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Selected problems of frequency analysis for time varying, discrete-time systems using singular value decomposition and discrete Fourier transform

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Abstract

The paper develops tools and methods for linear time-varying, discrete-time systems analysis. It consists of two parts. The first one gives a theoretical background. There are definitions and numerical algorithms for approximating frequency characteristics. The main method is based on singular value decomposition, discrete Fourier transform and power density spectrum approach. The second part of the paper contains numerical examples. Six different models have been analyzed. Three for frequency characteristics approximation and the other three for evaluating the degree of system non-stationarity. Some of the models are time-variant and some are time-invariant. For better evaluation, results yielded by the proposed method are compared with classical Bode characteristics.

1. Introduction

Frequency methods present one of the most important tools for linear, time-invariant (LTI) systems analysis, nevertheless well-developed concepts and analytic methods of time-invariant systems are not applicable to linear time-variant systems (LTV). Also the studies of stability and dynamics problems are different due to the finite time horizon. The analysis for a set with infinite length uses essentially a frequency domain description. The tool, which is used to describe a system by its frequency-dependent gain, is quite useful, but use for plants considered on a finite time interval is not known. This work presents tools for time-variant systems using approximation of classic, frequency methods. In the paper, a new approach for time-variant systems is presented using frequency methods.

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One of the first attempts to analyze LTV systems in the frequency domain was made by Zadeh [1,2]. The time-varying transfer function has been defined by extending the Laplace transform to the varying impulse response. However, in general, no closed form of Zadeh's transfer function is known. Furthermore, the use of Zadeh's transfer function in a limited time variation seems problematic since the definition assumes that the variation continues infinitely. The singularity or varying pole of Zadeh's transfer function was used to study the stability of time-varying systems by D'Angelo [3]. Recent works on frequency aspects for LTV systems focus on modal analysis. Ideas of varying eigenvalues or varying natural frequencies have been used without a rigorous definition by Bogoliubov and Mitropolsky [4], Basseville et al. [5], Maia and Silva [6]. The concept of pseudo-modal parameters (PMP) was introduced and described by Liu [7–10]. The PMP are related to the eigenvalues of the varying discrete-time state transition matrices by analogy to time-invariant systems. The PMP offer a description for dynamic properties of LTV systems using a compact set of parameters.

An analysis of singular value decomposition-discrete Fourier transform (SVD-DFT), which is introduced in this work, may be compared to the time-varying transfer function defined by Zadeh [1] or the PMP, described by Liu [7,9,10]. There are a few weaknesses and peculiarities of both methods. The SVD-DFT method gives Bode characteristics (magnitude- and phase-frequency responses), not only natural frequencies; nevertheless Bode characteristics are given by a finite set of frequencies (or singular vectors) and corresponding gains. Physical properties of the system are dependent not only on the poles but also on the zeros and on the gain of the system, which are neglected when using PMP. The products of SVD-DFT analysis are the characteristics (amplitude and phase-frequency responses). The results are that the information included in characteristics cannot be extracted for specific time samples. Extracting the exact nature of the variation is sometimes easier using PMP. Of course choice of the method depends not only on performance but also on habits. An important advantage of the SVD-DFT method is that the characteristics calculated for LTI systems are almost identical with classic Bode diagrams. Moreover, as the result of analysis, a coefficient of variability of the system can be computed. If the set of the system matrices has been taken by system identification [9], the coefficient gives information, whether the system is LTI, LTV or non-linear.

2. Model description

To describe dynamic time varying discrete-time systems, difference equations with time-dependent coefficients or a generalized description employing state equations with time-dependent matrices are used. Input–output relationships in real systems are always featured by a non-zero time delay. In such a case system matrix $\mathbf{D}(k) \equiv \mathbf{0}$ and term $\mathbf{D}(k) \cdot \mathbf{v}_p(k)$ in Eq. (2) can be omitted. State equations take the following form:

$$\mathbf{x}_p(k+1) = \mathbf{A}(k) \cdot \mathbf{x}_p(k) + \mathbf{B}(k) \cdot \mathbf{v}_p(k), \tag{1}$$

$$\mathbf{y}_p(k) = \mathbf{C}(k) \cdot \mathbf{x}_p(k) + \mathbf{D}(k) \cdot \mathbf{v}_p(k), \quad k \in \mathbf{N}, \quad \mathbf{x}_p(0) = \mathbf{0}, \tag{2}$$

where $\{\mathbf{x}_p(k) \in \mathbf{R}^n, k \in \{0, \ldots, N-1\}\}$ is nominal state, $\{\mathbf{v}_p(k) \in \mathbf{R}^m, k \in \{0, \ldots, N-1\}\}$ is nominal control, $\{\mathbf{y}_p(k) \in \mathbf{R}^p, k \in \{0, \ldots, N-1\}\}$ is nominal output, and $\{\mathbf{A}(k) \in \mathbf{R}^{n\times n}, \mathbf{B}(k) \in \mathbf{R}^{n\times m}, \mathbf{C}(k) \in \mathbf{R}^{p\times n},\ k \in \{0, \ldots, N-1\}\}$ are system matrices.

Alternatively, the system model may be described with the help of operators. Then Eqs. (1–2) can be given in the following form:

$$\mathbf{y}_p(k) = (\hat{\mathbf{C}}\hat{\mathbf{N}}\mathbf{x}_0)(k) + (\hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}}\mathbf{v}_p)(k) \quad \text{or} \quad \hat{\mathbf{y}} = \hat{\mathbf{C}} \cdot \hat{\mathbf{N}} \cdot \mathbf{x}_0 + \hat{\mathbf{C}} \cdot \hat{\mathbf{L}} \cdot \hat{\mathbf{B}} \cdot \hat{\mathbf{v}}. \tag{3}$$

The system natural response ($\hat{\mathbf{v}} = \mathbf{0}$) is determined by the $\hat{\mathbf{y}}_0 = \hat{\mathbf{C}} \cdot \hat{\mathbf{N}} \cdot \mathbf{x}_0$ term, and the system response at zero initial conditions is determined by the $\hat{\mathbf{y}}_v = \hat{\mathbf{C}} \cdot \hat{\mathbf{L}} \cdot \hat{\mathbf{B}} \cdot \hat{\mathbf{v}}$ term. In order that system (3) be equivalent to system (1–2), operators $\hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}\hat{\mathbf{N}}$ must be defined in one of the two equivalent notations: either an evolutionary one, where operators are written by means of sums and products [11] or a matrix-based one, where each of the operators can be presented in terms of matrices. Matrix-based definitions of the system operators are expressed as follows:

$$\hat{\mathbf{L}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{A}(1) & \mathbf{I} & \mathbf{0} & \vdots & \vdots \\ \vdots & \ddots & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}(N-2)\cdots\mathbf{A}(1) & \cdots & \mathbf{A}(N-2) & \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} \mathbf{I} \\ \mathbf{A}(0) \\ \vdots \\ \mathbf{A}(N-2)\cdots\mathbf{A}(0) \end{bmatrix}, \tag{4}$$

$$\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}(N-1) \end{bmatrix}, \quad \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C}(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}(N-1) \end{bmatrix}, \tag{5}$$

where operators $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ have block diagonal form. State $\mathbf{x}_p(\cdot)$, output $\mathbf{y}_p(\cdot)$ and input $\mathbf{v}_p(\cdot)$ have the following notation:

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_p(0) \\ \vdots \\ \mathbf{x}_p(N-1) \end{bmatrix}, \quad \hat{\mathbf{y}} = \begin{bmatrix} \mathbf{y}_p(0) \\ \vdots \\ \mathbf{y}_p(N-1) \end{bmatrix}, \quad \hat{\mathbf{v}} = \begin{bmatrix} \mathbf{v}_p(0) \\ \vdots \\ \mathbf{v}_p(N-1) \end{bmatrix}. \tag{6}$$

The operator $\hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}}$ is a compact, Hilbert–Schmidt operator from l_2 into l_2 and actually maps bounded signals $\mathbf{u}(k) \in \mathcal{U} = l_2[0, N]$ into signals $y \in \mathcal{Y}$.

