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On the free vibrations of a rectangular plate with two opposite sides simply supported and the other sides attached to linear springs

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Abstract

In this paper an initial-boundary value problem for a plate equation will be studied. This initial-boundary value problem can be regarded as a rather simple model describing free oscillations of a suspension bridge. The suspension bridge is modelled as a rectangular plate with two opposite sides simply supported and the other sides attached to linear springs. An adapted version of the method of separation of variables is used to find the eigenfrequencies for this plate configuration.

1. Introduction

Plates of various geometries, i.e., circular, annular, rectangular, polygonal, etc., and of orthotropic material are extensively used in engineering applications. These plates are widely used in modern aerospace technology, naval structural engineering, aircraft structures, and so on. A lot of literature exists for the free vibrations of rectangular plates. In most of these papers the classical theory for isotropic, homogeneous, thin plates with uniform thickness is used and the differential equation to describe the vibrations of the plate is given by

$$D(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) + \rho \frac{\partial^2 u}{\partial t^2} = 0, \tag{1}$$

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where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, E is Young's modulus, ν is Poisson's ratio with $0<\nu<1$, ρ is the mass density per unit area of the plate surface, h is the thickness of the plate, t is time, and $u(x,y,t)$ is the displacement of the plate in the z-direction. The majority of literature deals with classical boundary conditions representing clamped, simply supported, or free edges, and only a small number deals with edges which are restrained against translation or/and rotation, or with other non-classical boundary conditions. It has been observed for rectangular plates by Leissa in Refs. [1–3] that there exist 21 distinct cases which involve all possible combinations of classical boundary conditions. For six cases having two opposite sides simply supported, it is well known that exact solutions exist which are, in fact, the extensions of Voight's work. In Ref. [2] Leissa gives a survey of research on rectangular plate problems up to 1970. For a further overview up to the beginning of this century the reader is referred to Refs. [4–9].

One of the most commonly used methods in free vibration analysis of plates is the Rayleigh–Ritz energy technique, where appropriate functions associated with various boundary conditions are chosen to describe the lateral deflection of the deformed plates. The chosen functions almost always do not satisfy the governing differential equation. Also, the functions may or may not satisfy all of the boundary conditions. Thus, the results obtained by the Rayleigh–Ritz method are approximate. Gorman in Refs. [4,5] succeeded in solving approximately free vibration problems of plates for various geometries and boundary conditions. Compared to the Rayleigh–Ritz method, the superposition technique in Refs. [4,5] allows one to obtain an analytical form of the solution which satisfies the governing differential equation and the boundary conditions. Sakata and Hosokawa [6] studied the forced and free vibration of clamped orthotropic plates by using a double trigonometric series. During the last 40 years the free vibrations of rectangular plates were studied intensively (see, for instance, Refs. [4–9], and the references in those papers). In this paper the free vibrations of a rectangular plate with two opposite edges simply supported, and linear springs densely attached to the two other edges will be studied. This boundary support will lead to boundary conditions which seem not to be studied in the existing literature. Flexible structures, like tall buildings and suspension bridges, are subjected to oscillations due to windforces or various other causes. Simple models which describe these oscillations are given in the form of weakly non-linear second and fourth order partial differential equations, as can be seen in Refs. [10–16]. Usually asymptotic methods can be used to construct approximations for the solutions of these wave or beam equations. In Ref. [16] a survey of the literature on oscillations in suspension bridges is given. A simple way to model the behavior of a suspension bridge is to describe it as a vibrating one-dimensional beam with simply supported ends. In Ref. [16] the other two dimensions are not taken into account because the dimensions of the bridge in these directions are assumed to be small compared to the length of the bridge. When the width of the bridge is taken into account a plate equation like (1) is obtained. To study, for instance, wind-induced oscillations of suspension bridges one can of course use plate equations to describe the displacements of the deck of the bridge. However, to investigate these weakly non-linear wind-induced vibrations it is necessary first to know the related linear vibrations of the rectangular plate with the boundary conditions as described before and as indicated in Fig. 1. For that reason these linear vibrations will be studied in this paper. Using the results as obtained in this paper one can start to investigate the weakly non-linear vibrations of a plate in a windfield as model for the wind-induced oscillations of a suspension bridge.

