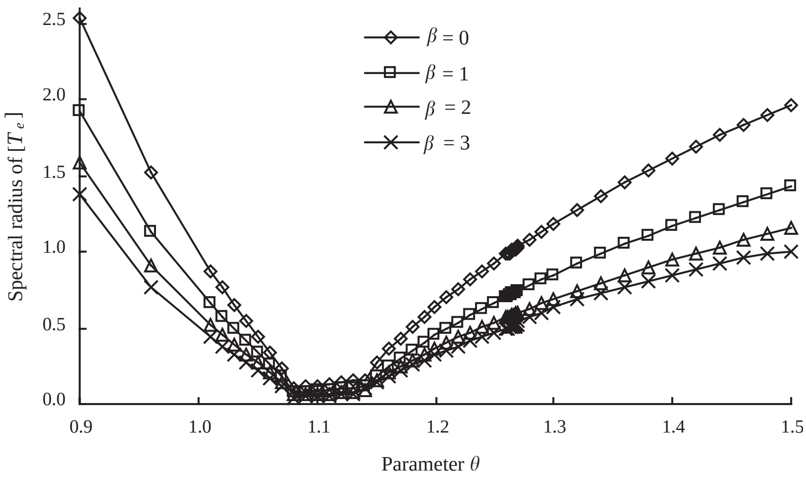

Fig. 1. Mixed method: spectral radius of $[T_e]$ when $\Delta t/T$ approaches infinity.

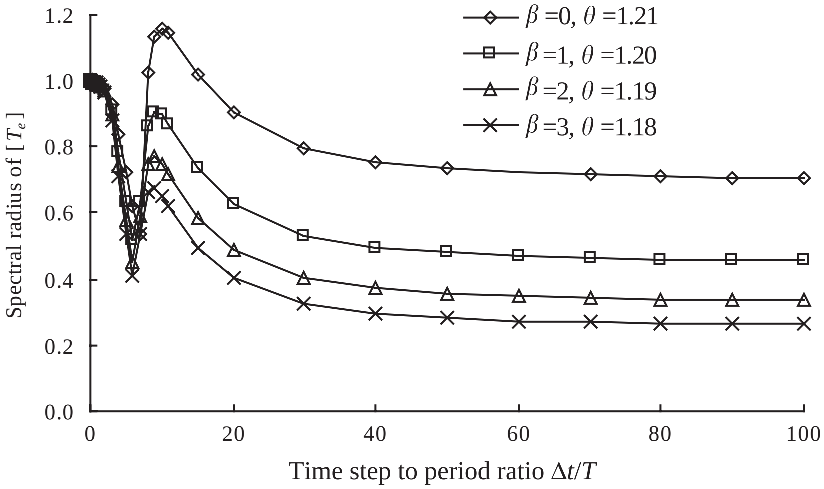

Fig. 2. Mixed method: spectral radius of $[T_e]$ against time step to period ratio $\Delta t/T$.