3. Distribution theorems

Elements of frequency analysis introduced here are based mainly on singular value decomposition (SVD) of the system operators. Such a decomposition presents a generalization of the classic SVD of matrices [12]. This is possible because operators defined for discrete-time system over a finite time horizon are finite dimensional. For such systems the time horizon is a product of sampling period of the system and total number of samples.

As in linear algebra, SVD decomposes the operator into corresponding sets of singular values σ_i, singular input vectors $\mathbf{v}_i$ and singular output vectors $\mathbf{u}_i$. Any complex or real matrix $\mathbf{X}$ may be written as a product of three matrices $\mathbf{X} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{V}^*$, where $\mathbf{\Sigma} = diag(\sigma_i)$ is a diagonal matrix, and orthonormal matrices **U, V** are composed of column vectors $\mathbf{u}_i$ and $\mathbf{v}_i$, respectively.

Two theorems, which will be used further to define frequency characteristics for time varying systems, are given below.

3.1. Theorem on a decomposed system

Theorem 1. *Response* $\mathbf{y}_v = \sigma_i \cdot \mathbf{u}_i$ *of a singular value decomposed* $\mathbf{U} \cdot \mathbf{S} \cdot \mathbf{V}^{\mathrm{T}} = \hat{\mathbf{C}} \cdot \hat{\mathbf{L}} \cdot \hat{\mathbf{B}}$ *system having been excited by the input* $\hat{\mathbf{v}} = \mathbf{v}_i$ *defined by the ith column of the* **V** *matrix, is equal to the product of ith singular value and ith column of the* **U** *matrix.*

Theorem 2. *Natural response* $\mathbf{y}_0 = \sigma_i \cdot \mathbf{u}_i$ *of a singular value decomposed* $\mathbf{U}_0 \cdot \mathbf{S}_0 \cdot \mathbf{V}_0^{\mathrm{T}} = \hat{\mathbf{C}} \cdot \hat{\mathbf{N}}$ *system at* $\mathbf{x}_0 = \mathbf{v}_i$ *initial conditions defined by the ith column of the* $\mathbf{V}_0$ *matrix, is equal to the product of ith singular value and ith column of the* $\mathbf{U}_0$ *matrix.*

Proofs of Theorems 1 and 2 follow from orthonormality of **U**, **V** matrices and from properties of SVD. The control vector **v** and initial conditions $\mathbf{x}_0$ for which the gain and energy capacity are the biggest are given by the first column of the **V** ($\mathbf{V}_0$) matrix. The control vector **v** and initial conditions $\mathbf{x}_0$ for which gain and energy capacity are the smallest are given by the last column of the **V** ($\mathbf{V}_0$) matrix.

4. Transform theorems

To derive relationships for frequency responses of time variant discrete-time systems one has to invoke the power spectral density function, for which it holds

$$S_y(\omega_k) = |G(\omega_k)|^2 \cdot S_v(\omega_k), \tag{7}$$

where $S_y(\omega_k)$, $S_v(\omega_k)$ are output and input spectral densities, respectively.

A frequency response $|G(\omega_k)|$ can be determined in a unique way if input and output spectral densities of the system are known. The following theorem can be proved by making use of SVD.

Theorem 3. *Discrete power spectral density for any orthonormal matrix originated from singular value decomposition is equal to* 1 *if counted as a sum of power spectral densities of individual matrix columns* $\{\mathbf{V} = \{\mathrm{v}_{ij}\}, i, j = 1, \ldots, N\}$:

$$S_v(\omega_k) = \sum_{j=1}^{N} S_j(\omega_k) = \frac{1}{N} \sum_{j=1}^{N} |DFT_k[\mathbf{v}_j]|^2$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left| \sum_{n=1}^{N} v_{ni} \mathrm{e}^{-\mathrm{j}2\pi(k-1)(n-1)/N} \right|^2 = \mathbf{1}, \tag{8}$$

where $\omega_k = k/(2T_p N)$, T_p *is the sampling period.*

Proof of the theorem follows directly from the orthonormality of the SVD matrix [12] and from unitary properties of the DFT transform. The following equation holds true then:

$$|DFT_k[\mathbf{v}_j]|^2 = \mathbf{1}, \tag{9}$$

hence

$$S(\omega_k) = \frac{1}{N}\sum^{N} 1 = 1. \tag{10}$$

Thus the theorem is proved.

Theorem 4. *The output power spectral density can be evaluated as a sum of power spectral densities of individual columns of a matrix defined as a product of* $\mathbf{U}\cdot\mathbf{S}$ *matrices. This can be written in the following way*:

$$S_y(\omega_k) = \frac{1}{N}\sum_{j=1}^{N} |DFT_k[\mathbf{u}_j \cdot s_{jj}]|^2 = \frac{1}{N}\sum_{i=1}^{N}\left|\sum_{n=1}^{N} u_{ni}\sigma_i \mathrm{e}^{-\mathrm{j}2\pi(k-1)(n-1)/N}\right|^2, \tag{11}$$

where $\omega_k = k/(2T_pN)$, T_p *is the sampling period*, $\sigma_i = s_{ii} - i$ *is the singular value of* $\mathbf{U}\cdot\mathbf{S}\cdot\mathbf{V}^{\mathrm{T}} = \hat{\mathbf{C}}\cdot\hat{\mathbf{L}}\cdot\hat{\mathbf{B}}$ *decomposition.*

Proof of the theorem follows directly from SVD properties, especially from orthonormality of $\mathbf{U}$, $\mathbf{V}$ matrices.

5. Bode characteristics approximation

Bode characteristics include the magnitude-frequency response $|G(\omega_k)|$ and the phase-frequency response $\varphi(\omega_k) = \arg(G(\omega_k))$. The former can be obtained by substituting Eq. (8) into Eq. (7). Taking square roots from both sides, one gets

$$|G(\omega_k)| = \sqrt{S_y(\omega_k)}. \tag{12}$$

After substituting Eq. (11), one obtains

$$|G(\omega_k)| = \sqrt{\frac{1}{N}\sum\nolimits_{j=1}^{N} \sigma_j^2 |DFT_k[\mathbf{u}_j]|^2} = \sqrt{\frac{1}{N}\sum\nolimits_{i=1}^{N}\left|\sigma_i \sum\nolimits_{n=1}^{N} u_{ni}\mathrm{e}^{-\mathrm{j}2\pi(k-1)(n-1)/N}\right|^2}, \tag{13}$$

which uniquely defines the magnitude-frequency response.