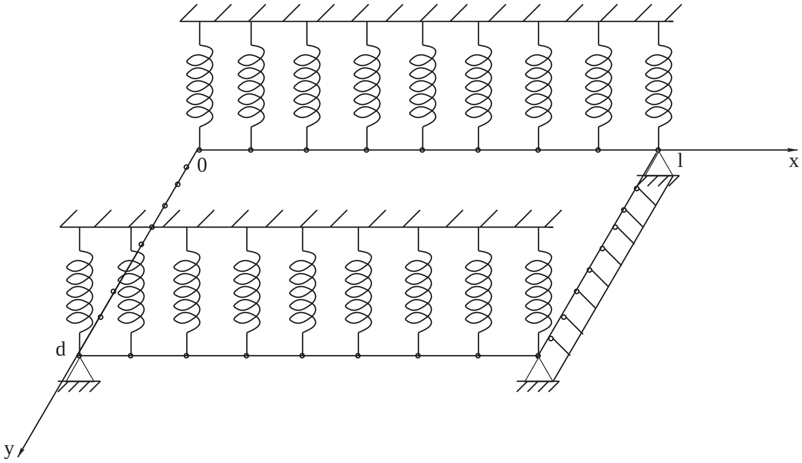

Fig. 1. A model of a suspension bridge.

2. The mathematical analysis of the problem

In this section the following initial-boundary value problem for the displacement function $u(x, y, t)$ will be considered:

$$u_{tt} + D_1(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) = 0, \quad 0<x<l,\ 0<y<d,\ t>0, \tag{2}$$

$$u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y), \quad 0<x<l,\ 0<y<d, \tag{3}$$

$$u(0, y, t) = u(l, y, t) = u_{xx}(0, y, t) = u_{xx}(l, y, t) = 0, \quad 0<y<d, \tag{4}$$

$$D(u_{yyy} + (2 - \nu)u_{xxy}) = -p^2 u \quad \text{for } y = 0,\ 0<x<l, \tag{5}$$

$$D(u_{yyy} + (2 - \nu)u_{xxy}) = p^2 u \quad \text{for } y = d,\ 0<x<l, \tag{6}$$

$$u_{yy} + \nu u_{xx} = 0 \quad \text{for } y = 0,\ y = d,\ 0<x<l, \tag{7}$$

where $D_1 = D/\rho$, and where p^2u in Eqs. (5)–(6) represents the linear restoring force of the springs acting on the boundaries of the plate at $y = 0$ and d (see also Fig. 1). The usually applied plate sign convention for the shearing force acting on an element of the plate explains the difference in sign of the term p^2u in the boundary conditions (5)–(6). It is assumed that there are no distributed bending moments acting along the edges of plate at $y = 0$ and d (see Eq. (7)). It is also assumed that the plate is simply supported at the edges $x = 0$ and l (see Eq. (4)). The initial displacement and the initial velocity of the plate in z-direction are given by $u_0(x, y)$ and $u_1(x, y)$, respectively (see Eq. (3)). The method of separation of variables will be used to find non-trivial solutions of the boundary value problem (2), (4)–(7), that is, non-trivial solutions in the form

$$T(t)v(x, y) \tag{8}$$

will be constructed for the boundary value problem (2), (4)–(7). By substituting Eq. (8) into Eq. (2) and by dividing the so-obtained equation by $T(t)v(x,y)$, it follows that

$$\frac{T''}{T} + D_1 \frac{v_{xxxx} + 2v_{xxyy} + v_{yyyy}}{v} = 0.$$

From this equation the following two equations are obtained:

$$\frac{T''}{T} = -\alpha D_1, \tag{9}$$

$$v_{xxxx} + 2v_{xxyy} + v_{yyyy} = \alpha v, \tag{10}$$

where $\alpha \in \mathbb{C}$ is a separation parameter. From the boundary conditions (4)–(7), it follows that v has to satisfy

$$v(0,y) = v(l,y) = v_{xx}(0,y) = v_{xx}(l,y) = 0, \quad 0<y<d, \tag{11}$$

$$D(v_{yyy} + (2-\nu)v_{xxy}) = -p^2 v \quad \text{for } y = 0,\ 0<x<l, \tag{12}$$

$$D(v_{yyy} + (2-\nu)v_{xxy}) = p^2 v \quad \text{for } y = d,\ 0<x<l, \tag{13}$$

$$v_{yy} + \nu v_{xx} = 0 \quad \text{for } y = 0 \text{ and } y = d,\ 0<x<l. \tag{14}$$

