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Letter to the Editor

Comments on "Free vibration of skew Mindlin plates by *p*-version of F.E.M."

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Recently, Woo et al. [1] applied the *p*-version finite element method to the free vibration analyses of skew Mindlin plates. The method can predict many modes using just one element and converges faster than that of the *h*-version methods for the same number of degrees of freedom. The problem of shear locking is also overcome. But the discerning reader will find that most of results in Ref. [1] are smaller while others are bigger than those of Liew et al. [2] who used a global approach named *pb*-2 Rayleigh–Ritz method for very accurate solutions. In fact, analytic integration [5] is available for skew plate problems using higher order *p*-elements. The main purpose of the paper is to show that numerical integration for highly oscillating shape functions will soften the stiffness of the element in general and that the monotonic convergence of the predicted natural frequencies cannot be guaranteed. With analytic integration, the problems are eliminated.

It is well known that numerical integration errors influence the results computed by *p*-version elements and the problem becomes obvious for highly oscillating shape functions such as the higher order Legendre polynomials. Numerical integration softens the stiffness. This is the reason that most of results in Ref. [1] are smaller than those of Liew et al. [2]. The numerical quadrature should only be used to predict several lowest frequencies with few shape functions [3].

Two numerical examples are chosen to substantiate the above viewpoints. Consider a simply supported square plate as the first example. The exact solutions are available that one can compare the computed natural frequencies using analytic integration, results of Woo et al. [1] and Liew et al. [2] with the exact solutions. The second example is the comparison of three methods for CFFF (clamped–free–free–free) skew plates with different skew angles. Only one *p*-element with *p*-lever = 7 are used in all computations. From Table 1, one can see that the thin plate results of Woo et al. [1] are in good agreement with the exact solutions but the thick plate results are all smaller than the exact solutions and the results of Liew et al. [2]. The difference between the results of Woo et al. [1] and the exact solutions [4] is increasing with an increasing number of *p*-lever. On the other hand, the analytic *p*-version finite element is well-behaved when compared

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Table 1
Frequency parameters $\lambda = (\omega b^2/\pi^2)\sqrt{\rho h/D}$ for simply supported square plates

h/b	Mode	Methods				
		Present	Woo [1]	Liew [2]	Exact[a]	Exact[b]
0.001	1	2.0000	1.9999	2.0000	2.0000	2.0000
	2	5.0144	4.9999	5.0000	5.0000	5.0000
	3	5.0144	5.0000	5.0000	5.0000	5.0000
	4	8.0211	7.9999	8.0000	8.0000	7.9999
	5	10.144	10.000	9.9999	10.000	9.9998
0.2	1	1.7679(0.00)	1.7637(−0.04)	1.7679		1.7679
	2	3.8678(0.06)	3.8497(−0.41)	3.8656		3.8656
	3	3.8678(0.06)	3.8497(−0.41)	3.8656		3.8656
	4	5.5911(0.06)	5.4989(−1.59)	5.5879		5.5879
	5	6.6505(0.76)	6.5681(−0.49)	6.6006		6.6006

($a/b = 1.0$, $\kappa = 0.83333$, $\nu = 0.3$, $\beta = 0°$, figures in brackets are percentage errors).
[a] Kirchhoff thin plate theory [4].
[b] Mindlin thick plate theory [4].

with the exact solutions whatever the plates are "thin" or "thick". The convergence is always from above. The same problem can also be observed in Table 2. Two series shape functions are used in the present solutions with analytic integration. From Table 2, it can be observed that the solutions using Legendre orthogonal polynomials as shape functions [6] are more accurate than those using shape functions in Ref. [1]. Most results of Woo et al. [1] are smaller than those of Liew et al. [2]. Though the difference between them is small in the table, it will be increasingly large for modal frequencies higher than five. Higher order hierarchical Legendre polynomials will make the element matrices ill-conditioned [5]. To study the ill-conditioning of the element matrices, the condition number **CN** of the matrix $\mathbf{I}^{00}$ is plotted in Fig. 1. The coefficients I_{ij}^{00} $(i, j = 1, 2, \ldots, p)$ in matrix $\mathbf{I}^{00}$ are used in forming the stiffness and mass matrices of the element, and they are defined by

$$I_{ij}^{00} = \int_{-1}^{1} f_i f_j \, \mathrm{d}\xi, \tag{1}$$

where f are the shape functions using Legendre orthogonal polynomials. Ref. [7] gives the definition of condition number **CN**:

$$\mathbf{CN} = \frac{\omega_{max}}{\omega_{min}}, \tag{2}$$

where ω_{min} and ω_{min} are, respectively, the largest and smallest eigenvalues of the matrix. When the p-lever = 57, ω_{min} and the condition number **CN** of the matrix are negative, and the high order polynomials lead to ill-conditioning stiffness and mass matrices.