By analogy with the latter, the phase-frequency response can be written as

$$\begin{aligned}\varphi(\omega_k) &= \arg\left(\sum_{j=1}^{N} \sigma_j \frac{DFT_k[\mathbf{u}_j]}{DFT_k[\mathbf{v}_j]}\right)\\ &= \arg\left(\sum_{i=1}^{N}\left(\sigma_i \frac{\sum_{n=1}^{N} u_{ni}\mathrm{e}^{-\mathrm{j}2\pi(k-1)(n-1)/N}}{\sum_{n=1}^{N} v_{ni}\mathrm{e}^{-\mathrm{j}2\pi(k-1)(n-1)/N}}\right)\right).\end{aligned} \tag{14}$$

Singular values σ_i in Eqs. (13)–(14) play their part as weight functions.

The derived relationships hold true for both time invariant and time variant systems. Characteristics obtained in the way shown for time invariant systems at a finite time horizon are close to Bode characteristics obtained in the classic way by substituting $z = \exp(j\omega T_p)$ into discrete transfer function. It is difficult to draw a comparison for time variant systems. The transfer function defined by Zadeh [1,2] and improved by his followers [13–17] is determined over an infinite time horizon. It is somewhat easier to compare the obtained results with those obtained while analyzing pseudomodal parameters. Nonetheless, apart from pole shifting, shifting of zeros and system gain may also occur.

6. Degree of system time variability

Dynamic linear systems can be variable either in the frequency domain (linear time invariant systems—LTI systems) or in the time domain (linear frequency invariant systems—LFI systems). The main difference between LTI and LFI systems can be recognized by comparison of the output function. The output function of an LTI system is a time domain convolution or frequency domain multiplication, whereas the output function of LFI system is a time domain multiplication or frequency domain convolution. Essentially, LFI is a static system with a time variant gain. In the most general case a dynamic linear system can be variable in both frequency and time domain. From the mathematical point of view a dichotomous classification into time or frequency variant systems is clearly defined. However, from the practical viewpoint determining the degree of time non-stationarity on a continuous scale is more significant than such a dichotomous classification (variant–invariant). This is due to the fact that scattering of system parameter values does not necessarily imply a change in system properties as a whole; if not, the changes can be insignificant.

To illustrate the problem of determining the degree of time variability, consider a system with a periodically time variant gain. The system is given in the state space as

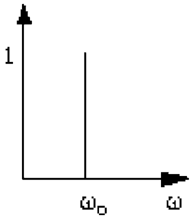
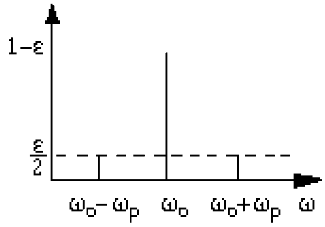
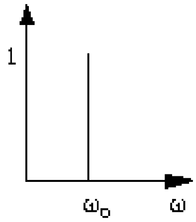
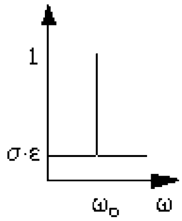
$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}(k) \cdot \mathbf{x}(k) + \mathbf{B}(k) \cdot \mathbf{u}(k), \\ \mathbf{y}(k) &= \mathbf{C}(k) \cdot \mathbf{x}(k) + \mathbf{D}(k) \cdot \mathbf{u}(k). \end{aligned} \tag{15}$$

The system at hand is a single-input single-output one. Thus system matrices are scalars. A simple analysis can be performed if $\mathbf{A}(k) \equiv 0$, $\mathbf{D}(k) \equiv 0$. In such a case, a typical time-delay system must be used. Depending on the values the $\mathbf{B}$ and $\mathbf{C}$ matrices take, the following four cases given in Table 1 may be distinguished.

In this example, the degree of time non-stationarity is dependent on the ε parameter. The greater is ε, the greater is the degree of system time non-stationarity. As $\varepsilon \to 0$ the system response tends asymptotically to the response produced by a time invariant system. The output frequency spectrum varies with the degree of system time variability. In the output spectrum side bands with $\omega_0 - \omega_p, \omega_0 + \omega_p$ frequencies appear and, in addition, the amplitude of the main band ω_0 diminishes. These phenomena are caused by the system variability modulation.

A little different effect is produced by white noise modulation. The noise modulated by any other signal still remains the noise, only the parameters are changed. In the quasi-non-stationary system the noise has been modulated by a sinusoidal input. As a result, a white noise with another variance is produced. A change in variance entails a change in standard deviation, in proportion to which is the value of the zero-frequency component in the system output magnitude-frequency

Table 1
Scattering of the signal spectrum for various systems

	Stationary	Non-stationary	Quasi-stationary	Quasi-non-stationary
$B(k)$	1	$1-\varepsilon+\varepsilon\cos(\omega_p kT_s)$ $0\leqslant\varepsilon\leqslant 1$	$1-\varepsilon+\varepsilon\cos(\omega_p kT_s)$ $0\leqslant\varepsilon\leqslant 1$	$1+\varepsilon\,\mathrm{randn}$
$C(k)$	1	1	$\dfrac{1}{1-\varepsilon+\varepsilon\cos(\omega_p(k-1)T_s)}$	1
Output spectra				

response. Let $x_{noise} = N(0,1)$ denote a normal distribution noise with parameters $E(x) = 0$, $V(x) = 1$, and $v \neq x_{noise}$ denote an arbitrary input signal. Then the modulator output $y = x_{noise}v$ is a noise with following parameters: $E(y) = 0$ and $V(y) = P(v)$, where

$$E(y) = \frac{1}{2\tau}\int_{-\tau}^{\tau} y(t)\,\mathrm{d}t, \quad V(y) = \frac{1}{2\tau}\int_{-\tau}^{\tau}(y(t)-E(y))^2\,\mathrm{d}t,$$
$$P(v) = \frac{1}{2\tau}\int_{-\tau}^{\tau} v^2(t)\,\mathrm{d}t. \tag{16}$$

For periodic signals τ is equal to the period $\tau = T$, and for other signals τ tends to infinity ($\tau \to \infty$). Standard deviation is equal to the square root of variance. For a quasi-non-stationary system and sinusoidal input $v(t) = \sin(\omega_0 t)$ it holds:

$$\sigma = \sqrt{V(y)} = \sqrt{P(v)} = \sqrt{\frac{1}{4\pi}\int_{-2\pi}^{2\pi}\sin^2(\omega_0 t)\,\mathrm{d}t} = \frac{\sqrt{2}}{2} \cong 0.7. \tag{17}$$

These relationships can be easily verified numerically.

Time and frequency variability for continuous systems was dealt with in Refs. [13–17]. An approach has been proposed there that is based on the impulse response of a time variant system, which presents an outgrowth of the approach originated by Zadeh.

