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Computation of normal forms for high dimensional non-linear systems and application to non-planar non-linear oscillations of a cantilever beam

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Abstract

A new and efficient computation of the normal forms is developed in this paper for high dimensional non-linear systems, and the computational method is applied to non-planar non-linear oscillations of a cantilever beam. The method developed here has the advantage of directly calculating the coefficients of the normal forms and the associated near identity non-linear transformations for three different cases, that is, (1) the case of two pairs of pure imaginary eigenvalues; (2) the case of one non-semisimple double zero and a pair of pure imaginary eigenvalues; and (3) the case of two non-semisimple double zero eigenvalues. The final partial differential equations of various resonant cases appear in a canonical form whose solutions can be conveniently obtained using polynomial equations. With the aid of the Maple software, a symbolic program for computing the normal forms of high dimensional non-linear systems is given. Comparing the method developed here with other methods of computing the normal forms, it is understood that we may, respectively, obtain the normal forms, the coefficients of the normal forms and the associated near identity non-linear transformations for three resonant cases by using a same main Maple symbolic program. Moreover, the method is easy to apply to engineering problems. The normal forms of the averaged equations and their coefficients for non-planar non-linear oscillations of the cantilever beam under combined parametric and forcing excitations are calculated for two resonant cases.

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1. Introduction

Normal form theory is one of the basic methods for the study of non-linear dynamics such as the homoclinic and heteroclinic bifurcations. The theory of normal form is concerned with constructing a series of near identity non-linear transformations that make the non-linear systems as simple as possible. With the aid of normal form theory, we may obtain a set of simpler differential equations, which is topologically equivalent to the original systems. Being "simpler" means that some non-linear terms may be eliminated from the original differential equations. The main attention of the paper is focused on developing a new and efficient computation of the normal forms for higher dimensional non-linear systems based on the adjoint operator method [1]. The method is then applied to obtain the normal forms of the averaged equations for non-planar non-linear oscillations of a parametrically and externally exited cantilever beam.

In engineering problems and applications of non-linear dynamics, research for degenerate bifurcations of codimension 2, 3 and 4 is connected with research for the normal forms in higher dimensional non-linear systems. To study degenerate bifurcations of higher codimension and the Silnikov-type homoclinic or heteroclinic bifurcations in a practical non-linear system, the method of multiple scales or the averaging method can be utilized first to obtain the averaged equations of non-autonomous non-linear systems. Then, normal form theory and universal unfolding are used to simplify the aforementioned averaged equations. Therefore, we need to compute the normal forms of high dimensional averaged equations and to find the formulas of the explicit relationship between the coefficients of the normal forms and those of the averaged equations. Furthermore, we need to determine the universal unfolding or unfolding of the system and to find the relation between unfolding parameters and those of the original system. Finally, we use the global perturbation method to study the bifurcation characteristics of the unfolding and give the Silnikov-type homoclinic bifurcations or heteroclinic bifurcations. For a more complicated practical non-linear system, it is perhaps not easy to get the universal unfolding of the system. In this case, we may obtain an unfolding instead of the universal unfolding to study the bifurcation characteristics of the system, and also some interesting results in spite of possible incompletion of the results.

In the past three decades, the researchers have obtained great achievements in the study of the normal form. Up to now the five basic methods for the computation of the normal form have been proposed: the adjoint operator method [1], the matrix representation method [2–4], the representation theory of Lie algebra $sl(2, \mathbf{R})$ [5], the method of more Lie brackets [6–8], and the method of the symbolic computation [9,10]. Recent research is focused on the computation of the coefficients of the normal form and its further reduction. Dangelmayr and Guckenheimer [11] used the Macsyma to compute the normal forms and studied the degenerate bifurcations of codimension 3 and 4 for non-linear dynamical systems with four parameters in a planar vector field. Dumortier and Fiddelaers [12] used the Macsyma and Mathematica programs to compute the normal forms of higher order and analyzed the degenerate bifurcations of codimension 3 and 4 in a planar vector field. Combining the center manifold theory and the normal form theory into one step simultaneously, Yu et al. [9,10] obtained the formulae to compute the normal forms of the non-linear systems which have four-dimensional center manifold. They developed a symbolic computation by using the Maple program to compute explicit normal forms and associated near identity non-linear transformations based on the coefficients of the original differential equations.

Zhang [13] utilized the matrix representation method to compute the normal forms of higher order and studied the degenerate bifurcations of codimension 3 for a non-linear dynamical system with Z_2-symmetry. With the application of the adjoint operator method, Zhang and Chen [14] calculated higher order normal forms of non-linear dynamical systems with Z_2-asymmetry. Subsequently, Zhang and Yu [15] employed high order normal form to investigate codimension-3 degenerate bifurcations of a parametrically and externally excited mechanical system. Sri Namachchivaya et al. [16] computed the normal form for a generalized Hopf bifurcation with non-semisimple 1:1 resonance. Moreover, Yu [17] used a perturbation technique and computer algebra to compute the normal form of non-linear dynamical systems with two pairs of pure imaginary eigenvalues and analyzed double Hopf bifurcations.

Different from the aforementioned work which is focused on the computation of the normal form, the recent attempt has been made on the simplification of the normal forms. Baider and Sanders [18] made further reduction to the simplified Takens–Bogdanov normal form. New linear grading functions were introduced by Kokubu et al. [19] to study further reduction of the normal forms. Based on their work, Wang et al. [20] solved a special case of the remaining problem in the paper of Baider et al. [18]. Li et al. [21] employed the method of more Lie brackets to investigate the general form of the simplest normal form of Bogdanov–Takens singularities.

In this paper, a new and efficient computational method is developed for high dimensional non-linear systems for the first time and the general computing formula is derived. This new method is based on the adjoint operator method, the use of which leads directly to the explicit expressions of coefficients of the normal form. Compared with few existing methods to directly compute coefficients of the normal forms, the advantage of this newly developed method is that it is easy to apply to engineering applications, and the final partial differential equations of various resonant cases appear in a canonical form which is convenient to solve using the same main Maple symbolic programs. Based on the Maple symbolic programs developed here, the new method is applied to non-planar non-linear oscillations of a cantilever beam under combined parametric and forcing excitations in two resonant cases to obtain the normal forms of the averaged equations. For these two resonant cases, there, respectively, exist two pairs of pure imaginary eigenvalues as well as one non-semisimple double zero and a pair of pure imaginary eigenvalues in the averaged equations.

2. Normal form of non-linear system and adjoint operator method

Consider the non-linear system described by

$$\dot{x} = X(x) = Ax + f(x), \quad x \in \mathbf{R}^n, \tag{1a}$$

or

$$\dot{x} = X(x) = Ax + f^2(x) + f^3(x) + \cdots + f^k(x), \tag{1b}$$

where $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$, A is a $n \times n$ Jordan matrix, $f(x) = (f^2(x), f^3(x), \ldots, f^k(x))^{\mathrm{T}}$ is a vector in which the terms are formal power series for each degree$\geqslant 2$, $X(0) = 0$. We also write $f(x) = \sum_{k \geqslant 2} f^k(x)$, where $f^k(x)$ is a vector of homogeneous polynomials of k-degree, and $f^k(x) \in H_n^k$ which represents the linear space of k-degree homogeneous polynomials in n variables.

