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Letter to the Editor

Some comments on the calculation of the local flexibility of cracked shafts

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1. Introduction

It is well known since 1983 [1] that the local flexibility of a cracked shaft can be represented and calculated by a double integral of the strain energy density function over the crack surface. In the Letter to the Editor by Zou et al. [2], a modification of the method presented by Dimarogonas and Papadopoulos [1] is proposed. This modification refers to one of the upper boundaries of the double integral of the calculation of the local flexibility of the shaft due to the presence of the crack. In the present letter it is stated that this point has been taken into account since 1983. The algorithm calculating the local flexibility is also presented. In this case there is a discrepancy comparing with the results of Zou et al. [2]. The calculation of the compliance for crack depths greater than the radius is also presented and discussed.

2. Analysis

Define α as the crack depth of the circular cross-section of diameter $D = 2R$. The crack can be bounded in the x direction by $-b$ and $+b$, and in the y direction by 0 and α_x.

The boundary b can be calculated using Pythagorean Theorem from Fig. 1,

$$b = \sqrt{R^2 - (R - a)^2} \quad (1)$$

and in dimensionless form, defining $\bar{b} = b/R$ and $\bar{a} = a/R$

$$\bar{b} = \sqrt{1 - (1 - \bar{a})^2}. \quad (2)$$

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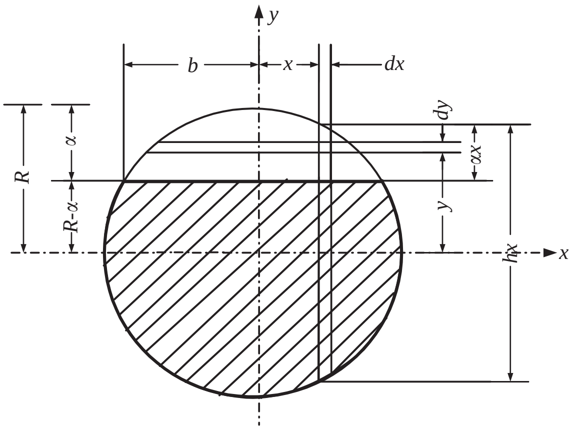

Fig. 1. The geometry of the cracked cross-section.

The height h_x, of the strip is calculated as

$$h_x = 2\sqrt{R^2 - x^2} \tag{3}$$

or in dimensionless form, defining $\bar{x} = x/R$ and $\bar{h}_x = h_x/R$, as

$$\bar{h}_x = 2\sqrt{1 - \bar{x}^2}. \tag{4}$$

The upper boundary of the crack at each x, can be calculated from

$$a_x = \frac{h_x}{2} - (R - a) = \sqrt{R^2 - x^2} - (R - a) \tag{5}$$

or in a form without dimensions, defining $\bar{a}_x = a_x/R$, $\bar{a} = a/R$ and using (4),

$$\bar{a}_x = \frac{\bar{h}_x}{2R} - \left(1 - \frac{a}{R}\right) = \sqrt{1 - \bar{x}^2} - (1 - \bar{a}). \tag{6}$$

The function F_2 can be taken from the next expression,

$$F_2\left(\frac{y}{h_x}\right) = \sqrt{\frac{\tan(\pi y/2h_x)}{\pi y/2h_x}} \frac{[0.923 + 0.199(1 - \sin(\pi y/(2h_x)))^4]}{\cos \pi y/(2h_x)}. \tag{7}$$

Then the double integral is taken as [1]

$$\frac{R^3 E}{(1 - \nu^2)} c_{55} = \int_{-b/R}^{b/R} \mathrm{d}\left(\frac{x}{R}\right) \int_0^{\alpha_x/R} \frac{32}{\pi}\left[1 - \left(\frac{x}{R}\right)^2\right] \frac{y}{R} F_2^2\left(\frac{y/R}{h_x/R}\right) \mathrm{d}\left(\frac{y}{R}\right) \tag{8}$$