This approach has been used successfully for analysis of time variant communication channels. The measure of potential time–frequency shifting, which system can impart is a spreading function. For example, asymmetrical spreading function or delay Doppler spread function introduced by Bello [13] is obtained:

$$S_H^{(1/2)}(\tau,\nu) = \int_t h(t, t-\tau)\mathrm{e}^{-\mathrm{i}2\pi\nu t}\,\mathrm{d}t. \tag{18}$$

For the spreading function to be evaluated, the knowledge of the set of system impulse responses $h(t, t-\tau)$ is needed, where t is the determined time instant, and $t-\tau$ is the point in time at which the impulse has been generated.

The generalized spreading function of an LTI system with kernel $h(t, t-\tau) = g(t-\tau) = g(\tau)$

$$S_H(\tau, v) = g(\tau) \cdot \delta(v) \tag{19}$$

and is concentrated along τ-axis reflecting the system can only cause time shifts. Dirac function is denoted by $\delta(v)$.

Employing the spreading function for system analysis makes it possible to determine the system variability in both time and frequency domains. To do this, however, the knowledge of system responses obtained with appropriate resolution over a wide time horizon is needed. An attempt made by the author to determine the spreading function for a discrete system defined over a finite time horizon failed. The numerical algorithm turned out to be unstable, and the results obtained were almost independent of actual changes in system parameters.

Another approach that may be useful to determine the system time variability is the modal analysis [6,7,9,10]. It yields a relatively great body of data the interpretation of which requires some knowledge and experience. The cardinal virtues of PMP are its numerical stability independent of the time horizon and a very wide field of application. Comparison of PMP and SVT-DFT is given in Table 2, where r is the system order, and N is the time horizon length.

Analysis of output amplitude spectra carried out by the author for different systems allows one to define certain functions that enable the system time variability to be measured. The test input should be chosen first. In the light of SVD properties, especially those revealed by Theorem 1, the optimal test signal is represented by the system singular vectors $\mathbf{v}_i$ with their corresponding weights σ_i. If so, the measure of the system non-stationarity may be defined as

(I) weighted main band attenuation,
(II) weighted relative distance between the side bands and the main band.

The less the output is affected by the parameter non-stationarity, the smaller is coefficient (I). Rate of parameter changes is of secondary importance here. The value of coefficient (II) depends on the rate of parameter changes. The slower the changes, the smaller coefficient (II).

Numerically, coefficient (I) can be evaluated as a sum of squared differences between consecutive discrete values for power spectral density of normed characteristic vectors for input

Table 2
PMP and SVD-DFT-based analyses compared from the viewpoint of complexity of the results yielded

	PMP analysis	SVD-DFT analysis
Numerical results	rN real values	N real values or $N/2$ complex values
Graphical results	r characteristics of N points every	2 characteristics of $N/2$ points every

and output spectra:

$$S_{var1} = \sum_{i=1}^{N} \frac{\sigma_i}{\sigma_1} \Big\| \, |DFT[\mathbf{v}_i]| - |DFT[\mathbf{u}_i]| \, \Big\|_2 = \sum_{i=1}^{N} \frac{\sigma_i}{\sigma_1} \sqrt{\sum\nolimits_{k=1}^{N} \left(|DFT_k[\mathbf{v}_i]| - |DFT_k[\mathbf{u}_i]|\right)^2}. \quad (20)$$

The formula for numerical computations is given below

$$S_{var1} = \sum_{i=1}^{N} \frac{\sigma_i}{\sigma_1} \sqrt{\sum\nolimits_{k=1}^{N} \left(\left| \sum\nolimits_{n=1}^{N} v_{ni} \mathrm{e}^{-\mathrm{j}2\pi(k-1)(n-1)/N} \right| - \left| \sum\nolimits_{n=1}^{N} u_{ni} \mathrm{e}^{-\mathrm{j}2\pi(k-1)(n-1)/N} \right| \right)^2}. \quad (21)$$

Coefficient (II) is evaluated from

$$S_{var2} = \Delta f \sum_{i=1}^{N} |k_{mv}(i) - k_{mu}(i)| \frac{\sigma_i}{\sigma_1} = \frac{1}{TN} \sum_{i=1}^{N} |k_{mv}(i) - k_{mu}(i)| \frac{\sigma_i}{\sigma_1}, \quad (22)$$

where $k_{mv}(i)$ is index of maximal value in vector $\mathbf{V}_i$, $k_{mu}(i)$ is index of maximal value in vector $\mathbf{U}_i$, resolution in frequency domain $\Delta f = 1/TN$ is normalization factor for Eq. (22). For time invariant systems all coefficients should be equal to zero.

In the next section, an attempt will be made to check whether the proposed relationships present a good measure of the system non-stationarity.

7. Numerical examples for bode characteristics approximation for time-varying systems

7.1. Oscillatory, minimum phase, time invariant system of the fourth order

The system is described in the state space. Sampling period is equal to $T_p = 0.5$ s. System matrices and computed zeros and poles for this system are

$$\mathbf{A} = \begin{bmatrix} 0.12 & 0.10 & -0.16 & 0.33 \\ 0.36 & 0.41 & -0.19 & 0.05 \\ 0.12 & -0.30 & 0.46 & 0.22 \\ 0.04 & 0.25 & 0.31 & 0.35 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -0.41 \\ -0.01 \\ 0.37 \\ 0.06 \end{bmatrix},$$

$$\mathbf{z} = \begin{bmatrix} 0.6811 \\ 0.27 + 0.18\mathrm{i} \\ 0.27 - 0.18\mathrm{i} \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} -0.037 + 0.27\mathrm{i} \\ -0.037 - 0.27\mathrm{i} \\ 0.6807 \\ 0.73 \end{bmatrix},$$

$$\mathbf{C} = [-0.85 \quad 0 \quad 0.66 \quad -0.85], \quad \mathbf{D} = [0]$$

Fig. 1 shows the magnitude-frequency response and the phase-frequency response obtained via the SVD-DFT method (solid line) and those obtained in the classic way by substituting $z = \mathrm{e}^{\mathrm{j}\omega T_p}$. The error in magnitude-frequency response deviation does not exceed 1 dB, and is caused by a