Analytical integration can easy be obtained for rectangular, skew and trapezoidal p-version elements [5,8]. These elements are enclosed by two pairs of opposite faces in which at least one pair parallel to each other. Then the two planar co-ordinates in the Jacobian of the elemental area

Table 2
Frequency parameters $\lambda = (\omega b^2/\pi^2)\sqrt{\rho h/D}$ for CFFF skew plates

β	Methods	Modes				
		1	2	3	4	5
0°	Present[a]	0.3384	0.7452	1.7807	2.2767	2.4234
	Present[b]	0.3384	0.7447	1.7807	2.2768	2.4209
	Woo [1]	0.3383	0.7432	1.7797	2.2712	2.4097
	Liew [2]	0.3382	0.7437	1.7779	2.2741	2.4163
15°	Present[a]	0.3482	0.7601	1.8331	2.1923	2.6369
	Present[b]	0.3482	0.7597	1.8330	2.1916	2.6355
	Woo [1]	0.3479	0.7579	1.8309	2.1863	2.6242
	Liew [2]	0.3479	0.7588	1.8299	2.1886	2.6309
30°	Present[a]	0.3776	0.8175	1.9825	2.1661	3.1048
	Present[b]	0.3774	0.8171	1.9818	2.1656	3.1033
	Woo [1]	0.3768	0.8146	1.9752	2.1606	3.0925
	Liew [2]	0.3768	0.8161	1.9772	2.1627	3.0974
45°	Present[a]	0.4246	0.9676	2.1160	2.3915	3.6922
	Present[b]	0.4241	0.9671	2.1136	2.3911	3.6870
	Woo [1]	0.4225	0.9644	2.1001	2.3855	3.6684
	Liew [2]	0.4226	0.9650	2.1059	2.3869	3.6789
60°	Present[a]	0.4831	1.3477	2.2651	2.9529	4.1954
	Present[b]	0.4816	1.3449	2.2561	2.9500	4.1755
	Woo [1]	0.4791	1.3415	2.2427	2.9414	4.1418
	Liew [2]	0.4781	1.3370	2.2387	2.9411	4.1599

($a/b = 1.0$, $h/b = 0.2$, $\kappa = 0.83333$, $v = 0.3$).
[a] Using shape functions from Ref. [1].
[b] Using shape functions from Ref. [6].

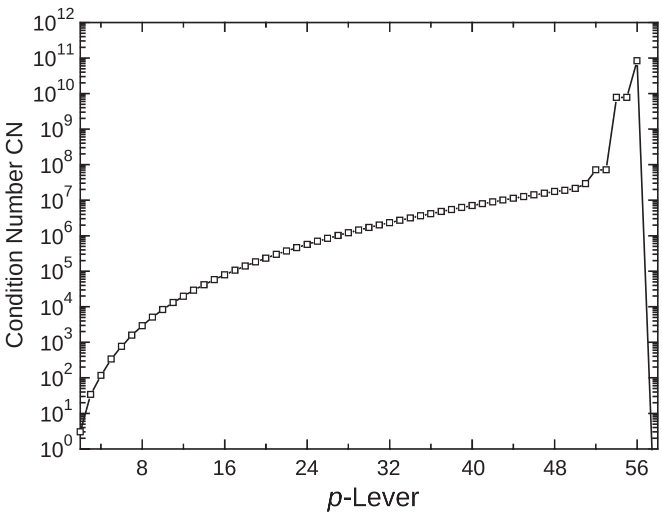

Fig. 1. Condition number of matrix $\mathbf{I}^{00}$.

are uncoupled and can be integrated independently. Unfortunately, the skew elements in Ref. [1] cannot be used in a general finite element analysis. But one can always break a triangle into three trapezoids by drawing three lines parallel to the edges from any point inside the triangle. So the range of application of trapezoidal elements is the same as that of the triangle elements. Table 3 presents the first five natural frequencies for an equilateral triangular plate (see Fig. 2) with different thickness and compared with those of Ritz method [9]. The same p-lever = 7 is used in the computation. The computed results by three trapezoidal elements using Legendre orthogonal