First, it will be shown that the non-trivial solutions of the boundary value problem (10)–(14), that is, the eigenfunctions of Eqs. (10)–(14) are mutually orthogonal on $0<x<l$ and $0<y<d$. Let $v_1(x,y)$ and $v_2(x,y)$ be two different eigenfunctions belonging to the different eigenvalues α_1 and α_2, respectively. Thus

$$\begin{aligned} v_{1xxxx} + 2v_{1xxyy} + v_{1yyyy} &= \alpha_1 v_1, \\ v_{2xxxx} + 2v_{2xxyy} + v_{2yyyy} &= \alpha_2 v_2, \end{aligned} \tag{15}$$

where both functions v_1 and v_2 satisfy the boundary conditions (11)–(14). It will be shown that

$$(\alpha_2 - \alpha_1) \int_0^d \int_0^l v_1 v_2 \, \mathrm{d}x \, \mathrm{d}y = 0. \tag{16}$$

Let the differential operator A be given by

$$A = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \tag{17}$$

and consider

$$\int_0^d \int_0^l (v_1 A v_2 - v_2 A v_1) \, \mathrm{d}x \, \mathrm{d}y. \tag{18}$$

By using Eq. (15) it follows that

$$\int_0^d \int_0^l (v_1 A v_2 - v_2 A v_1) \, \mathrm{d}x \, \mathrm{d}y = (\alpha_2 - \alpha_1) \int_0^d \int_0^l v_1 v_2 \, \mathrm{d}x \, \mathrm{d}y. \tag{19}$$

On the other hand, by two times integrating by parts it follows from Eq. (18) that

$$\begin{aligned}\int_0^d\int_0^l (v_1Av_2 - v_2Av_1)\,\mathrm{d}x\,\mathrm{d}y = &\int_0^d\int_0^l (v_{1xx}v_{2xx} + 2v_{1xy}v_{2xy} + v_{1yy}v_{2yy})\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_0^l (2v_1v_{2xxy} + v_1v_{2yyy} - v_{1y}v_{2yy})|_{y=0}^{d}\,\mathrm{d}x \\ &- \int_0^d\int_0^l (v_{1xx}v_{2xx} + 2v_{1xy}v_{2xy} + v_{1yy}v_{2yy})\,\mathrm{d}x\,\mathrm{d}y \\ &- \int_0^l (2v_2v_{1xxy} + v_2v_{1yyy} - v_{2y}v_{1yy})|_{y=0}^{d}\,\mathrm{d}x. \end{aligned} \tag{20}$$

Using the boundary conditions (14) it follows that the integral

$$\int_0^l (v_{2y}v_{1yy} - v_{1y}v_{2yy})|_{y=0}^{d}\,\mathrm{d}x = -\int_0^l \nu(v_{2y}v_{1xx} - v_{1y}v_{2xx})|_{y=0}^{d}\,\mathrm{d}x.$$

Integrating the integral two times by parts, and by using the boundary conditions (11)–(14) it follows that

$$\int_0^l (v_{2y}v_{1yy} - v_{1y}v_{2yy})|_{y=0}^{d}\,\mathrm{d}x = \int_0^l \nu(v_2v_{1xxy} - v_1v_{2xxy})|_{y=0}^{d}\,\mathrm{d}x.$$

After substituting the last expression and the boundary conditions (12), and (13) into Eq. (20), it finally follows that

$$\int_0^d\int_0^l (v_1Av_2 - v_2Av_1)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{21}$$

From Eqs. (19) and (21) it follows that Eq. (16) is true, and therefore v_1 and v_2 are orthogonal for $\alpha_1 \neq \alpha_2$.