We use a series of near identity non-linear transformations to reduce non-linear system (1) to a simpler form, that is, the normal form. Assuming that a near identity non-linear transformation is of the form

$$x = y + P^k(y), \quad P^k(y) \in H_n^k, \tag{2}$$

where

$$P^k(x) = (P_1^k(x), P_2^k(x), \ldots, P_n^k(x))^{\mathrm{T}}. \tag{3}$$

Then, we have

$$\dot{x} = (I + DP^k(y))\dot{y}, \tag{4}$$

and

$$(I + DP^k(y))^{-1} = I - DP^k(y) + O(\|y\|^{2k-2}), \tag{5}$$

where $DP^k(y)$ is the $n \times n$ Jacobian matrix

$$DP^k(y) = \begin{pmatrix} \dfrac{\partial P_1^k}{\partial y_1} & \dfrac{\partial P_1^y}{\partial y_2} & \cdots & \dfrac{\partial P_1^k}{\partial y_n} \\ \dfrac{\partial P_2^k}{\partial y_1} & \dfrac{\partial P_2^k}{\partial y_2} & \cdots & \dfrac{\partial P_2^k}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial P_n^k}{\partial y_1} & \dfrac{\partial P_n^k}{\partial y_2} & \cdots & \dfrac{\partial P_n^k}{\partial y_n} \end{pmatrix}. \tag{6}$$

Substituting Eqs. (2), (4) and (5) into Eq. (1) yields

$$\dot{y} = Ay + f^2(y) + \cdots + f^{k-1}(y) + \{f^k(y) - [DP^k(y)Ay - AP^k(y)]\} + O(\|y\|^{k+1}), \tag{7}$$

where

$$f^k(y + P^k(y)) = f^k(y) + O(\|y\|^{k+1}). \tag{8}$$

We define a linear operator as

$$ad_A^k : H_n^k \to H_n^k,$$

$$ad_A^k P^k(y) = DP^k(y)Ay - AP^k(y) = [Ay, P^k(y)], \quad P^k(y) \in H_n^k, \tag{9}$$

where $[\cdot,\cdot]$ is the Lie bracket, and the ad_A^k is called a homological operator. Let R^k be the image of ad_A^k, that is, $R^k = \mathrm{Im}\, ad_A^k$, and C^k be any complementary subspaces of R^k in H_n^k, $H_n^k = R^k \oplus C^k$. We assume $f^k(y) = h^k(y) + g^k(y)$, where $g^k(y) \in C^k$, $h^k(y) \in R^k$, and we may choose $P^k(y)$ such that

$$ad_A^k P^k(y) = h^k(y) = f^k(y) - g^k(y). \tag{10}$$

Eq. (10) is called a homological equation. Using a series of near identity non-linear transformations, then, Eq. (1) can be transformed after replacing y by x as

$$\dot{x} = Ax + g^2(x) + \cdots + g^N(x), \tag{11}$$

where $g^k(y) \in C^k$ for $k = 2, \ldots, N$.

We refer to Eq. (11) as the N-order normal form of non-linear system (1). The goal to compute its normal form is to choose a series of near identity non-linear transformations $P(x) = (P^2(x), P^3(x), \ldots, P^N(x))^{\mathrm{T}}$ such that non-linear system (1) will be in the form as simple as possible.

In the following, we give the procedure of computing the normal form with the aid of adjoint operator method [1]. From the above analysis, it is known that the key of computing the normal forms is to find C^k and a basis of C^k for $k = 2, \ldots, N$. The following analysis shows how to find a vertical complementary subspace of $\mathrm{Im}\, ad_A^k$ in H_n^k.

Assume that V is a finite dimensional inner product space, L is a linear operator in V and L^* is the adjoint operator of L. Then, we have

$$(1)\ \mathrm{Ker}\, L^* = (\mathrm{Im}\, L)^{\perp}, \quad (2)\ V = \mathrm{Im}\, L \oplus \mathrm{Ker}\, L^*, \tag{12}$$

where $\mathrm{Ker}\, L^*$ is the null space of L^*. The proof of the above result is given in Ref. [22].

It is known that if we may find the adjoint operator $(ad_A^k)^*$ of the linear operator ad_A^k, $\mathrm{Ker}(ad_A^k)^*$ is a vertical complementary subspace of $\mathrm{Im}\, ad_A^k$ in H_n^k. Based on the analysis given in Refs. [1,4], it is known that the operator $ad_{A^*}^k$ is the adjoint operator of ad_A^k, that is, $ad_{A^*}^k = (ad_A^k)^*$, where $A^* = \bar{A}^{\mathrm{T}}$ is the adjoint transposed matrix of A. Then, there is

$$ad_{A^*}^k P^k(x) = DP^k(x)A^* x - A^* P^k(x). \tag{13}$$

Therefore, it is found that $\mathrm{Ker}\, ad_{A^*}^k$ is a vertical complementary subspace of $\mathrm{Im}\, ad_A^k$ in H_n^k, i.e., $H_n^k = \mathrm{Im}\, ad_A^k \oplus \mathrm{Ker}\, ad_{A^*}^k$. It is clear that $\mathrm{Ker}\, ad_{A^*}^k$ is a subspace which consists of all k order vector polynomial solutions in n variables for the linear partial differential equation

$$ad_{A^*}^k P^k(x) = 0, \quad P^k(x) \in H_n^k, \tag{14}$$

or

$$DP^k(x)A^* x - A^* P^k(x) = 0. \tag{15}$$

Note that the differential equation (15) has the same form for different k, and therefore the computation of any order normal forms of non-linear systems will be identical.

3. Computation of normal forms and their coefficients

In engineering problems, up to now, we can only investigate the global bifurcations and chaotic dynamics of four-dimensional non-linear systems with the cubic terms by using the analytical approaches, for example, the global perturbation approach developed by Kovacic and Wiggins [23], and the energy-phase method given by Haller and Wiggins [24]. For higher-dimensional non-linear systems, it is very difficult to analytically treat the global bifurcations and chaotic dynamics. Therefore, let us focus on the four-dimensional generalized averaged systems with $Z_2 \oplus Z_2$-symmetry, which only involve 3 order non-linear terms ($k = 3$)

$$\dot{x} = X(x) = Ax + f^3(x), \quad x \in \mathbf{R}^4, \tag{16}$$

where $f^3(x) \in H_4^3$,

$$\begin{aligned} f^3(x) &= (f_1^3(x), f_2^3(x), f_3^3(x), f_4^3(x))^{\mathrm{T}} \\ &= \Bigg(\sum_{|m|=3} a_{m_1 m_2 m_3 m_4} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}, \sum_{|m|=3} b_{m_1 m_2 m_3 m_4} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}, \\ &\quad \sum_{|m|=3} c_{m_1 m_2 m_3 m_4} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}, \sum_{|m|=3} d_{m_1 m_2 m_3 m_4} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} \Bigg)^{\mathrm{T}} \end{aligned} \tag{17}$$

and $|m| = m_1 + m_2 + m_3 + m_4$.