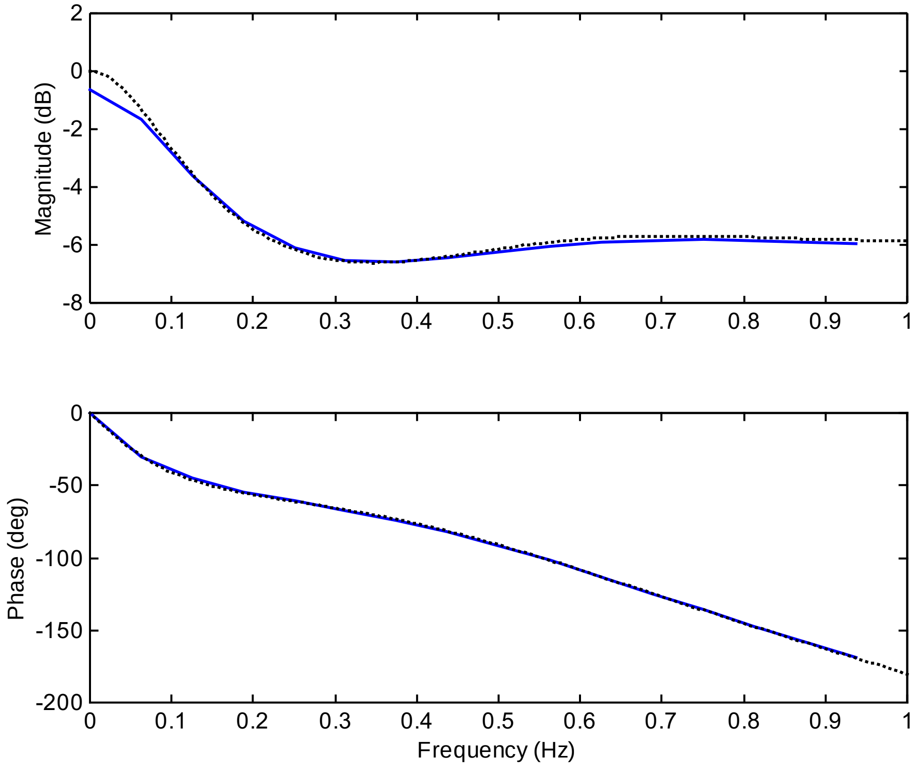


Fig. 1. Bode characteristics, determined using SVD-DFT method (solid line) and using classical method (dotted line).

short time horizon used for analysis which entails a low resolution in the frequency domain $\Delta f = 0.0625$ Hz corresponding to 16 points on the curve.

Computed value of non-stationarity coefficient $S_{var1} = 1.6 \cdot 10^{-13}$, $S_{var2} = 0$.

7.2. Oscillatory element with variable resonant frequency

The system is a discretized analogue oscillatory, variable structure element. Specifications and model parameters are assumed in the simulation:

- The oscillatory element is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 2\beta(t)\omega_0(t)\frac{dy(t)}{dt} + \omega_0^2(t)y(t) = k\omega_0^2(t)u(t),$$

- On the assumption of constant $\omega_0(t) = \omega_0$, $\beta(t) = \beta$ the corresponding transfer function is equal to

$$G_{osc}(s) = \frac{k\omega_0^2}{s^2 + 2\beta\omega_0 s + \omega_0^2},$$

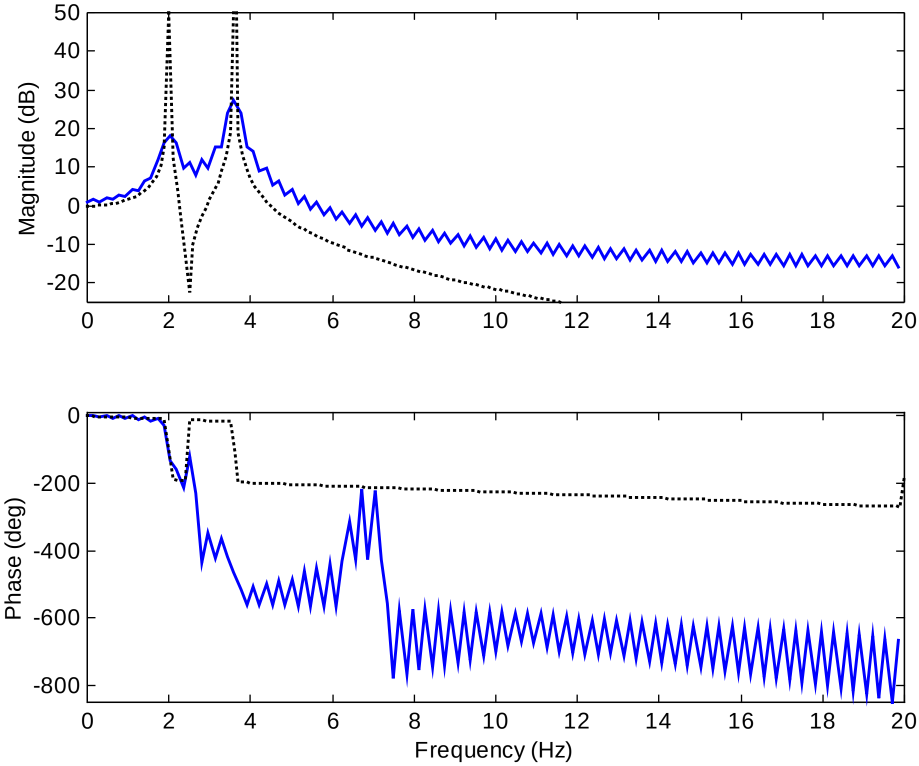


Fig. 2. Bode characteristics for damping factors $\beta_1 = 4 \times 10^{-4}$, $\beta_2 = 2.2 \times 10^{-4}$ determined using SVD-DFT method for time-varying system (solid line) and for average time-invariant transfer function $G_z(z)$ (dotted line).

- Sampling period $T_p = 0.025$ s,
- Simulation horizon $N = 256$ (time horizon 6.4 s, resolution in frequency domain $\Delta f = 0.039$ Hz),
- Gain $k = 1$,
- Resonant frequency before change ($t \leqslant 3.2$ s) $f_1 = 2$ Hz and after change ($t > 3.2$ s) $f_2 = 3.6$ Hz,
- Damping factors $\beta_1 = 4 \times 10^{-4}$, $\beta_2 = 2.2 \times 10^{-4}$ (simulation in Figs. 2–4) or $\beta_1 = \beta_2 = 0.08 \times 10^{-4}$ (simulation in Fig. 3),
- Time-invariant transfer function for averaged system $G_i(z) = \frac{1}{2}(G_1(z) + G_2(z))$ (G_1 and G_2 conversion has been done by the zero order hold method).

Resonant frequencies f_1, f_2 and damping factors β_1, β_2 may be considered as corresponding pseudomodal values at defined time instants.

In Figs. 2 and 3 Bode characteristics for an oscillatory system with variable resonant frequency are depicted. A reference frequency response obtained for an averaged reference system G_z by substituting $z = e^{j\omega T_p}$ is plotted with a dash line. Fig. 4 shows the system step response.

Computed value of non-stationarity coefficient $S_{var1} = 1.6$, $S_{var2} = 2.5$ for simulation in Fig. 2 and $S_{var1} = 2.9$, $S_{var2} = 4.4$ for simulation on Fig. 3.