Now it will be shown that the eigenvalue α is real. Let $v(x,y)$ be an eigenfunction belonging to the eigenvalue α, so $Av = \alpha v$. Consider $\overline{Av} = \overline{\alpha v}$. Then, replacing in Eqs. (19) and (21) the functions v_1 and v_2 by v and $\bar{v}$, respectively, and using the fact that $Av = \alpha v$ and $A\bar{v} = \bar{\alpha}\bar{v}$ in Eqs. (19) and (21), it similarly follows that

$$(\alpha - \bar{\alpha})\int_0^d\int_0^l v\bar{v}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{22}$$

Since v and $\bar{v}$ are eigenfunctions it follows that $\int_0^d\int_0^l v\bar{v}\,\mathrm{d}x\,\mathrm{d}y > 0$, and so it follows from Eq. (22) that $\alpha - \bar{\alpha} = 0$. It also can be shown elementarily that $\alpha > 0$ by considering $\int_0^d\int_0^l vAv\,\mathrm{d}x\,\mathrm{d}y$, where $v(x,y)$ is an eigenfunction belonging to the eigenvalue α, that is, $Av = \alpha v$. Firstly, it should be observed that

$$\int_0^d\int_0^l vAv\,\mathrm{d}x\,\mathrm{d}y = \alpha\int_0^d\int_0^l v^2(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{23}$$

and secondly, it follows (by integrating by parts, and by using the boundary conditions (11)–(14)) that

$$\begin{aligned}\int_0^d \int_0^l vAv\,\mathrm{d}x\,\mathrm{d}y &= \int_0^d \int_0^l v(v_{xxxx} + 2v_{xxyy} + v_{yyyy})\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^d \int_0^l (v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \bar{p}^2 \int_0^l (v^2(x,d) + v^2(x,0))\,\mathrm{d}x, \end{aligned} \tag{24}$$

where $\bar{p}^2 = p^2/D$. From Eqs. (23) and (24) it can readily be deduced that $\alpha > 0$.

To investigate further the boundary value problem (10)–(14) for $v(x,y)$ the method of separation of variables will be used again, that is, it is assumed that a non-trivial solution of the boundary-value problem (10)–(14) can be found in the form

$$X(x)Y(y). \tag{25}$$

By substituting Eq. (25) into Eq. (10), it follows that

$$\frac{\ddddot{X}}{X} + 2\frac{\ddot{X}}{X}\frac{Y''}{Y} + \frac{Y''''}{Y} = \alpha, \tag{26}$$

where $' = \partial(...)/\partial y$ and $^{\cdot} = \partial(...)/\partial x$. Generally, it is assumed that the variables in Eq. (26) cannot be separated because of the mixed term $2\ddot{X}/X\ Y''/Y$. However, using an adapted version of the method of separation of variables (see Refs. [17,18]), this equation can easily be separated by simply differentiating Eq. (26) with respect to x or y. For instance, if Eq. (26) is differentiated with respect to x, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ddddot{X}}{X}\right) + 2\frac{Y''}{Y}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ddot{X}}{X}\right) = 0,$$

and so,

$$\frac{Y''}{Y} = -\gamma, \tag{27}$$

where $\gamma \in \mathbb{C}$ is a separation parameter. From Eq. (27) it follows that $Y'''' = -\gamma Y'' = \gamma^2 Y$, and then it can be deduced from Eq. (26) that $X(x)$ and $Y(y)$ have to satisfy

$$\ddddot{X} - 2\,\gamma\ddot{X} + (\gamma^2 - \alpha)X = 0, \quad 0 < x < l, \tag{28}$$

$$Y'' = -\gamma Y, \quad 0 < y < d. \tag{29}$$

By substituting Eq. (25) into the boundary conditions (11)–(14) the usual boundary value problem for $X(x)$ and for $Y(y)$ is obtained. It turns out, however, that these boundary value problems only admit the trivial solution. The elementary calculations to obtain this result will be omitted.

So, differentiation with respect to x leads only to the trivial solution. However, if Eq. (26) is differentiated with respect to y it will turn out that non-trivial solutions can be found. When