In this case, Eq. (15) can become as

$$DP^3(x)A^* x - A^* P^3(x) = 0. \tag{18}$$

When the Jordan matrix A and the non-linear terms $f^3(x)$ in generalized averaged equations (16) are known, we are able to obtain, based on Eq. (18), the normal forms of system (16) and the coefficients of the normal forms associated with the coefficients of Eq. (16).

Without loss of generality, the three cases of the Jordan matrix A in four-dimensional non-linear systems are considered as follows:

(1) The Jordan matrix A has two pairs of pure imaginary eigenvalues;
(2) The Jordan matrix A has one non-semisimple double zero and a pair of pure imaginary eigenvalues;
(3) The Jordan matrix A has two non-semisimple double zero eigenvalues.

The forms of the Jordan matrix A in the aforementioned three cases can be represented as

$$A = \begin{bmatrix} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & \omega_2 & 0 \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \tag{19}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \tag{20}$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}. \tag{21}$$

Knowing the form of the Jordan matrix A, we may obtain a set of linear partial differential equations from Eq. (18) and find 3 order polynomial solutions in 4 variables. To achieve this, express DP^3 as

$$DP^3 = \left\{ \frac{\partial P^3}{\partial x} \right\}_{4\times 4} = \begin{bmatrix} \frac{\partial P_1^3}{\partial x_1} & \frac{\partial P_1^3}{\partial x_2} & \frac{\partial P_1^3}{\partial x_3} & \frac{\partial P_1^3}{\partial x_4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial P_4^3}{\partial x_1} & \frac{\partial P_4^3}{\partial x_2} & \frac{\partial P_4^3}{\partial x_3} & \frac{\partial P_4^3}{\partial x_4} \end{bmatrix}. \tag{22}$$

Considering the first case, we have

$$A^* = \begin{bmatrix} J_1^* & 0 \\ 0 & J_2^* \end{bmatrix} = \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix}. \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (18) yields

$$\begin{bmatrix} \frac{\partial P_1^3}{\partial x_1} & \frac{\partial P_1^3}{\partial x_2} & \frac{\partial P_1^3}{\partial x_3} & \frac{\partial P_1^3}{\partial x_4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial P_4^3}{\partial x_1} & \frac{\partial P_4^3}{\partial x_2} & \frac{\partial P_4^3}{\partial x_3} & \frac{\partial P_4^3}{\partial x_4} \end{bmatrix} \begin{bmatrix} J_1^* & 0 \\ 0 & J_2^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} J_1^* & 0 \\ 0 & J_2^* \end{bmatrix} \begin{bmatrix} P_1^3 \\ P_2^3 \\ P_3^3 \\ P_4^3 \end{bmatrix} = 0. \tag{24}$$

Simplifying the above equation, we obtain

$$\omega_1 x_2 \frac{\partial P_1^3}{\partial x_1} - \omega_1 x_1 \frac{\partial P_1^3}{\partial x_2} + \omega_2 x_4 \frac{\partial P_1^3}{\partial x_3} - \omega_2 x_3 \frac{\partial P_1^3}{\partial x_4} - \omega_1 P_2^3 = 0, \tag{25a}$$

$$\omega_1 x_2 \frac{\partial P_2^3}{\partial x_1} - \omega_1 x_1 \frac{\partial P_2^3}{\partial x_2} + \omega_2 x_4 \frac{\partial P_2^3}{\partial x_3} - \omega_2 x_3 \frac{\partial P_2^3}{\partial x_4} + \omega_1 P_1^3 = 0, \tag{25b}$$

$$\omega_1 x_2 \frac{\partial P_3^3}{\partial x_1} - \omega_1 x_1 \frac{\partial P_3^3}{\partial x_2} + \omega_2 x_4 \frac{\partial P_3^3}{\partial x_3} - \omega_2 x_3 \frac{\partial P_3^3}{\partial x_4} - \omega_2 P_4^3 = 0, \tag{25c}$$

$$\omega_1 x_2 \frac{\partial P_4^3}{\partial x_1} - \omega_1 x_1 \frac{\partial P_4^3}{\partial x_2} + \omega_2 x_4 \frac{\partial P_4^3}{\partial x_3} - \omega_2 x_3 \frac{\partial P_4^3}{\partial x_4} + \omega_2 P_3^3 = 0. \tag{25d}$$

For the second case, we have

$$A^* = \begin{bmatrix} J_1^* & 0 \\ 0 & J_2^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}. \tag{26}$$

Substituting Eqs. (22) and (26) into Eq. (18) and simplifying the equation obtained here yields

$$x_1 \frac{\partial P_1^3}{\partial x_2} + \omega x_4 \frac{\partial P_1^3}{\partial x_3} - \omega x_3 \frac{\partial P_1^3}{\partial x_4} = 0, \tag{27a}$$

$$x_1\frac{\partial P_2^3}{\partial x_2}+\omega x_4\frac{\partial P_2^3}{\partial x_3}-\omega x_3\frac{\partial P_2^3}{\partial x_4}-P_1^3=0, \tag{27b}$$

$$x_1\frac{\partial P_3^3}{\partial x_2}+\omega x_4\frac{\partial P_3^3}{\partial x_3}-\omega x_3\frac{\partial P_3^3}{\partial x_4}-\omega P_4^3=0, \tag{27c}$$

$$x_1\frac{\partial P_4^3}{\partial x_2}+\omega x_4\frac{\partial P_4^3}{\partial x_3}-\omega x_3\frac{\partial P_4^3}{\partial x_4}+\omega P_3^3=0. \tag{27d}$$

Finally, the third case leads to

$$A^*=\begin{bmatrix} J_1^* & 0 \\ 0 & J_2^* \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{28}$$

Similarly, we obtain

$$x_1\frac{\partial P_1^3}{\partial x_2}+x_3\frac{\partial P_1^3}{\partial x_4}=0, \tag{29a}$$

$$x_1\frac{\partial P_2^3}{\partial x_2}+x_3\frac{\partial P_2^3}{\partial x_4}-P_1^3=0, \tag{29b}$$

$$x_1\frac{\partial P_3^3}{\partial x_2}+x_3\frac{\partial P_3^3}{\partial x_4}=0, \tag{29c}$$

$$x_1\frac{\partial P_4^3}{\partial x_2}+x_3\frac{\partial P_4^3}{\partial x_4}-P_3^3=0. \tag{29d}$$

To compute 3 order normal forms of Eq. (16), we only need to find all 3 order polynomial solutions of the sets of partial differential Eqs. (25), (27) and (29). In Ref. [1], the characteristic system of the set of partial differential equations and the independent first integrals of the characteristic system were used to find the polynomial solutions of the set of partial differential equations. In this paper, a different method is developed to obtain all 3 order polynomial solutions of the sets of partial differential equations. To achieve this, we introduce 20 monomials in the linear space H_4^3 as follows:

$$\begin{aligned}X=[&x_1^3,x_2^3,x_3^3,x_4^3,x_1^2x_2,x_1^2x_3,x_1^2x_4,x_2^2x_1,x_2^2x_3,x_2^2x_4,x_3^2x_1,x_3^2x_2,\\ &x_3^2x_4,x_4^2x_1,x_4^2x_2,x_4^2x_3,x_1x_2x_3,x_1x_2x_4,x_1x_3x_4,x_2x_3x_4]=\{x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}\}.\end{aligned} \tag{30}$$

It is clear that any 3 degree polynomials in H_4^3 can be represented by the combination of these monomials in Eq. (30).