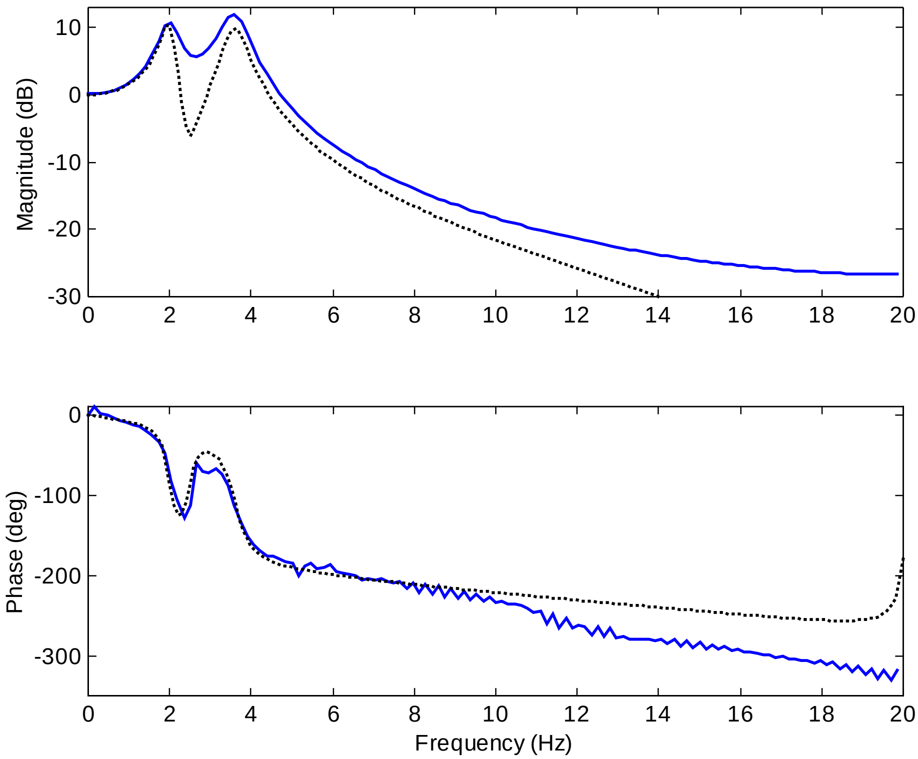


Fig. 3. Bode characteristics for damping factors $\beta_1 = \beta_2 = 0.08 \times 10^{-4}$ determined using SVD-DFT method for time-varying system (solid line) and for average time-invariant transfer function $G_z(z)$ (dotted line).

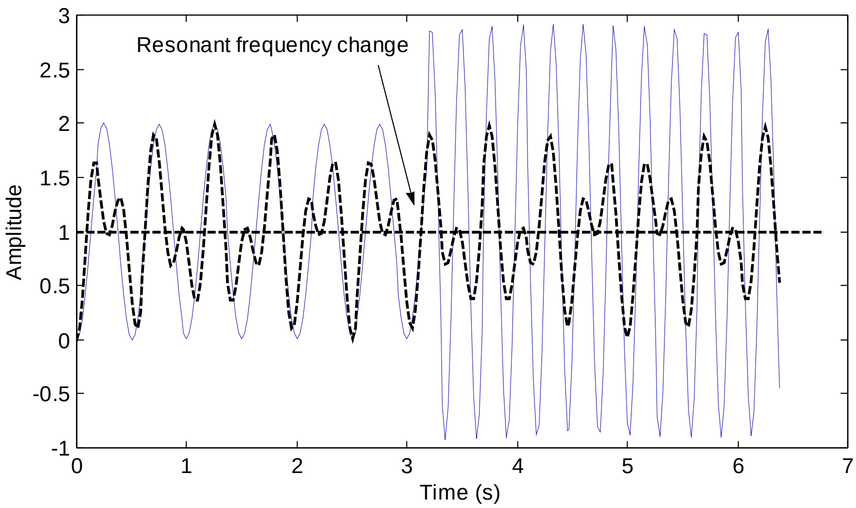


Fig. 4. Step response for damping factors $\beta_1 = 4 \times 10^{-4}$, $\beta_2 = 2.2 \times 10^{-4}$, for time-varying system (solid line) and for average time-invariant transfer function $G_i(z)$ (dashed line).

7.3. Variable structure, minimal phase, time varying system of fourth order

The system is described as zeros–poles model. Sampling period is equal to $T_p = 0.1$ s, time horizon is $N = 32$ steps. The structure has been changed after N_p steps.

System before change:

$$\mathbf{z} = \begin{bmatrix} 0.34 + 0.08\mathrm{i} \\ 0.34 - 0.08\mathrm{i} \\ -0.43 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 0.11 + 0.75\mathrm{i} \\ 0.11 - 0.75\mathrm{i} \\ 0.27 + 0.09\mathrm{i} \\ 0.27 - 0.09\mathrm{i} \end{bmatrix}.$$

System after N_p steps:

$$\mathbf{z} = \begin{bmatrix} 0.34 + 0.0008\mathrm{i} \\ 0.34 - 0.0008\mathrm{i} \\ -0.43 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 0.11 + 0.008\mathrm{i} \\ 0.11 - 0.008\mathrm{i} \\ 0.27 + 0.0009\mathrm{i} \\ 0.27 - 0.0009\mathrm{i} \end{bmatrix}.$$

Since system realization is of the lowest order, so values of the system pseudomodes may be considered as equal to those of poles given above at specific time instants.

In Fig. 5 the magnitude- and phase-frequency responses obtained for two different time instants at which the system structure varies are plotted. The system is oscillatory before it has changed, and approximately oscillation-free after the change. In the case that the structure is changed after three steps the oscillation-free character is prevailing (three steps of the oscillatory character, 29 steps of the oscillation-free character), which can be concluded from the solid line plot. In the case that the structure is changed after 15 steps the oscillatory character of the plot can be seen (dash line plot).

Computed value of non-stationarity coefficient $S_{var1} = 1.5$, $S_{var2} = 1.3$ for $N_p = 3$ and $S_{var1} = 1.1$, $S_{var2} = 0.4$ for $N_p = 15$.

8. Numerical examples for evaluating the degree of system non-stationarity

The main concern in the examples having been analyzed in the former section was with evaluation of SVD-DFT frequency responses for different systems and comparison, as far as possible, with classic Bode characteristics. However, in the light of how the variability coefficients (20,22) are defined, an analysis of the influence the ε parameter changes of a non-stationary system exert on the whole system properties may be of interest. To carry out a numerical analysis four models of systems with a specified magnitude of parameter variations that has been presented and discussed in Section 6 are used. As a measure of non-stationarity definitions (20)–(22), and for the purpose of comparison the graphical shape of step responses have been taken. In this section, coefficients of Eqs. (20), (22) versus ε parameter are evaluated for similar systems by way of simulation. For purposes of simulation it is assumed that the system is also non-stationary in the frequency domain and represents a time lag $A(k) = 0.1$. The remaining system parameters are the same as given in Table 1.