Eq. (26) is differentiated with respect to y, it follows that

$$2\frac{\ddot{X}}{X}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{Y''}{Y}\right)+\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{Y''''}{Y}\right)=0,$$

which can be easily separated, yielding

$$\frac{\ddot{X}}{X}=-\beta, \tag{30}$$

where $\beta\in\mathbb{C}$ is a separation parameter. From Eq. (30) it follows that $\ddot{\ddot{X}}=-\beta\ddot{X}=\beta^2 X$, and then it can be deduced from Eq. (26) and the boundary conditions (12)–(14) that $Y(y)$ has to satisfy

$$Y''''-2\beta Y''+(\beta^2-\alpha)Y=0 \tag{31}$$

subject to the boundary conditions

$$D(Y'''-(2-\nu)\beta Y')=-p^2 Y \quad \text{for } y=0, \tag{32}$$

$$D(Y'''-(2-\nu)\beta Y')=p^2 Y \quad \text{for } y=d, \tag{33}$$

$$Y''-\beta\nu Y=0 \quad \text{for } y=0,d. \tag{34}$$

It follows from Eq. (11) that $X(x)$ also has to satisfy

$$X(0)=X(l)=\ddot{X}(0)=\ddot{X}(l)=0. \tag{35}$$

The non-trivial solutions of the differential equation (30) subject to the boundary conditions (35) are given by

$$X(x)=\gamma_n\sin\left(\sqrt{\beta_n}x\right),\quad \beta_n=\left(\frac{n\pi}{l}\right)^2 \tag{36}$$

with $n\in\mathbb{Z}^+$, and where γ_n is an arbitrary constant. The characteristic equation for ODE (31) now becomes

$$k^4-2\beta_n k^2+\beta_n^2-\alpha=0 \;\Leftrightarrow\; (k^2-\beta_n)^2=\alpha. \tag{37}$$

In this section it has already been shown that $\alpha>0$. So, only the following three cases have to be considered in Eq. (37)

$$\alpha>\beta_n^2,\quad 0<\alpha<\beta_n^2 \quad\text{and}\quad \alpha=\beta_n^2.$$

2.1. The case $\alpha>\beta_n^2$

The solutions of the characteristic equation (37) in this case will be

$$\sqrt{\sqrt{\alpha}+\beta_n},\quad -\sqrt{\sqrt{\alpha}+\beta_n},\quad \mathrm{i}\sqrt{\sqrt{\alpha}-\beta_n}\quad\text{and}\quad -\mathrm{i}\sqrt{\sqrt{\alpha}-\beta_n}$$

and the solution of the differential equation (31) can be written in the form

$$
\begin{aligned}
Y(y) = {} & C_1 \cosh\left(\sqrt{\sqrt{\alpha}+\beta_n}y\right) + C_2 \sinh\left(\sqrt{\sqrt{\alpha}+\beta_n}y\right) \\
& + C_3 \cos\left(\sqrt{\sqrt{\alpha}-\beta_n}y\right) + C_4 \sin\left(\sqrt{\sqrt{\alpha}-\beta_n}y\right),
\end{aligned}
\tag{38}
$$

where C_1, C_2, C_3, and C_4 are constants of integration.

By substituting Eq. (38) into the four boundary conditions (32)–(34) a system of four equations for C_1, C_2, C_3, and C_4 is obtained. To find non-trivial solutions for $Y(y)$ the determinant of the corresponding coefficient matrix should be set equal to zero, that is,

$$
\begin{vmatrix}
a^2 - v\beta_n & 0 & -(r^2 + v\beta_n) & 0 \\
(a^2 - v\beta_n)\cosh(ad) & (a^2 - v\beta_n)\sinh(ad) & -(r^2 + v\beta_n)\cos(rd) & -(r^2 + v\beta_n)\sin(rd) \\
\bar{p}^2 & f & \bar{p}^2 & -g \\
f\sinh(ad) - \bar{p}^2\cosh(ad) & f\cosh(ad) - \bar{p}^2\sinh(ad) & g\sin(rd) - \bar{p}^2\cos(rd) & -g\cos(rd) - \bar{p}^2\sin(rd)
\end{vmatrix} = 0,
\tag{39}
$$

where $a = \sqrt{\sqrt{\alpha}+\beta_n}$, $r = \sqrt{\sqrt{\alpha}-\beta_n}$, $f = a(a^2 - (2-v)\beta_n)$, and $g = r(r^2 + (2-v)\beta_n)$. From Eq. (39) the eigenvalues α can be calculated. The eigenvalues α depend on the parameters n, $\bar{p}^2$, v, the length l, and the width d of the rectangular plate. When the parameter $\bar{p}^2$ tends to zero, the boundary conditions correspond to the case for the plate with two opposite edges simply supported and the other two free. When the parameter $\bar{p}^2$ tends to infinity the boundary conditions correspond to the case for a plate with all edges simply supported.