Utilizing Eq. (30), non-linear term $f^3(x)$ in Eq. (16) can be rewritten as

$$f^3(x)=[C_{4\times 20}]X^{\mathrm{T}}, \tag{31}$$

where the matrix $[C_{4\times 20}]$ is determined by generalized averaged Eq. (16), X^{T} is the transposed matrix of X.

In addition, based on Eq. (3), let

$$
\begin{aligned}
P^3 &= [P_1^3, P_2^3, P_3^3, P_4^3]^{\mathrm{T}} \\
&= \left[\sum_{|m|=3} d_{1m}x^m, \sum_{|m|=3} d_{2m}x^m, \sum_{|m|=3} d_{3m}x^m, \sum_{|m|=3} d_{4m}x^m\right]^{\mathrm{T}} = [D_{4\times 20}]X^{\mathrm{T}},
\end{aligned} \tag{32}
$$

where $m = m_1m_2m_3m_4$, $x^m = x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$, and $[D_{4\times 20}]$ represents a matrix which involves 80 unknown coefficients.

Based on the different A given in Eqs. (19)–(21), substituting Eq. (32) into Eqs. (25), (27) and (29) respectively, we may obtain the three sets of 3-degree non-linear algebraic equations with respect to $x \in \mathbf{R}^4$.

(1) For the case of two pairs of pure imaginary eigenvalues, we have

$$\left(\omega_1 x_2 \frac{\partial}{\partial x_1} - \omega_1 x_1 \frac{\partial}{\partial x_2} + \omega_2 x_4 \frac{\partial}{\partial x_3} - \omega_2 x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{1m}x^m - \omega_1 \sum_{|m|=3} d_{2m}x^m = 0, \tag{33a}$$

$$\left(\omega_1 x_2 \frac{\partial}{\partial x_1} - \omega_1 x_1 \frac{\partial}{\partial x_2} + \omega_2 x_4 \frac{\partial}{\partial x_3} - \omega_2 x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{2m}x^m + \omega_1 \sum_{|m|=3} d_{1m}x^m = 0, \tag{33b}$$

$$\left(\omega_1 x_2 \frac{\partial}{\partial x_1} - \omega_1 x_1 \frac{\partial}{\partial x_2} + \omega_2 x_4 \frac{\partial}{\partial x_3} - \omega_2 x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{3m}x^m - \omega_2 \sum_{|m|=3} d_{4m}x^m = 0, \tag{33c}$$

$$\left(\omega_1 x_2 \frac{\partial}{\partial x_1} - \omega_1 x_1 \frac{\partial}{\partial x_2} + \omega_2 x_4 \frac{\partial}{\partial x_3} - \omega_2 x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{4m}x^m + \omega_2 \sum_{|m|=3} d_{3m}x^m = 0. \tag{33d}$$

(2) For the case of one non-semisimple double zero and a pair of pure imaginary eigenvalues, we obtain

$$\left(x_1 \frac{\partial}{\partial x_2} + \omega x_4 \frac{\partial}{\partial x_3} - \omega x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{1m}x^m = 0, \tag{34a}$$

$$\left(x_1 \frac{\partial}{\partial x_2} + \omega x_4 \frac{\partial}{\partial x_3} - \omega x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{2m}x^m - \sum_{|m|=3} d_{1m}x^m = 0, \tag{34b}$$

$$\left(x_1 \frac{\partial}{\partial x_2} + \omega x_4 \frac{\partial}{\partial x_3} - \omega x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{3m}x^m - \omega \sum_{|m|=3} d_{4m}x^m = 0, \tag{34c}$$

$$\left(x_1 \frac{\partial}{\partial x_2} + \omega x_4 \frac{\partial}{\partial x_3} - \omega x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{4m}x^m + \omega \sum_{|m|=3} d_{3m}x^m = 0. \tag{34d}$$

(3) For the case of two non-semisimple double zero eigenvalues, we have

$$\left(x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}\right) \sum_{|m|=3} d_{1m}x^m = 0, \tag{35a}$$

$$\left(x_1\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{2m}x^m-\sum_{|m|=3} d_{1m}x^m=0, \tag{35b}$$

$$\left(x_1\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{3m}x^m=0, \tag{35c}$$

$$\left(x_1\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{4m}x^m-\sum_{|m|=3} d_{3m}x^m=0. \tag{35d}$$

Balancing the coefficients of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}(\sum_i^4 m_i=3)$ on the left side and the right side of Eqs. (33)–(35), the three sets of linear algebraic equations, each of which respectively involves 80 equations, can be obtained. Solving each set of 80-dimensional linear algebraic equations, we are able to present three bases of $\operatorname{Ker} ad^k_{A*}$ in H_4^3 and three matrices $[\bar{D}_{4\times 20}]_j$ $(j=1,2,3)$ in which all coefficients of matrices $[D_{4\times 20}]$ are determined. Therefore, we have

$$g^3(x)=[\bar{D}_{4\times 20}]_j X^{\mathrm{T}},\quad j=1,2,3. \tag{36}$$

Next, substituting Eq. (32) into Eq. (9) $(k=3,\ n=4)$ yields the following three equations.

(1) For the case of two pairs of pure imaginary eigenvalues, we obtain

$$ad^3_{\mathrm{A}}P^3(x)=DP^3(x)Ax-AP^3(x)$$
$$=\begin{pmatrix}\left(-\omega_1x_2\frac{\partial}{\partial x_1}+\omega_1x_1\frac{\partial}{\partial x_2}-\omega_2x_4\frac{\partial}{\partial x_3}+\omega_2x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{1m}x^m+\omega_1\sum_{|m|=3} d_{2m}x^m\\ \left(-\omega_1x_2\frac{\partial}{\partial x_1}+\omega_1x_1\frac{\partial}{\partial x_2}-\omega_2x_4\frac{\partial}{\partial x_3}+\omega_2x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{2m}x^m-\omega_1\sum_{|m|=3} d_{1m}x^m\\ \left(-\omega_1x_2\frac{\partial}{\partial x_1}+\omega_1x_1\frac{\partial}{\partial x_2}-\omega_2x_4\frac{\partial}{\partial x_3}+\omega_2x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{3m}x^m+\omega_2\sum_{|m|=3} d_{4m}x^m\\ \left(-\omega_1x_2\frac{\partial}{\partial x_1}+\omega_1x_1\frac{\partial}{\partial x_2}-\omega_2x_4\frac{\partial}{\partial x_3}+\omega_2x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{4m}x^m-\omega_2\sum_{|m|=3} d_{3m}x^m\end{pmatrix}. \tag{37}$$