Table 3
Frequency parameters $\lambda = (\omega b^2/(2\pi))\sqrt{\rho h/D}$ for CFF equilateral triangular plates

h/b	Methods	Modes					
		1	2	3	4	5	6
0.001	Present	1.420	5.585	6.125	14.26	14.82	17.16
	Ref. [9]	1.420	5.585	6.125	14.26	14.82	17.15
0.05	Present	1.404	5.387	5.929	13.50	13.86	15.99
	Ref. [9]	1.404	5.387	5.929	13.50	13.86	15.99
0.10	Present	1.376	4.999	5.540	12.00	12.12	13.85
	Ref. [9]	1.376	4.999	5.540	12.00	12.12	13.85
0.15	Present	1.339	4.529	5.066	10.34	10.43	11.77
	Ref. [9]	1.339	4.529	5.066	10.34	10.43	11.77
0.20	Present	1.295	4.051	4.586	8.802	9.060	10.06
	Ref. [9]	1.295	4.051	4.586	8.802	9.060	10.06

($\kappa = 0.823$, $v = 0.3$).

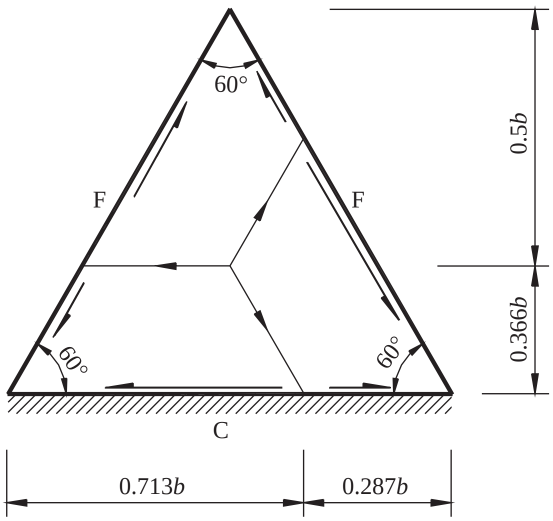

Fig. 2. Geometric sizes and mesh for a C–F–F triangular plate.

polynomials as shape functions are in excellent agreement with those in Ref. [9]. So the trapezoidal p-version elements with analytic quadrature can be applied to the vibration analysis of plates with arbitrary polygonal shapes without numerical integration.

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References

[1] K.S. Woo, C.H. Hong, P.K. Basu, C.G. Seo, Free vibration of skew Mindlin plates by p-verstion of F.E.M, *Journal of Sound and Vibration* 268 (4) (2003) 637–656.
[2] K.M. Liew, Y. Xiang, S. Kitipornchai, C.M. Wang, Vibration of thick skew plates based on Mindlin shear deformation plate theory, *Journal of Sound and Vibration* 168 (1) (1993) 36–69.
[3] A. Houmat, A triangular Fourier p-element for the analysis of membrane vibrations, *Journal of Sound and Vibration* 230 (1) (2000) 31–43.
[4] C.M. Wang, J.N. Reddy, K.H. Lee, *Shear Deformable Beams and Plates: Relationships with Classical Solutions*, Elsevier, Oxford, UK, 2000.
[5] A.Y.T. Leung, B. Zhu, J.J. Zheng, H. Yang, Analytic trapezoidal Fourier p-element for vibrating plane problems, *Journal of Sound and Vibration* 271 (1 + 2) (2004) 67–81.
[6] A. Houmat, Hierarchical finite element analysis of the vibration of membranes, *Journal of Sound and Vibration* 201 (4) (1997) 465–472.
[7] R.D. Cook, D.S. Malkus, M.E. Plesha, *Concepts and Application of Finite Element Analysis*, 4th Edition, Wiley, New York, 1989.
[8] A.Y.T. Leung, B. Zhu, J.J. Zheng, H. Yang, A trapezoidal Fourier p-element for membrane vibrations, *Thin-walled Structures* 41 (5) (2003) 479–491.
[9] S. Kitipornchai, K. M. Liew, Y. Xiang, C.M. Wang, Free flexural vibration of triangular Mindlin plates, *International Journal of Mechanical Sciences* 35 (2) (1993) 89–102.