One calculates numerically from Eq. (39) some eigenvalues α for some values of the parameters. Some of the numerical approximations for α up to 50 000 are given in Tables 1 and 2.

2.2. The case $\alpha < \beta_n^2$

In this case the solutions of the characteristic equation (37) will be

$$
\sqrt{\sqrt{\alpha}+\beta_n}, \quad -\sqrt{\sqrt{\alpha}+\beta_n}, \quad \sqrt{\beta_n - \sqrt{\alpha}}, \quad -\sqrt{\beta_n - \sqrt{\alpha}}
$$

and the solution of the differential equation (31) can be written in the form

$$
\begin{aligned}
Y(y) = {} & G_1 \cosh\left(\sqrt{\sqrt{\alpha}+\beta_n}y\right) + G_2 \sinh\left(\sqrt{\sqrt{\alpha}+\beta_n}y\right) \\
& + G_3 \cosh\left(\sqrt{\beta_n - \sqrt{\alpha}}y\right) + G_4 \sinh\left(\sqrt{\beta_n - \sqrt{\alpha}}y\right),
\end{aligned}
\tag{40}
$$

where G_1, G_2, G_3, and G_4 are constants of integration.

By substituting Eq. (38) into the four boundary conditions (32)–(34) a system of four equations for G_1, G_2, G_3, and G_4 is obtained. To find non-trivial solutions for $Y(y)$ the determinant of the

Table 1
Approximations of the eigenvalues α

n						
$v = 0.3,\ l = 10,\ d = 1,\ \bar{p}^2 = 1$						
1	1.9861	7.6513	516.1518	3829.3326	14657.3213	40001.3080
2	2.1484	12.7612	538.9518	3882.7911	14752.4921	40149.9413
3	2.7739	21.6547	577.3297	3972.2310	14911.4787	40398.0399
4	4.4052	34.8982	631.8441	4098.1669	15134.8325	40746.1682
5	7.8156	53.2853	703.2675	4261.3209	15423.3239	41195.1144
6	14.0147	77.8385	792.5844	4462.6248	15777.9403	41745.8900
7	24.2497	109.8092	900.9942	4703.2228	16199.8847	42399.7278
8	40.0047	150.6790	1029.9158	4984.4739	16690.5743	43158.0810
9		202.1601	1180.9932	5307.9560	17251.6399	44022.6214
10		266.1959	1356.1002	5675.4693	17884.9256	44995.2379
$v = 0.43, l = 10, d = 1, \bar{p}^2 = 1$						
1	1.9893	7.3471	515.1976	3827.7181	14655.0633	39998.4052
2	2.1504	11.5408	535.1470	3876.7395	14743.4633	40138.3321
3	2.7397	18.8953	568.8103	3957.7389	14891.1757	40371.9266
4	4.2548	29.9595	616.7939	4072.4616	15098.7686	40699.7629
5	7.4164	45.4983	679.9250	4221.2733	15367.0360	41122.6440
$v = 0.3, l = 10, d = 1, \bar{p}^2 = 1$						
6	13.1786	66.5027	759.2433	4405.1593	15696.9969	41641.6012
7		94.1809	855.9993	4625.3260	16089.8944	42257.8931
8		129.9639	971.6561	4883.2029	16547.1949	42973.0051
9		175.5054	1067.8944	5180.4432	17070.5871	43788.6480
10		232.6826	1266.6159	5518.9257	17661.9817	44706.7578
$v = 0.5, l = 10, d = 1, \bar{p}^2 = 1$						
1	1.9909	7.1830	514.6837	3826.8486	14653.8474	39999.8422
2	2.1494	10.8793	533.0951	3872.8643	14738.6010	40132.0807
3	2.7106	17.3876	564.2076	3949.9290	14880.2403	40357.8637
4	4.1396	27.2307	608.6426	4058.6003	15079.3394	40674.7693
5	7.1170	41.1420	667.2432	4199.6589	15336.7011	41083.6059
6		60.0667	741.0620	4374.1082	15653.3555	41585.4120
7		85.1638	831.3573	4583.1744	16030.5590	42181.4549
8		117.8072	939.5938	4828.3068	16649.7934	42873.2305
9		159.5872	1067.4469	5111.1785	16972.7648	43662.4623
10		212.3121	1216.8074	5433.6866	17541.4029	44551.3210
$v = 0.4, l = 10, d = 1, \bar{p}^2 = 10$						
1	17.2169	60.0325	589.4929	3901.0656	14728.0124	40071.2956
2	17.7689	64.4837	609.6992	3950.7358	14817.9624	40213.2310
3	18.9723	72.2716	643.8309	4033.8802	14968.2555	40450.1715
4	21.2800	83.9501	692.5434	4151.0405	15179.4562	40782.6888
5	25.3670	100.2956	756.7155	4302.9762	15452.3532	41211.5827
6	32.1501	122.3071	837.4325	4490.6657	15787.9589	41737.8801
7	42.7963	151.2070	935.9796	4715.3078	16187.5082	42362.8341
8	58.7241	188.4419	1053.8436	4978.3231	16652.4577	43087.9233