(2) For the case of one non-semisimple double zero and a pair of pure imaginary eigenvalues, we have

$$ad^3_{\mathrm{A}}P^3(x)=DP^3(x)Ax-AP^3(x)$$
$$=\begin{pmatrix}\left(x_2\frac{\partial}{\partial x_1}-\omega x_4\frac{\partial}{\partial x_3}+\omega x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{1m}x^m-\omega_1\sum_{|m|=3} d_{2m}x^m\\ \left(x_2\frac{\partial}{\partial x_1}-\omega x_4\frac{\partial}{\partial x_3}+\omega x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{2m}x^m\\ \left(x_2\frac{\partial}{\partial x_1}-\omega x_4\frac{\partial}{\partial x_3}+\omega x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{3m}x^m+\omega\sum_{|m|=3} d_{4m}x^m\\ \left(x_2\frac{\partial}{\partial x_1}-\omega x_4\frac{\partial}{\partial x_3}+\omega x_3\frac{\partial}{\partial x_4}\right)\sum_{|m|=3} d_{4m}x^m-\omega\sum_{|m|=3} d_{3m}x^m\end{pmatrix}. \tag{38}$$

(3) For the case of two non-semisimple double zero eigenvalues, we obtain

$$ad_{\mathrm{A}}^3 P^3(x) = DP^3(x)Ax - AP^3(x)$$
$$= \begin{pmatrix} \left(x_2\dfrac{\partial}{\partial x_1} + x_4\dfrac{\partial}{\partial x_3}\right)\sum_{|m|=3} d_{1m}x^m - \sum_{|m|=3} d_{2m}x^m \\ \left(x_2\dfrac{\partial}{\partial x_1} + x_4\dfrac{\partial}{\partial x_3}\right)\sum_{|m|=3} d_{2m}x^m \\ \left(x_2\dfrac{\partial}{\partial x_1} + x_4\dfrac{\partial}{\partial x_3}\right)\sum_{|m|=3} d_{3m}x^m - \sum_{|m|=3} d_{4m}x^m \\ \left(x_2\dfrac{\partial}{\partial x_1} + x_4\dfrac{\partial}{\partial x_3}\right)\sum_{|m|=3} d_{4m}x^m \end{pmatrix}. \quad (39)$$

Simplifying Eqs. (37)–(39) leads to

$$h^3(x) = [\bar{K}_{4\times 20}]_j X^{\mathrm{T}}, \quad j = 1,2,3. \quad (40)$$

Finally, substitution of Eqs. (31), (36) and (40) into Eq. (10) ($k = 3,\ n = 4$) results in

$$[C_{4\times 20}]X^{\mathrm{T}} = [\bar{D}_{4\times 20}]_j X^{\mathrm{T}} + [\bar{K}_{4\times 20}]_j X^{\mathrm{T}}, \quad j = 1,2,3. \quad (41)$$

Balancing the coefficients of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ ($\sum_i^4 m_i = 3$) on the left side and the right hand side of Eq. (41), a set of linear algebra equations, which involves 80 equations, can be obtained as

$$[C_{4\times 20}] = [\bar{D}_{4\times 20}]_j + [\bar{K}_{4\times 20}]_j, \quad j = 1,2,3. \quad (42)$$

Solving Eq. (42), we may obtain the coefficients of the normal forms associated with the coefficients of the generalized averaged Eqs. (16) and near identity non-linear transformation.

The deriving procedure and formulae given in the aforementioned analysis can be directly utilized to obtain Maple symbolic computation program. The outline of the program is given as follows.

1. Based on the original equation, create the matrix $[C_{4\times 20}]$.
2. Give near identity non-linear transformation $P^3(x) = [D_{4\times 20}]X^{\mathrm{T}}$.
3. Substitute A^*, $P^3(x)$ and $DP^3(x)$ into Eq. (18). Balance the coefficients of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ ($\sum_i^4 m_i = 3$) and get the matrices $[\bar{D}_{4\times 20}]_j$ ($j = 1,2,3$).
4. Obtain $g^3(x) = [\bar{D}_{4\times 20}]_j X^{\mathrm{T}},\ j = 1,2,3$.
5. Substitute A, $P^3(x)$ and $DP^3(x)$ into Eq. (9) and get $h^3(x) = [\bar{K}_{4\times 20}]_j X^{\mathrm{T}},\ j = 1,2,3$.
6. Balance the coefficients of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ ($\sum_i^4 m_i = 3$) on the left side and the right hand side of Eq. (41) and solve Eq. (42).

The Maple source codes for computing the normal forms by applying the above procedures are presented in Appendix A.

In the original work on the adjoint method, Elphick et al. [1] did not give the explicit formulae of the relationship between the coefficients of the original system and those of the normal form. They did not also demonstrate how to compute the coefficients of the normal form in terms of the coefficients of the original system. Sri Namachchivaya et al. [16] obtained the 4 leading complex coefficients of the normal form by using the coefficients of the original system. They did not give all coefficients in the normal form. Comparing the method developed here with other methods given in Refs. [9,17,25], it is observed that we may, respectively, obtain the normal forms, the coefficients of the normal forms and the associated near identity non-linear transformations for three resonant cases by using the same main Maple symbolic program. Therefore, it is more convenient to utilize the approach developed here to compute the normal form of the averaged equations for different resonant cases. It is also found that in Refs. [9,17] the researchers only investigated the cases in which the center manifold is four-dimensional. In addition, Leung and Zhang only computed the normal forms of non-linear systems with two-, three- and four-dimensional center manifolds by using the Mathematica language in paper [25]. They did not calculate the normal form on center manifold of dimensional number > 4.

In the next section, the two examples will be employed for non-planar non-linear oscillations of a cantilever beam to demonstrate the procedure of computing the normal forms with the aid of the method developed above and the Maple program.

4. Application to non-planar motions of a cantilever beam

In order to conveniently investigate the global bifurcations and chaotic dynamics in non-planar non-linear oscillations of a cantilever beam under combined parametric and forcing excitations, we need to reduce the averaged equations for non-planar non-linear oscillations of the cantilever beam to a simpler normal form.

We consider a cantilever beam with length L, and mass m per unit length subjected to a harmonic axial excitation at free end. Assume that the beam considered here is the Euler–Bernoulli beam. A Cartesian co-ordinate system, $Oxyz$, is adopted which is located in the symmetric plane of the cantilever beam. The s denotes the curve co-ordinate along the elastic axis before deformation. ξ, η and ς are the principal axes of the cross-section for the cantilever beam at position s. The symbols $v(s,t)$ and $w(s,t)$ denote the displacements of a point in the middle line of the cantilever beam in the y and z directions, respectively. The harmonic axial excitation may be expressed in the form $2F_1\cos\Omega_1 t$. The transverse excitations in the y and z directions are represented in the forms $2F_2(s)\cos\Omega_2 t$ and $2F_3(s)\cos\Omega_2 t$, respectively. The non-dimensional governing equations of non-planar non-linear motion for the cantilever beam under combined parametric and forcing excitations are of the following form [26]:

$$\begin{aligned}\ddot{v} + c\dot{v} + \beta_y v^{iv} + F_1\cos(\Omega_1 t)v'' = {} & (1-\beta_y)\left[w''\int_1^s v''w''\,\mathrm{d}s - w'''\int_0^s v''w'\,\mathrm{d}s\right]' \\ & - \frac{1}{\beta_\gamma}(1-\beta_y)^2\left[w''\int_0^s\int_1^s v''w''\,\mathrm{d}s\,\mathrm{d}s\right]'' - \beta_y[v'(v'v''+w'w'')']' \\ & - \frac{1}{2}\left[v'\int_1^s \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\int_0^s (v'^2+w'^2)\,\mathrm{d}s\right\}\mathrm{d}s\right]' - F_1\cos(\Omega_1 t)[v'(v'^2+w'^2)]' \\ & + F_2(s)\cos(\Omega_2 t), \end{aligned} \tag{43a}$$