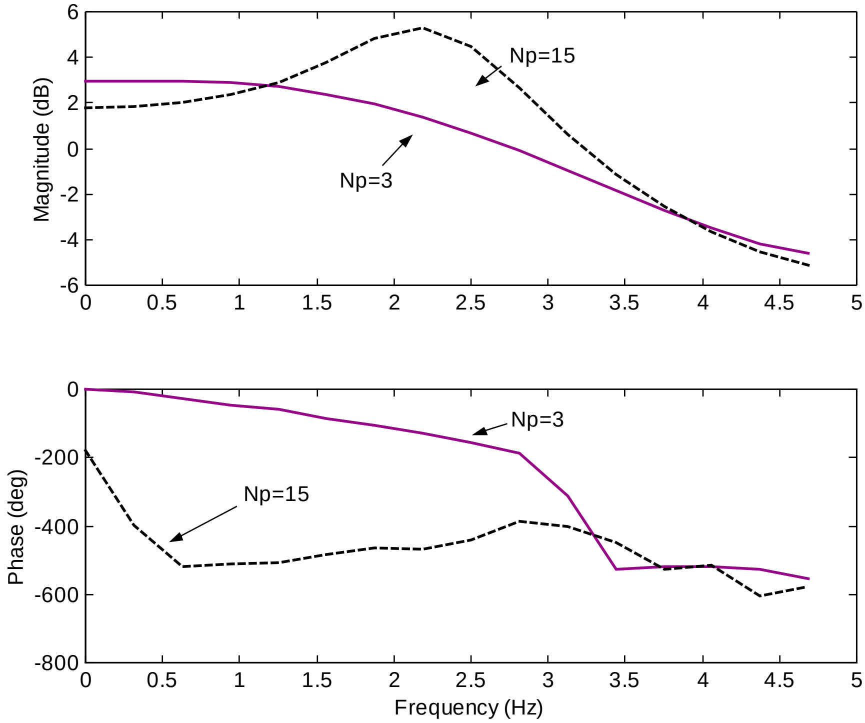


Fig. 5. Bode characteristics, determined using SVD-DFT method for variable structure system. Solid line—change structure point $N_p = 3$, and dashed line—change structure point $N_p = 15$.

8.1. Non-stationary system (cosinusoidal)

System coefficients are as follows:

$$A(k) = 0.1, \quad B(k) = 1 - \varepsilon + \varepsilon\cos(\omega_p k T_s), \quad C(k) = 1, \quad D(k) = 0.$$

It is assumed here that ε varies within the range $\varepsilon \in \langle 10^{-2}, 1 \rangle$, from where 10 values of ε have been chosen to evaluate the system variability coefficient. The results obtained are summarized in Fig. 6. Results yielded by Eq. (21) are marked with "x". For $\varepsilon < 10^{-6}$ the coefficient amounts to $S_{var1} = 0$. Results yielded by Eq. (22) are marked with "+". For $\varepsilon < 10^{-1}$ S_{var2} takes the value 0.

From Fig. 6 it may be concluded that S_{var1} is more stable numerically, even at short time horizons. Feasibility of S_{var2} evaluation is conditioned largely by the resolution obtained in the frequency domain. Requirements to be met here are much higher than those for S_{var1}.

Step responses corresponding to 10 highest values of ε are depicted in Fig. 7.

8.2. Quasi-stationary system

System coefficients are as follows:

$$A(k) = 0.1, \quad B(k) = 1 - \varepsilon + \varepsilon\cos(\omega_p k T_s),$$

$$C(k) = \frac{1}{1 - \varepsilon + \varepsilon\cos(\omega_p (k-1) T_s)}, \quad D(k) = 0.$$

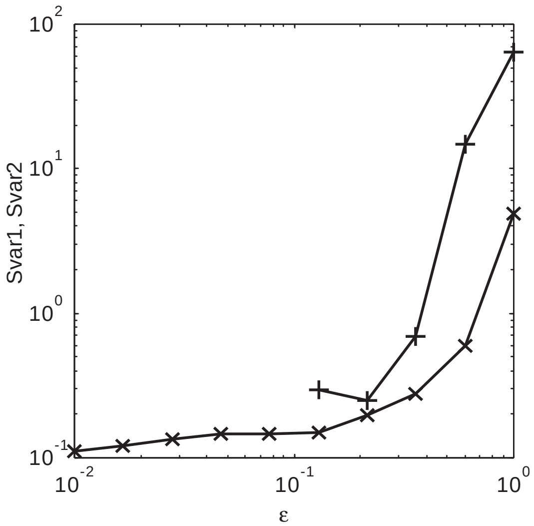


Fig. 6. Variability coefficient versus ε for a non-stationary system (cosinusoidal).

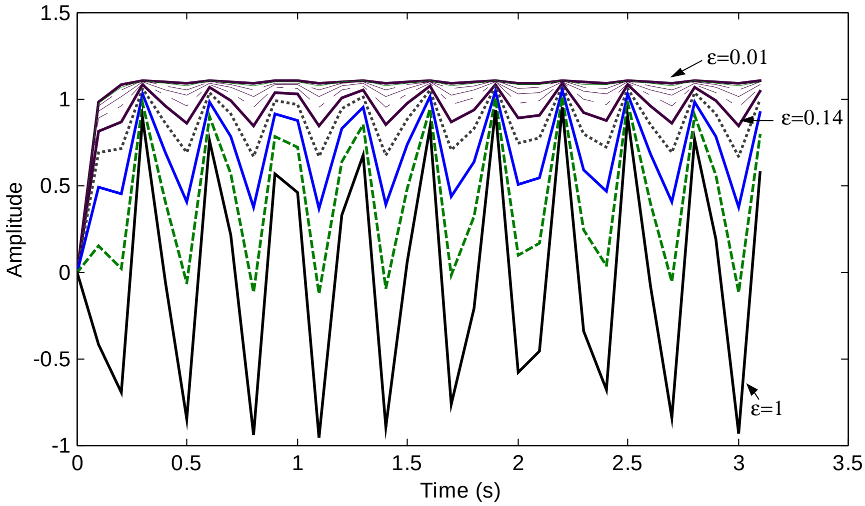


Fig. 7. Step responses obtained for 10 biggest values of ε (non-stationary system).

It is assumed here that ε varies within the range $\varepsilon \in \langle 10^{-1}, 1 \rangle$, from where 10 values of ε have been chosen to evaluate the system variability coefficient. The results obtained are summarized in Fig. 8. Results yielded by Eq. (21) are marked with "x". Results yielded by Eq. (22) are marked with "+". For $\varepsilon < 0.6$ S_{var2} takes the value 0.

From Fig. 8 it may be concluded that S_{var1} is more stable numerically, even at short time horizons. Feasibility of S_{var2} evaluation is conditioned largely by the resolution obtained in the frequency domain. Also, requirements to be met here are much higher than those for S_{var1}.

Step responses corresponding to ten highest values of ε are depicted in Fig. 9.

8.3. Quasi-non-stationary

System coefficients are as follows:

$A(k) = 0.1, B(k) = 1 + \varepsilon\, randn,\ C(k) = 1, D(k) = 0$, where *randn* is random noise described by normal distribution with parameters $N(0, 1)$.

It is assumed here that ε varies within the range $\varepsilon \in \langle 10^{-5}, 1 \rangle$, from where 10 values of ε have been chosen to evaluate the system variability coefficient. The results obtained are summarized in

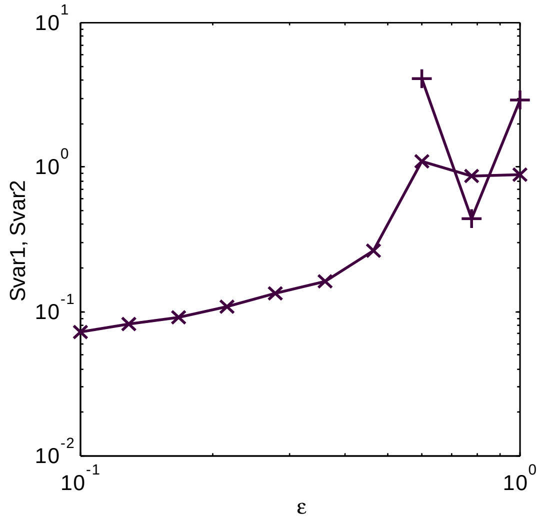


Fig. 8. Variability coefficient versus ε for a quasi-stationary system.