and in non-dimensional form, if $\bar{c}_{55} = [R^3 E/(1 - \nu^2)] c_{55}$

$$\bar{c}_{55} = \int_{-\bar{b}}^{\bar{b}} \mathrm{d}\bar{x} \int_0^{\bar{\alpha}_x} \frac{32}{\pi}[1 - \bar{x}^2]\, \bar{y} F_2^2\left(\frac{\bar{y}}{\bar{h}_x}\right) \mathrm{d}\bar{y}. \tag{9}$$

```
fig = {};
Do[hx = 2 Sqrt[1 - x^2];
  b = Sqrt[1 - (1 - α)^2];
  ax = Sqrt[1 - x^2] - (1 - α);
  F2 = Sqrt[(Tan[π*y/(2 hx)])/(π*y/(2 hx)) * (0.923 + 0.199*(1 - Sin[π*y/(2 hx)])^4)/Cos[π*y/(2 hx)]];   (*Valid for every y/hx*)
  f = 32/π * (1 - x^2) * y * F2^2;
  in = NIntegrate[f, {x, -b, b}, {y, 0, ax}];
  fig = Append[fig, {in, α}], {α, 0.04, 1.00, 0.04}];
fig
<< Graphics`Graphics`
LogLinearListPlot[fig, PlotStyle → PointSize[0.018], GridLines → Automatic,
  Frame → True];
```

Fig. 2. The algorithm in Mathematica used for the double integral calculation.

Table 1
The dimensionless compliance $\bar{c}_{55}$ as a function of the relative crack depth a/R

α/R	$\bar{c}_{55}$	α/R	$\bar{c}_{55}$
0.04	0.002973	0.52	1.340520
0.08	0.016190	0.56	1.600490
0.12	0.043041	0.60	1.892200
0.16	0.085426	0.64	2.219100
0.20	0.144614	0.68	2.585270
0.24	0.221576	0.72	2.995620
0.28	0.317167	0.76	3.456030
0.32	0.432249	0.80	3.973550
0.36	0.567780	0.84	4.556690
0.40	0.724880	0.88	5.215780
0.44	0.904893	0.92	5.963420
0.48	1.109450	0.96	6.815160
		1.00	7.790390

The calculation of the above double integral (9) using the following procedure in Mathematica is presented in Fig. 2. All parameters of the algorithm are in dimensionless form. That means that $h_x, x, b, \alpha, a_x, y$ of the algorithm are equivalent to $\bar{h}_x$, $\bar{x}$, $\bar{b}$, $\bar{a}$, $\bar{\alpha}_x$, and $\bar{y}$ of relation (9).

Using the above algorithm the dimensionless compliance $\bar{c}_{55}$ of Eq. (9) has been computed and presented in Table 1.

In Fig. 3 the diagram of $\bar{c}_{55}$ is plotted, where $E' = E/(1 - v^2)$. In the same picture the results in Ref. [2] are also presented for comparison purposes.

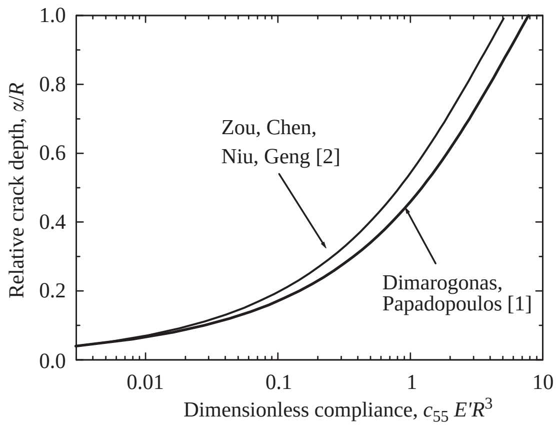

Fig. 3. The dimensionless compliance $\bar{c}_{55}$ as a function of the crack depth a/R [1,2].

3. The modification made by Zou et al.

Zou et al. in their Letter to the Editor [2] state that the boundary of integration a_x taken in Ref. [1] is a and not $a_x = \sqrt{R^2 - x^2} - (R - a)$ see Fig. 1.

This is not true because it is very clearly stated in the referred paper [1] since 1983 at page 585 that:

"*The solution for a strip with width* dξ *and depth* $\eta = \alpha + \sqrt{R^2 - \xi^2} - R$ *will be used*", where ξ corresponds to our x and η corresponds to a_x.

The above expression for the boundary a_x has been also used in other papers [3–5], as well as in Ref. [6]. For example in Ref. [5, p. 2] one can read that "*The values of SIF are available for an elementary rectangular strip* (*Fig.* 2) *with width* dξ, *height* $h = 2(R^2 - \xi^2)^{1/2}$ *and crack depth* $\eta = h^2 - (R - \alpha)$". The last relation should be written as $\eta = h/2 - (R - \alpha)$.