$$\begin{aligned}\ddot{w} + c\dot{w} + w^{iv} + F_1\cos(\Omega_1 t)w'' = {} & -(1-\beta_y)\left[v''\int_1^s v''w''\,\mathrm{d}s - v'''\int_0^s w''v'\,\mathrm{d}s\right]' \\ & - \frac{1}{\beta_\gamma}(1-\beta_y)^2\left[v''\int_0^s\int_1^s v''w''\,\mathrm{d}s\,\mathrm{d}s\right]'' - [w'(v'v''+w'w'')']' \end{aligned}$$

$$-\frac{1}{2}\left[w'\int_1^s \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\int_0^s (v'^2+w'^2)\,\mathrm{d}s\right\}\mathrm{d}s\right]' - F_1\cos(\Omega_1 t)[w'(v'^2+w'^2)]'$$
$$+\ F_3(s)\cos(\Omega_2 t), \tag{43b}$$

where the dots and primes, respectively, represent partial differentiation with respect to t and x, c is the damping coefficient, and β_y is the ratio between the in-plane and out-of-plane principal flexural stiffnesses, that is, $\beta_y = D_\varsigma/D_\eta$.

The boundary conditions are

$$v(0,t) = w(0,t) = v'(0,t) = w'(0,t) = 0, \tag{44a}$$

$$v''(1,t) = w''(1,t) = v'''(1,t) = w'''(1,t) = 0. \tag{44b}$$

In the following analysis, we apply the Galerkin procedure to Eq. (43) to obtain a two-degree-of-freedom (d.o.f.) non-linear system with parametric and forcing excitations. A planar and a non-planar flexural mode for the cantilever beam are considered as

$$v(s,t) = y(t)G(s), \tag{45a}$$

$$w(s,t) = z(t)G(s), \tag{45b}$$

where the function $G(s)$ is a mode of the transverse free vibration for linear cantilever beam and is of the following form:

$$G(s) = \cosh(rs) - \cos(rs) - [(\cosh(r)+\cos(r))/(\sinh(r)+\sin(r))][\sinh(rs)-\sin(rs)]. \tag{46}$$

The linear mode $G(s)$ satisfies the differential equation

$$G'''' - r^4 G = 0, \tag{47}$$

and

$$G(0) = G'(0) = G''(1) = G'''(1) = 0. \tag{48}$$

r is determined by the characteristic equation

$$\cosh(r)\cos(r) + 1 = 0. \tag{49}$$

Introduce the time variable $\hat{t} = r^2 t$. For convenience of the following analysis, we drop the hat. Substituting Eq. (45) into Eq. (43), multiplying Eq. (43) by $G(s)$ and integrating to s from 0 to 1, a 2-d.o.f. non-linear system with parametric and forcing excitations is obtained as

$$\ddot{y} + \beta_y y = -\hat{c}\dot{y} + 2\alpha_1 F_1\cos(\Omega_1 t)y - \alpha_2 y(y\ddot{y} + \dot{y}^2 + z\ddot{z} + \dot{z}^2) - \alpha_3\beta_y y^3$$
$$-\left[\beta_y\alpha_3 + (1-\beta_y)\alpha_4 - \frac{1}{\beta_y}(1-\beta_y)^2\alpha_5\right]yz^2 + 2\alpha_6 F_1\cos(\Omega_1 t)(y^3 + yz^2) + f_1\cos\Omega t, \tag{50a}$$

$$\ddot{z} + z = -\hat{c}\dot{z} + 2\alpha_1 F_1\cos(\Omega t)z - \alpha_2 z(y\ddot{y} + \dot{y}^2 + z\ddot{z} + \dot{z}^2) - \alpha_3 z^3$$
$$+\left[(1-\beta_y)\alpha_4 + \frac{1}{\beta_y}(1-\beta_y)^2\alpha_5 - \beta_y\alpha_3\right]zy^2 + 2\alpha_6 F_1\cos(\Omega_1 t)(z^3 + zy^2) + f_2\cos\Omega t, \tag{50b}$$

where the dots denote partial differentiation with respect to $\hat{t}$, and

$$\hat{c} = \frac{c}{r^2},\ \alpha_1 = -\frac{1}{r^4}\int_0^1 GG''\,\mathrm{d}s,\ \alpha_2 = \int_0^1 G\left[G'\int_1^s\int_0^s G'^2\,\mathrm{d}s\,\mathrm{d}s\right]'\mathrm{d}s,\ \alpha_3 = \frac{1}{r^4}\int_0^1 G[G'(G'G'')']'\mathrm{d}s,$$

$$\alpha_4 = -\frac{1}{r^4}\int_0^1 G\left[G''\int_1^s G''^2\,\mathrm{d}s - G'''\int_0^s G'G''\,\mathrm{d}s\right]'\mathrm{d}s,\ \alpha_5 = -\frac{1}{r^4}\int_0^1 G\left[G''\int_0^s\int_1^s G''^2\,\mathrm{d}s\,\mathrm{d}s\right]''\mathrm{d}s,$$

$$\alpha_6 = -\frac{1}{2r^4}\int_0^1 G(G'^3)'\,\mathrm{d}s,\ f_1 = \frac{1}{r^4}\int_0^1 GF_2\,\mathrm{d}s,\ f_2 = \frac{1}{r^4}\int_0^1 GF_3\,\mathrm{d}s. \quad (51)$$

In this section, we will use the results obtained above to give the normal forms of the averaged equations for non-planar non-linear oscillations of the cantilever beam under combined parametric and forcing excitations in two resonant cases. In other paper [27], based on the normal forms of the averaged equations for non-planar non-linear oscillations of the cantilever beam under combined parametric and forcing excitations presented here, we will analyze the global bifurcations and chaotic dynamics for non-planar non-linear oscillations of the cantilever beam.

4.1. An example for two pairs of pure imaginary eigenvalues

First, principal parametric resonance-1/2 subharmonic resonance and 1:1 internal resonance are considered. From Eq. (50), it is found that there is $\omega_2^2 = 1$. Therefore, when the ratio $\beta_y = \omega_1^2 \approx 1$, there is the relation of 1:1 internal resonance in Eq. (50). The resonant relations are expressed as

$$\Omega_2^2 = \Omega_1^2, \quad \beta_y = \omega_1^2 = \tfrac{1}{4}\Omega_1^2 + \varepsilon\sigma_1, \quad \omega_2^2 = \tfrac{1}{4}\Omega_1^2 + \varepsilon\sigma_2, \quad (52)$$

where σ_1 and σ_2 are two detuning parameters.