Table 1 (*continued*)

n						
9	81.5997	235.6833	1192.7194	5281.3557	17184.4577	43914.8507
10	113.3323	294.8280	1354.5169	5626.2753	17785.4892	44845.5433
$v = 0.46, l = 10, d = 1, \bar{p}^2 = 100$						
1	69.6196	475.0370	1368.5399	4705.0375	15489.0254	40811.9114
2	73.0771	480.9335	1385.0898	4750.7959	15575.3462	40949.6487
3	79.0445	491.0580	1413.1931	4827.4891	15719.6021	41179.5956
4	87.8447	505.8623	1453.6163	4935.7557	15922.3743	41502.3286
5	99.9592	525.9896	1507.4054	5076.4798	16184.4744	41918.6541
6	116.0529	552.2844	1575.8641	5250.7839	16506.9433	42429.6078
7	137.0004	585.8012	1660.5304	5460.0203	16891.0493	43036.4538
8	163.9098	627.8138	1763.1549	5705.7649	17338.2866	43740.6841
9	198.1431	679.8229	1885.6846	5989.8110	17850.3741	44544.0172
10	241.3291	743.5613	2030.2517	6314.1662	18429.2540	45448.3979

Table 2
Approximations of the eigenvalues α for $v = 0.3,\ l = 10$

n	$d = 0.1, \bar{p}^2 = 1$		$d = 0.1, \bar{p}^2 = 10$	
1	20.0095	225.8181	199.9985	765.8022
2	20.1454	723.3859	200.1480	1263.3647
3	20.7266	1553.0419	200.7735	2093.0118
4	22.2851	2715.3508	202.3940	3255.3083
5	25.5660	4211.1029	205.7544	4751.0444
6	31.5277	6041.3146	211.8131	6581.2367
7	41.3424	8207.2282	221.7421	8747.1272
8	56.3963	10710.3116	236.9272	11250.1842
9	78.2898	13552.2590	258.9689	14092.1016

corresponding coefficient matrix should be set equal to zero, that is,

$$\begin{vmatrix} a^2 - v\beta_n & 0 & (c^2 - v\beta_n) & 0 \\ (a^2 - v\beta_n)\cosh(ad) & (a^2 - v\beta_n)\sinh(ad) & (c^2 - v\beta_n)\cosh(cd) & (c^2 - v\beta_n)\sinh(cd) \\ \bar{p}^2 & f & \bar{p}^2 & h \\ f\sinh(ad) - \bar{p}^2\cosh(ad) & f\cosh(ad) - \bar{p}^2\sinh(ad) & h\sinh(cd) - \bar{p}^2\cosh(cd) & h\cosh(cd) - \bar{p}^2\sinh(cd) \end{vmatrix} = 0, \tag{41}$$

where $a = \sqrt{\sqrt{\alpha} + \beta_n}$, $c = \beta_n - \sqrt{\sqrt{\alpha}}$, $f = a(a^2 - (2 - v)\beta_n)$, and $h = c(c^2 - (2 - v)\beta_n)$. The eigenvalues α can be calculated from Eq. (41). Some numerical approximations of α are given in Table 3 for some values of the parameters.