As it could be observed, the value of the dimensionless $\bar{c}_{55}$ for $a/R = 1$, is 7.790390 while in Ref. [2] is 4.996. There is a discrepancy of $(7.790390 - 4.996)/7.790390 \times 100 = 35.9\%$. The results presented by Zou et al. [2] are shown in the same Fig. 3.

At this point it should be noted that in Ref. [1] the value of $\bar{c}_{55}$ = about 10 that corresponds to $a/R = 1$ is due to a programming inadvertence. The power of 4 in the formula of F_2 was put in Ref. [1] before parentheses (see Fig. 2). This should be revised.

4. Singularity observed for crack depths greater than R

The local compliance at a point of a shaft due to a crack by its definition is zero for the case without crack (if $a = 0$ then $\bar{c}_{55} = 0$) and infinite for the imaginary case where the crack is equal to the diameter of the shaft (if $a = D$ then $\bar{c}_{55} = \infty$).

In the last case there is a difficulty using the double integral, because the functions used for the calculation of the stress intensity factor present a singularity when x is equal to b as it is observed and presented in Ref. [7]. If one tries to calculate the double integral from $-b$ to b then a singularity error occurs for values of crack depth above R. This singularity was referred to by

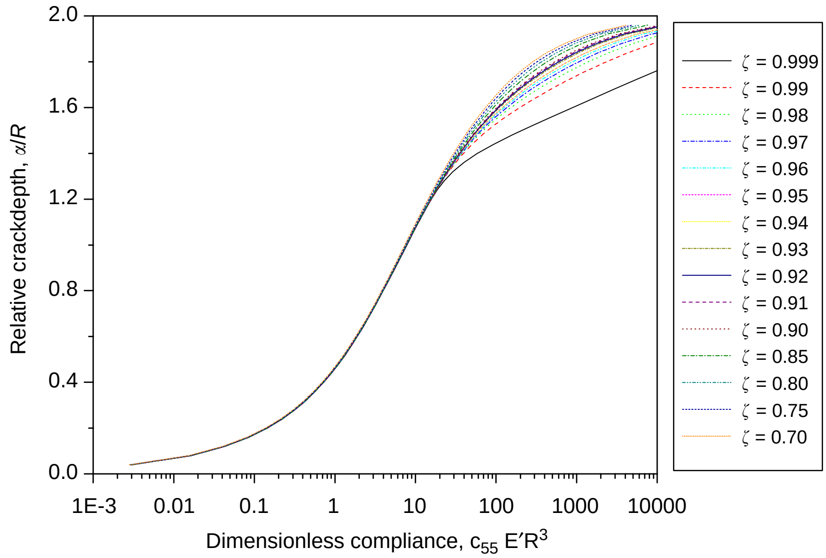

Fig. 4. The compliance for the region $0<a<2R$.

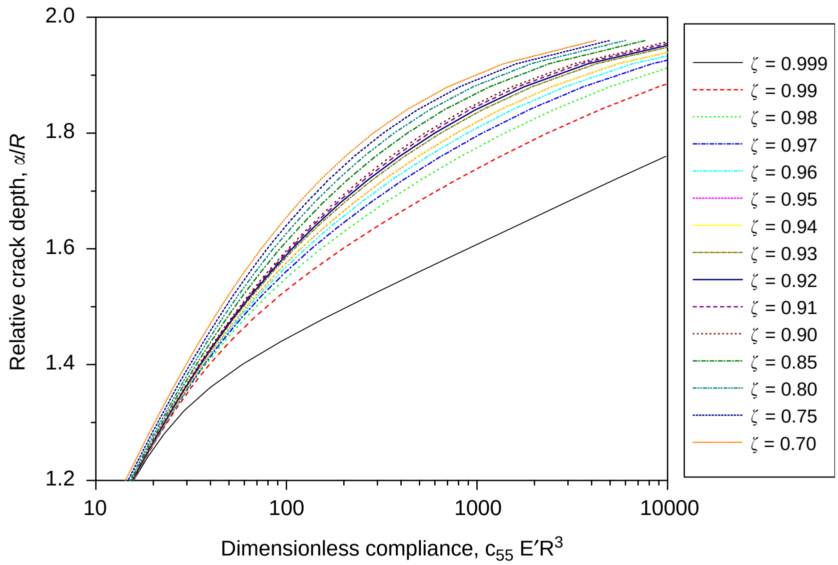

Fig. 5. Zoom of previous Fig. 4.