Using the method of multiple scales, the averaged equations for non-planar non-linear oscillations of the flexible cantilever beam under combined parametric and forcing excitations are of the form

$$\begin{aligned}\dot{x}_1 = &-\tfrac{1}{2}cx_1 - \tfrac{1}{2}(\sigma_1 + \alpha_1 F_1)x_2 + \tfrac{1}{8}(2\alpha_2 - 3\alpha_3)x_2(x_2^2 + x_1^2 + x_4^2)\\ &- \tfrac{1}{8}(2\alpha_2 + \alpha_3)x_2x_3^2 + (\tfrac{1}{2}\alpha_2 - \tfrac{1}{4}\alpha_3)x_1x_3x_4, \end{aligned} \quad (53a)$$

$$\begin{aligned}\dot{x}_2 = &-\tfrac{1}{2}cx_2 + \tfrac{1}{2}(\sigma_1 - \alpha_1 F_1)x_1 - \tfrac{1}{8}(2\alpha_2 - 3\alpha_3)x_1(x_2^2 + x_1^2 + x_3^2)\\ &+ \tfrac{1}{8}(2\alpha_2 + \alpha_3)x_1x_4^2 - (\tfrac{1}{2}\alpha_2 - \tfrac{1}{4}\alpha_3)x_2x_3x_4, \end{aligned} \quad (53b)$$

$$\begin{aligned}\dot{x}_3 = &-\tfrac{1}{2}cx_3 - \tfrac{1}{2}(\sigma_2 + \alpha_1 F_1)x_4 + \tfrac{1}{8}(2\alpha_2 - 3\alpha_3)x_4(x_4^2 + x_3^2 + x_2^2)\\ &- \tfrac{1}{8}(2\alpha_2 + \alpha_3)x_1^2x_4 + (\tfrac{1}{2}\alpha_2 - \tfrac{1}{4}\alpha_3)x_1x_2x_3, \end{aligned} \quad (53c)$$

$$\begin{aligned}\dot{x}_4 = &-\tfrac{1}{2}cx_4 + \tfrac{1}{2}(\sigma_2 - \alpha_1 F_1)x_3 - \tfrac{1}{8}(2\alpha_2 - 3\alpha_3)x_3(x_4^2 + x_3^2 + x_1^2)\\ &+ \tfrac{1}{8}(2\alpha_2 + \alpha_3)x_2^2x_3 - (\tfrac{1}{2}\alpha_2 - \tfrac{1}{4}\alpha_3)x_1x_2x_4. \end{aligned} \quad (53d)$$

It is noticed that the averaged Eqs. (53) have the $Z_2 \oplus Z_2$ and D_4 symmetries. Therefore, these symmetries are also held in the normal form. It is known that system (53) has a trivial zero solution $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ at which the Jacobian matrix can be written as

$$J = D_x X = \begin{bmatrix} -\frac{1}{2}c & -\frac{1}{2}(\sigma_1 + \alpha_1 F_1) & 0 & 0 \\ \frac{1}{2}(\sigma_1 - \alpha_1 F_1) & -\frac{1}{2}c & 0 & 0 \\ 0 & 0 & -\frac{1}{2}c & -\frac{1}{2}(\sigma_2 + \alpha_1 F_1) \\ 0 & 0 & \frac{1}{2}(\sigma_2 - \alpha_1 F_1) & -\frac{1}{2}c \end{bmatrix}. \tag{54}$$

The characteristic equation corresponding to the trivial zero solution is

$$(\lambda^2 + 2c\lambda + c^2 + \sigma_1^2 - f_0^2)(\lambda^2 + 2c\lambda + c^2 + \sigma_2^2 - f_0^2) = 0, \tag{55}$$

where $f_0 = \alpha_1 F_1$.

Let

$$\Delta_1 = c^2 + \sigma_1^2 - f_0^2, \quad \Delta_2 = c^2 + \sigma_2^2 - f_0^2. \tag{56}$$

When $c = 0$, $\Delta_1 = \sigma_1^2 - f_0^2 > 0$ and $\Delta_2 = \sigma_2^2 - f_0^2 > 0$ are simultaneously satisfied, system (53) has two pairs of pure imaginary eigenvalues

$$\lambda_{1,2} = \pm i\bar{\omega}_1, \quad \lambda_{3,4} = \pm i\bar{\omega}_2, \tag{57}$$

where $\bar{\omega}_1^2 = \sigma_1^2 - f_0^2$, $\bar{\omega}_2^2 = \sigma_2^2 - f_0^2$.

In the case of 1:1 internal resonance, there is the relation $\bar{\omega}_1 \approx \bar{\omega}_2$. Considering the excitation amplitude f_0 as a parameter, the averaged equation (53), which does not have the parameters, becomes

$$\begin{aligned}
\dot{x}_1 &= -\tfrac{1}{2}\sigma_1 x_2 + \tfrac{1}{8}(2\alpha_2 - 3\alpha_3)x_2(x_2^2 + x_1^2 + x_4^2) - \tfrac{1}{8}(2\alpha_2 + \alpha_3)x_2 x_3^2 + \tfrac{1}{2}(\alpha_2 - \tfrac{1}{2}\alpha_3)x_1 x_3 x_4, \\
\dot{x}_2 &= \tfrac{1}{2}\sigma_1 x_1 - \tfrac{1}{8}(2\alpha_2 - 3\alpha_3)x_1(x_2^2 + x_1^2 + x_3^2) + \tfrac{1}{8}(2\alpha_2 + \alpha_3)x_1 x_4^2 - \tfrac{1}{2}(\alpha_2 - \tfrac{1}{2}\alpha_3)x_2 x_3 x_4, \\
\dot{x}_3 &= -\tfrac{1}{2}\sigma_2 x_4 + \tfrac{1}{8}(2\alpha_2 - 3\alpha_3)x_4(x_4^2 + x_3^2 + x_2^2) - \tfrac{1}{8}(2\alpha_2 + \alpha_3)x_1^2 x_4 + \tfrac{1}{2}(\alpha_2 - \tfrac{1}{2}\alpha_3)x_1 x_2 x_3, \\
\dot{x}_4 &= \tfrac{1}{2}\sigma_2 x_3 - \tfrac{1}{8}(2\alpha_2 - 3\alpha_3)x_3(x_4^2 + x_3^2 + x_1^2) + \tfrac{1}{8}(2\alpha_2 + \alpha_3)x_2^2 x_3 - \tfrac{1}{2}(\alpha_2 - \tfrac{1}{2}\alpha_3)x_1 x_2 x_4.
\end{aligned} \tag{58}$$

System (58) may be rewritten as

$$\dot{x} = Ax + f^3(x), \tag{59}$$

where

$$A = \begin{pmatrix} 0 & -\frac{1}{2}\sigma_1 & 0 & 0 \\ \frac{1}{2}\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sigma_2 \\ 0 & 0 & \frac{1}{2}\sigma_2 & 0 \end{pmatrix}. \tag{60}$$

The adjoint transposed matrix of A is of the form

$$A^* = \begin{pmatrix} 0 & \frac{1}{2}\sigma_1 & 0 & 0 \\ -\frac{1}{2}\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sigma_2 \\ 0 & 0 & -\frac{1}{2}\sigma_2 & 0 \end{pmatrix}. \tag{61}$$