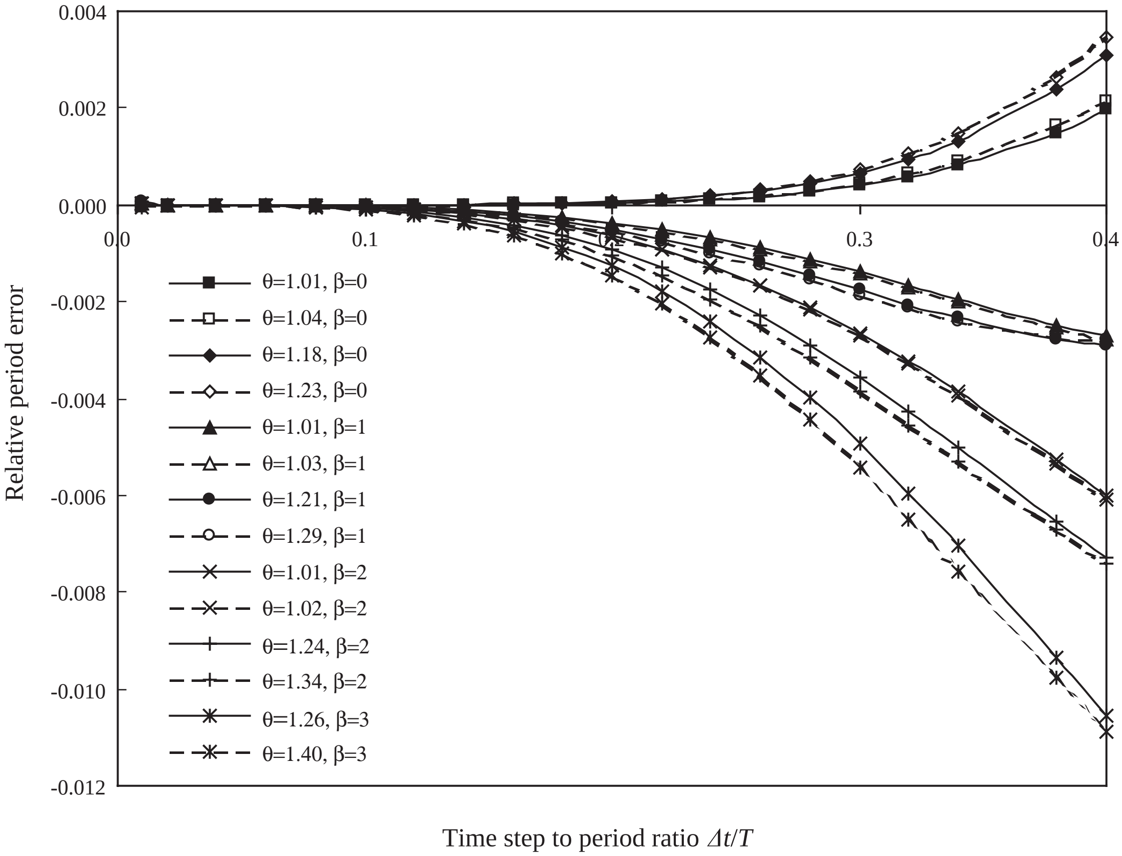

Fig. 3. Mixed method: relative period error versus time step to period ratio ($\zeta = 0.0$).

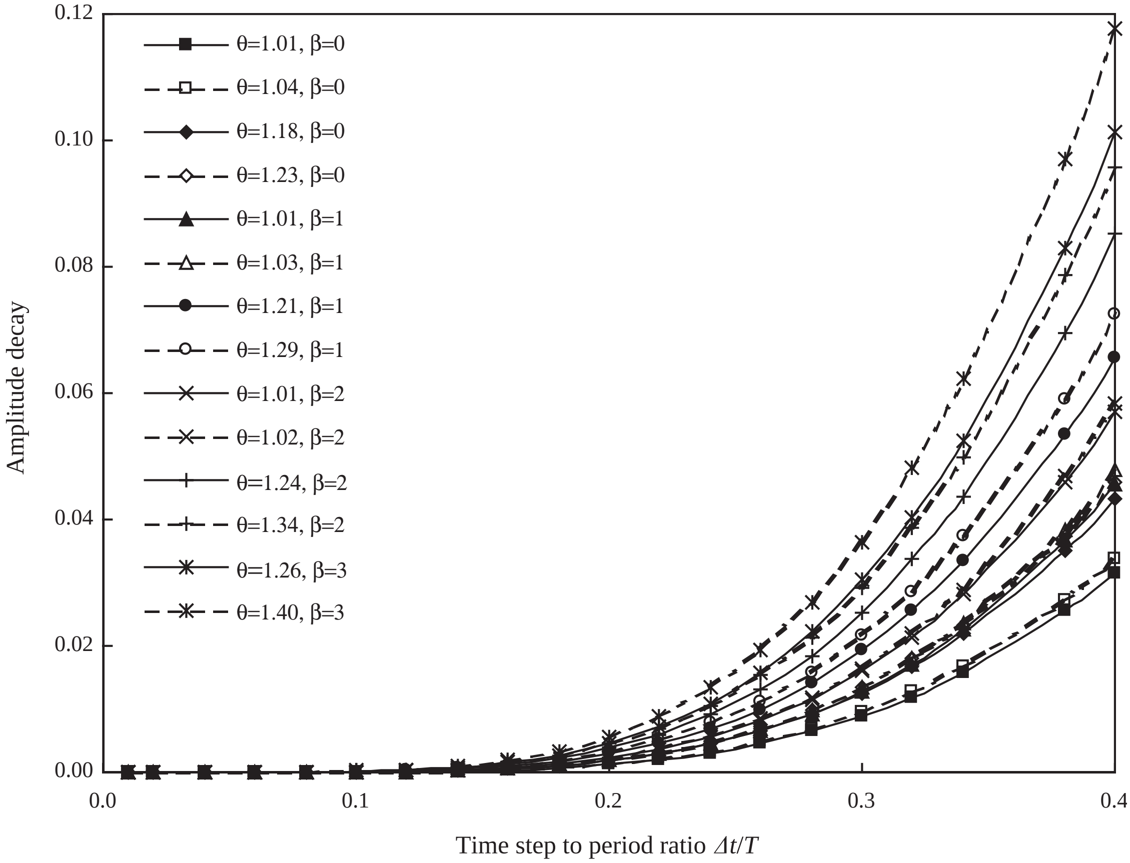

Fig. 4. Mixed method: amplitude decay versus time step to period ratio ($\zeta = 0.0$).

tends to infinity. The following parameters are therefore suggested:

$$\begin{aligned}
&1.01 \leqslant \theta \leqslant 1.04, \quad 1.18 \leqslant \theta \leqslant 1.23 \quad \text{for } \beta = 0,\\
&1.01 \leqslant \theta \leqslant 1.03, \quad 1.21 \leqslant \theta \leqslant 1.29 \quad \text{for } \beta = 1,\\
&1.01 \leqslant \theta \leqslant 1.02, \quad 1.24 \leqslant \theta \leqslant 1.34 \quad \text{for } \beta = 2,\\
&1.26 \leqslant \theta \leqslant 1.40 \qquad\qquad\qquad\qquad\quad \text{for } \beta = 3.
\end{aligned} \tag{23}$$