Abraham and Brandon [7] as well as in Dimarogonas' reply [8] where it was suggested to select a ζ factor to make the integration near but not on the boundaries b and $-b$,

$$\zeta = \frac{Integration\ \ Length}{Crack\ \ Length}, \quad \text{where } 0<\zeta<1. \tag{10}$$

In Fig. 4 the compliance $\bar{c}_{55}$ is computed for different ζ, and a zoom of Fig. 4 is shown in Fig. 5.

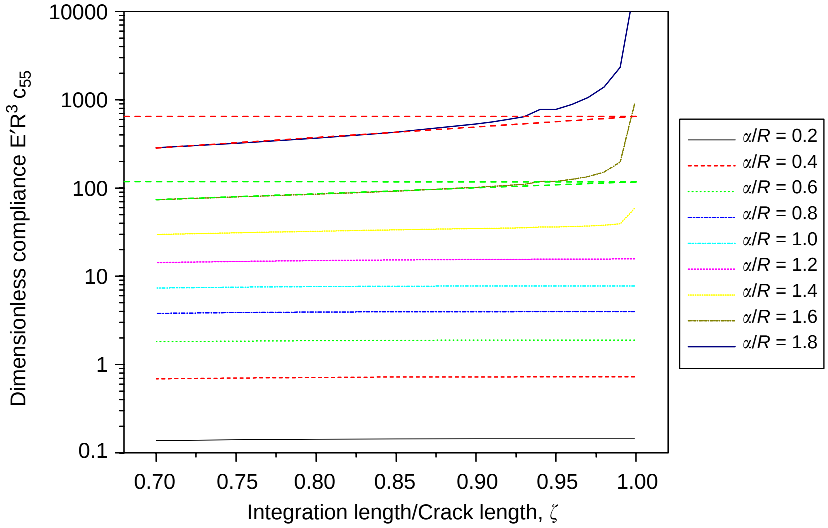

Fig. 6. Calculation of the compliance as function of ζ.

A slope difference of the curves in the region above $\alpha/R = 1.2$ and near 1.4 for values of $\zeta > 0.90$ can be observed. This happens because the compliance for $\alpha > R$ diverges for values near $\zeta = 1$. As observed by Dimarogonas [5, Fig. 2], the value of the ζ near the inflexion point of the compliance due to the increase of ζ, should be used. Such a diagram was computed and presented here (Fig. 6) for reasons of completeness. A value of $0.90 \leqslant \zeta \leqslant 0.95$ is proposed for the calculation of the compliance for crack depths $\alpha > R$. Of course for the curves corresponding to $\zeta < 0.90$, it could be observed that they are almost parallel and there is no sudden slope change. This happens because for such values there is no sudden increase in the compliance due to the singularity.

If a value of $\zeta = 0.93$ is chosen, that means in the algorithm of Fig. 2 the boundaries of integration along the x-axis are $-0.93b$ to $0.93b$, then the calculation gives for $\alpha/R = 1.8$, a value of 643.5 for the compliance. If one then extends in Fig. 6 (dashed red line) the straight part of the upper line in Fig. 6 (corresponding to $\alpha/R = 1.8$), to intersect the compliance axis, the intersection corresponds to about 650 which is near the computed value. The same procedure is repeated for the next value of $\alpha/R = 1.6$. Calculating the compliance for $\zeta = 0.93$ and $\alpha/R = 1.6$ the value of 110.19 is obtained, and extending the straight part of the corresponding line a value of about 110 can be observed in the diagram (green-dashed line).

It is obvious that in this case one can suppose that the part of the sudden change of the slope is due to the singularity and does not reflect the real compliance.

5. Conclusions

- The local flexibility is computed using the well-known result from Ref. [1,3] regarding the double integral relations over the crack surface using the appropriate boundaries. The assertion that the boundaries used for the double integration were not the appropriate ones is arbitrary. For clarity reasons the "Mathematica" code used, to calculate the compliance, is included.

- The local flexibility of a crack of depth equal to the diameter of the shaft is infinite. Thus when the crack depth approaches the value of the diameter, the compliance function approaches infinity asymptotically.
- A singularity appears when the crack depth is greater than the radius. A ζ value in the range $(0.90 \leqslant \zeta \leqslant 0.95)$ should then be used in order to avoid the singularity and estimate a realistic value for the compliance in such extreme cases of very deep cracks.

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