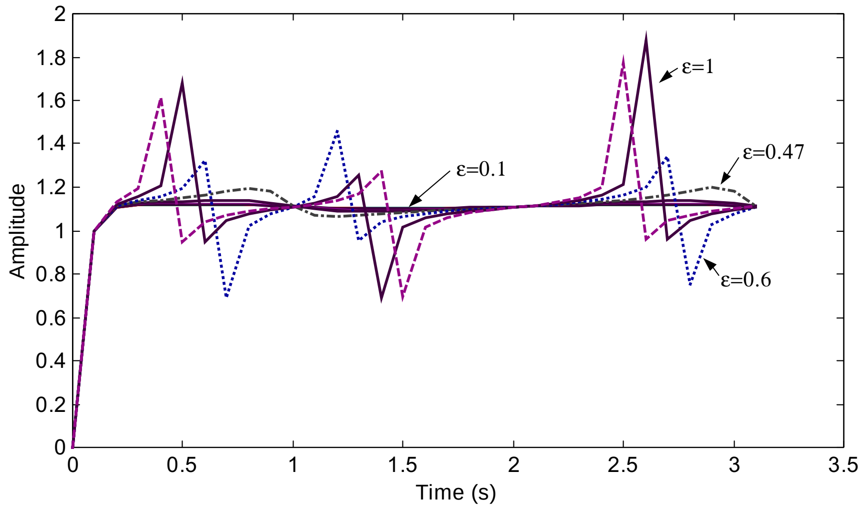


Fig. 9. Step responses obtained for 10 biggest values of ε (quasi-stationary system).

Fig. 10. Results yielded by Eq. (21) are marked with "x". Results yielded by Eq. (22) are marked with "+". For $\varepsilon < 0.07$ S_{var2} takes the value 0.

From Fig. 10 it may be concluded that S_{var1} is more stable numerically, even at short time horizons, as for previous examples. Feasibility of S_{var2} evaluation is conditioned largely by the resolution obtained in the frequency domain. Requirements to be met here are much higher than those for S_{var1}.

Step responses corresponding to 10 highest values of ε are depicted in Fig. 11.

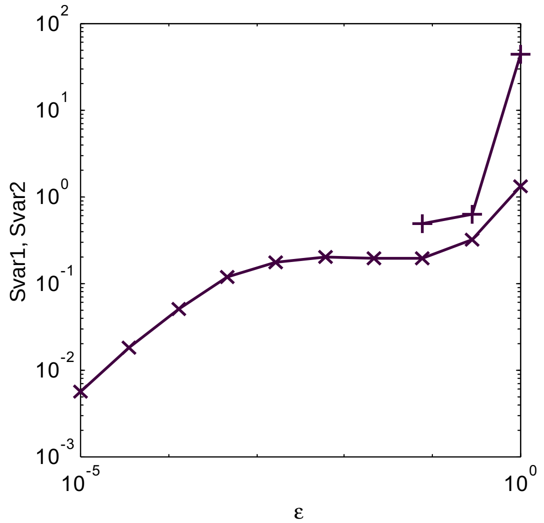


Fig. 10. Variability coefficient versus ε for a quasi-non-stationary system.

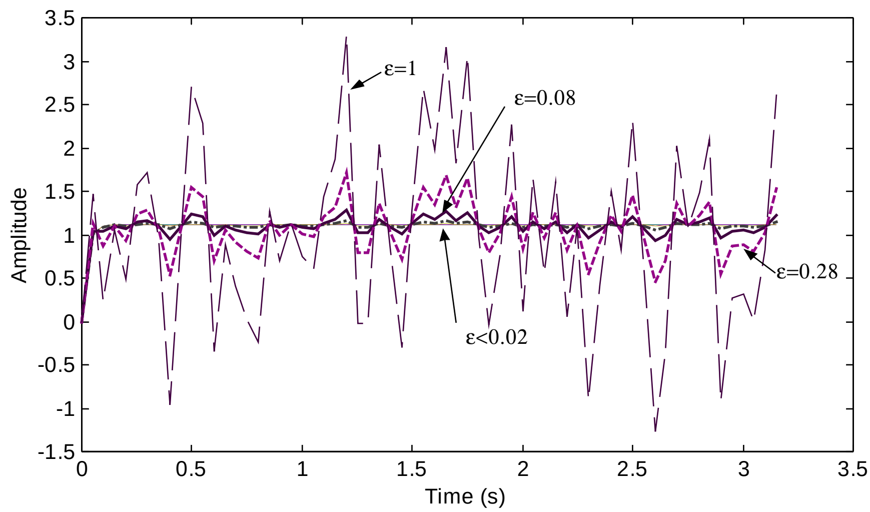


Fig. 11. Step responses obtained for 10 biggest values of ε (quasi-non-stationary system).

9. Conclusion

The study has achieved two tasks: (1) to extend the well-defined frequency analysis and Bode characteristics concepts to characterize the dynamics of an LTV system and (2) to introduce a new method for estimating the non-stationarity degree of the system.

For the first task, a discrete-time, state-space model and operator notation have been used to represent an LTV system. The paper has explored how the SVD and DFT in connection with operator notation can be used to describe global properties of an LTV system. It is shown that the SVD-DFT analysis preserve certain characteristics of conventional frequency analysis defined for LTI systems.

For the second task, the paper has shown that the time-variability coefficient of the system can be defined using data from the SVD-DFT analysis of the system. The key to the proposed method would be to modify input–output spectra of LTV system. The quantity of the modification can be a measure of time-variability of the system.

A few numerical examples with one LTI and two LTV–variable structure systems have been used as illustration for the first task. Also three examples with numerical data have illustrated how the variability coefficient can help understand the degree of time variations due to different parameter values in the system model.

As part of the study some weaknesses of the proposed methods have been revealed. This raises some open questions that will determine further studies.

(a) The obtained characteristics for stationary non-minimum-phase systems differ significantly from their counterparts obtained in a classic way. Would the proposed methods be applicable to non-minimum-phase systems? (see also Ref. [18]).
(b) The coefficient of system variability S_{var2} exhibits a certain numerical instability.
(c) The coefficient of system variability S_{var1} changes its properties whenever any of the poles goes beyond the unit circle. Would it be possible to generalize the observations made to arbitrary systems and to define, on this basis, stability for discrete-time systems determined over a finite time horizon? The response of such a system is always bounded.
(d) Which of the properties exhibited by Bode characteristics hold true for approximated characteristics for non-stationary systems?
(e) In which applications will the performed evaluation of magnitude- and phase-frequency response and variability coefficient for a non-stationary system be entirely sufficient?

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