Table 3
Approximations of the eigenvalues α

n	$l=10,\ d=1,\ \bar{p}^2=1$					
	$v=0.3$	$v=0.4$	$v=0.5$	$v=0.6$	$v=0.8$	
5					5.0474	
6			12.5524	11.4163	8.1418	
7		23.1493	21.6909	19.4993	13.3306	
8		38.1964	35.6143	32.1065	21.4984	
9	62.9993	60.2307	56.2396	50.7482	33.7101	
10	95.1879	91.1683	85.3165	77.1611	51.2067	
11	138.7581	133.1580	124.9248	113.3053	75.3987	136.9501
12	196.1310	188.5811	177.3735	161.3620	107.8584	178.9353
13	269.9607	260.0502	245.1999	223.7315	150.3134	231.0134
14	363.1342	350.4097	331.1689	303.0322	204.6407	294.8865
15	478.7715	462.7357	438.2735	402.1001	272.8618	372.4121
16	620.2261	600.3362	569.7342	523.9882	357.1389	465.6039
17	791.0843	766.7508	729.9995	671.9661	459.7719	576.6301
18	995.1661	965.7514	919.7457	849.5199	583.1959	707.8135
19	1236.5247	1201.3419	1145.8771	1060.3514	729.9795	861.6300

2.3. The case $\alpha = \beta_n^2$

In this case the characteristic equation (37) becomes

$$k^2(k^2 - 2\beta_n) = 0 \tag{42}$$

and its solutions are

$$k_{1,2} = 0, \quad k_{3,4} = \pm\sqrt{2\beta_n}.$$

The solution of the differential equation (31) is then given by

$$Y(y) = S_1 + S_2 y + S_3 \cosh\left(\sqrt{2\beta_n}y\right) + S_4 \sinh\left(\sqrt{2\beta_n}y\right), \tag{43}$$

where S_1, S_2, S_3, and S_4 are constants of integration. As in the previous two cases the following determinant is similarly obtained when looking for non-trivial solutions of the boundary value problem for $Y(y)$ (where $Y(y)$ is given by Eq. (43)). As in the two previous cases using boundary conditions (32)–(34) the system of the four equations for the determination of eigenvalues is received. This system has a non-trivial solution when the determinant of the coefficient matrix for the unknown quantities $S_i = 0,\ i = 1,2,3,4$ is equal to zero. In this case the determinant has the following form:

$$\begin{vmatrix} -v\beta_n & 0 & (2-v)\beta_n & 0 \\ -v\beta_n & -v\beta_n d & (2-v)\beta_n\cosh(b_1 d) & (2-v)\beta_n\sinh(b_1 d) \\ \bar{p}^2 & -(2-v)\beta_n & \bar{p}^2 & v\beta_n b_1 \\ -\bar{p}^2 & -(\bar{p}^2 d+(2-v)\beta_n) & v\beta_n b_1\sinh(b_1 d)-\bar{p}^2\cosh(b_1 d) & v\beta_n b_1\cosh(b_1 d)-\bar{p}^2\sinh(b_1 d) \end{vmatrix} = 0, \tag{44}$$

where $b_1 = \sqrt{2\beta_n}$. Solutions exist for some special values of the parameters. For example for $l = 100$, $d = 0.1$, $\bar{p}^2 = 1$, $v = 0.6$ solutions for α exist for the first five modes. For $l = 100$, $d = 1$, $\bar{p}^2 = 1$, $v = 0.6$ solutions exist for the first three modes and these solutions for α will be exactly the same as for $d = 0.1$. This is due to the fact that $\alpha = \beta_n^2 = (n\pi/l)^4$ and that α depends only on n and l. The other parameters such as v and $\bar{p}^2$ will only determine the existence of non-trivial solutions $Y(y)$.

3. Conclusions and remarks

In this paper the free vibrations of a rectangular plate with two opposite sides simply supported and the other two densely attached to linear springs have been studied. This combination of boundary conditions seems not to be considered in the literature before. This rectangular plate model is one of the simplest models to describe a suspension bridge. For the rectangular plate model the relationship between the plate parameters and the frequencies has been obtained by using an adapted version of the method of separation of variables (see Ref. [18]). This result is important to investigate the wind-induced oscillations of a rectangular plate. The relationship between the plate parameters and the frequencies has been obtained analytically. For some values of the parameters numerical approximations of the frequencies are given.

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