The proposed algorithm can be shown to be fifth order accurate. The accuracy of the algorithm can be measured by the relative period error and the amplitude decay. Fig. 3 shows how the relative period error varies with the ratio $\Delta t/T$ for different combinations of parameters θ and β. It is observed that the smaller the parameters θ and β, the smaller the relative period error. Similarly the amplitude decay is plotted against the ratio $\Delta t/T$ in Fig. 4, which shows that the smaller the parameters θ and β, the smaller the amplitude decay. Compared with the conventional numerical integration schemes such as Houbolt method, Wilson θ method (WM) and Newmark β method (NM) [1], the proposed algorithm gives smaller relative period error and amplitude decay. This shows that the proposed method is an improvement over the conventional methods. Bearing in mind the desirability of stability and accuracy, and that the proposed method places a lot of weight on the latter part of the time step with values of θ close to unity, it is suggested that the parameters θ and β are chosen as follows:

$$\begin{aligned}&1.18 \leqslant \theta \leqslant 1.23 \quad \text{for } \beta = 0,\\ &1.21 \leqslant \theta \leqslant 1.29 \quad \text{for } \beta = 1,\\ &1.24 \leqslant \theta \leqslant 1.34 \quad \text{for } \beta = 2,\\ &1.26 \leqslant \theta \leqslant 1.40 \quad \text{for } \beta = 3.\end{aligned} \tag{24}$$

Extension of the proposed MM is fairly straightforward, and it requires the derivation of the corresponding coefficient matrices.

3. Numerical example

An s.d.o.f. system with damping ratio $\zeta = 0.05$ and undamped circular natural frequency $\omega = 1.0$ can be described by the following equation:

$$\ddot{x}(t) + 2\omega\zeta\dot{x}(t) + \omega^2 x(t) = f(t), \quad x(0) = 0.0, \quad \dot{x}(0) = 0.0. \tag{25}$$

Note that the units are omitted for convenience. The applied loading $f(t)$ is

$$f(t) = \begin{cases} \sin(\pi t), & 0 \leqslant t \leqslant 1.0, \\ 0, & t \geqslant 1.0. \end{cases} \tag{26}$$

Table 1
Displacement responses of the s.d.o.f. system due to a half-sinusoidal loading

Scheme	Δt	Time t					Max. error (%)
		0.2	0.4	0.6	0.8	1.0	
Exact		0.00407805	0.0303926	0.0913166	0.183735	0.289484	—
MM	0.01	0.00407805	0.0303926	0.0913166	0.183735	0.289484	0.000
	0.2	0.00407863	0.0303943	0.0913190	0.183739	0.289488	0.014
PTSIM	0.001	0.00407805	0.0303926	0.0913166	0.183735	0.289484	0.000
	0.01	0.00407772	0.0303901	0.0913091	0.183720	0.289461	0.008
	0.05	0.00406947	0.0303298	0.0911285	0.183357	0.288889	0.210
	0.20	0.00389127	0.0293407	0.0882162	0.177613	0.279920	4.580
NM	0.001	0.00407810	0.0303926	0.0913166	0.183735	0.289484	0.001
	0.01	0.00408242	0.0303969	0.0913142	0.183720	0.289453	0.107
	0.05	0.00418686	0.0305004	0.0912551	0.183349	0.288701	2.668
	0.20	0.00576260	0.0320355	0.0902718	0.177561	0.277025	41.31
WM	0.001	0.00407805	0.0303926	0.0913166	0.183735	0.289485	0.000
	0.01	0.00407763	0.0303896	0.0913079	0.183718	0.289457	0.010
	0.05	0.00406736	0.0303192	0.0911024	0.183311	0.288826	0.262
	0.20	0.00385433	0.0291349	0.0878084	0.176909	0.278951	5.485

The displacement responses for $0.2 \leqslant t \leqslant 1.0$ computed by five different methods are listed in Table 1. The methods used include the MM, the PTSIM [8], the NM [1] and the WM [1]. The numerical results are then compared with the available exact solution. Different values of time step Δt have been used to investigate the capabilities of various methods. For the MM, the parameters $\beta = 0$ and $\theta = 1.18$ are used. The parameters $\alpha = 0.5$ and $\delta = 0.25$ have been used in the NM. The parameter $\theta = 1.40$ has been used in the WM. It is observed that the proposed method performs better.

4. Conclusions

The MM presented here uses a combination of the weighted residual method and the collocation method. The dynamic response within a time interval is interpolated between the dynamic responses at the end points using a fifth order polynomial. The optimum selection of the working parameters is discussed, and recommendations are given for their choice to ensure unconditional stability in computation. The algorithm is fifth order accurate and the dissipation is controllable. The accuracy of the method presented is studied and compared with those of other commonly used schemes.

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