Executing the Maple program given in Appendix A, the 3 order normal form of the averaged equation (58) is obtained as

$$\dot{y}_1 = -\tfrac{1}{2}\sigma_1 y_2 + (\tfrac{1}{4}\alpha_2 - \tfrac{3}{8}\alpha_3)y_2(y_1^2 + y_2^2) - \tfrac{3}{16}\alpha_3 y_2(y_3^2 + y_4^2), \tag{62a}$$

$$\dot{y}_2 = \tfrac{1}{2}\sigma_1 y_1 - (\tfrac{1}{4}\alpha_2 - \tfrac{3}{8}\alpha_3)y_1(y_1^2 + y_2^2) + \tfrac{3}{16}\alpha_3 y_1(y_3^2 + y_4^2), \tag{62b}$$

$$\dot{y}_3 = -\tfrac{1}{2}\sigma_2 y_4 + (\tfrac{3}{16}\alpha_2 - \tfrac{9}{32}\alpha_3)y_4(y_3^2 + y_4^2) - \tfrac{1}{4}\alpha_3 y_4(y_1^2 + y_2^2), \tag{62c}$$

$$\dot{y}_4 = \tfrac{1}{2}\sigma_2 y_3 - (\tfrac{3}{16}\alpha_2 - \tfrac{9}{32}\alpha_3)y_3(y_3^2 + y_4^2) + \tfrac{1}{4}\alpha_3 y_3(y_1^2 + y_2^2). \tag{62d}$$

The non-linear transformation used in the above computing procedure is of the form

$$\begin{aligned} x_1 = {} & y_1 + \frac{1}{2}\frac{\sigma_1^2\sigma_2(2\alpha_2 - 3\alpha_3)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_1^2 y_3 + \frac{1}{2}\frac{\sigma_1^2\sigma_2(2\alpha_2 + \alpha_3)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_1^2 y_4 \\ & + \frac{1}{4}\frac{\sigma_2(2\alpha_2 - 3\alpha_3)(7\sigma_1^2 - \sigma_2^2)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_2^2 y_3 + \frac{1}{4}\frac{(7\sigma_1^2 - \sigma_2^2)(2\alpha_2 + \alpha_3)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_2^2 y_4 \\ & + \frac{(-4\sigma_1\alpha_2 - 3\sigma_2\alpha_3)(7\sigma_1^2 - \sigma_2^2)}{8\sigma_1\sigma_2} y_1 y_3^2 + \frac{(2\sigma_1\sigma_2\alpha_3 - 2\sigma_1^2\alpha_2 + 3\sigma_2^2\alpha_3 - 3\sigma_1^2\alpha_3)}{8\sigma_1(\sigma_1^2 - \sigma_2^2)} y_1 y_4^2 \\ & + \frac{\sigma_1(6\sigma_1^2\alpha_2 + 3\sigma_1^2\alpha_3 - \sigma_2^2\alpha_3 - 2\sigma_2^2\alpha_2)}{2(9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4)} y_1 y_2 y_3 - \frac{\sigma_1(3\sigma_1^2 - \sigma_2^2)(2\alpha_2 - 3\alpha_3)}{2(9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4)} y_1 y_2 y_4 \\ & + \frac{1}{4}\frac{\sigma_1(\sigma_1\alpha_2 - \sigma_2\alpha_3)}{\sigma_2(\sigma_1^2 - \sigma_2^2)} y_2 y_3 y_4, \end{aligned} \tag{63a}$$

$$\begin{aligned} x_2 = {} & y_2 - \frac{3}{2}\frac{\sigma_1^3(2\alpha_2 + \alpha_3)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_1^2 y_3 + \frac{3}{2}\frac{\sigma_1^3(2\alpha_2 - 3\alpha_3)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_1^2 y_4 \\ & - \frac{1}{4}\frac{\sigma_1(2\alpha_2 + \alpha_3)(3\sigma_1^2 - \sigma_2^2)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_2^2 y_3 + \frac{1}{4}\frac{\sigma_1(3\sigma_1^2 - \sigma_2^2)(2\alpha_2 - 3\alpha_3)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_2^2 y_4 \\ & + \frac{1}{4}\frac{(\sigma_2^2\alpha_3 - 2\sigma_1^2\alpha_2 + 2\sigma_2^2\alpha_2 - \sigma_1\sigma_2\alpha_2)}{\sigma_2(\sigma_1^2 - \sigma_2^2)} y_2 y_3^2 - \frac{\sigma_1^2\sigma_2(2\alpha_2 - 3\alpha_3)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_1 y_2 y_3 \\ & - \frac{\sigma_1^2\sigma_2(2\alpha_2 + \alpha_3)}{9\sigma_1^4 - 10\sigma_2^2\sigma_1^2 + \sigma_2^4} y_1 y_2 y_4 - \frac{1}{4}\frac{(\sigma_1\sigma_2\alpha_3 + \sigma_1^2\alpha_2 - 2\sigma_2^2\alpha_2)}{\sigma_2(\sigma_1^2 - \sigma_2^2)} y_1 y_3 y_4, \end{aligned} \tag{63b}$$

$$\begin{aligned} x_3 = {} & y_3 + \frac{(-2\alpha_2 + 3\alpha_3)}{8\sigma_2} y_3^3 + \frac{(2\alpha_2 - \alpha_3)}{4(\sigma_1 - \sigma_2)} y_2^2 y_3 \\ & + \frac{(-2\alpha_2 + 3\alpha_3)}{16\sigma_2} y_3 y_4^2 - \frac{(2\alpha_2 - \alpha_3)}{4(\sigma_1 - \sigma_2)} y_1 y_2 y_4, \end{aligned} \tag{63c}$$

$$x_4 = y_4 + \frac{1}{4}\frac{(2\alpha_2 - \alpha_3)}{\sigma_1 - \sigma_2} y_1^2 y_4 + \frac{1}{16}\frac{3(-2\alpha_2 + 3\alpha_3)}{\sigma_2} y_3^2 y_4 - \frac{1}{4}\frac{(2\alpha_2 - \alpha_3)}{\sigma_1 - \sigma_2} y_1 y_2 y_3. \tag{63d}$$

It is observed that normal form (62) is simpler than averaged equation (53). However, normal form (62) is topologically equivalent to averaged equation (53).

4.2. An example for a double zero and a pair of pure imaginary eigenvalues

In this section, we investigate the case of the ratio $\beta_y = \omega_1^2 \approx 1/4$. In this case, there is the relation of 1:2 internal resonance for Eq. (50). In addition, principal parametric resonance-1/2 subharmonic resonance for the first mode and fundamental parametric resonance–primary resonance for the second mode are considered. The resonant relations are represented as

$$\Omega_2^2 = \Omega_1^2, \quad \omega_1^2 = \beta_y = \tfrac{1}{4}\Omega_1^2 + \varepsilon\sigma_1, \quad 1 = \omega_2^2 = \Omega_1^2 + \varepsilon\sigma_2, \tag{64}$$

where σ_1 and σ_2 are two detuning parameters.

Using the method of multiple scales, the averaged equations for non-planar non-linear oscillations of the flexible cantilever beam under combined parametric and forcing excitations are of the form

$$\dot{x}_1 = -\tfrac{1}{2}cx_1 - \tfrac{1}{2}(\sigma_1 + \alpha_1 F_1)x_2 + \tfrac{1}{32}(2\alpha_2 - 3\alpha_3)x_2(x_2^2 + x_1^2) + \beta_1 x_2(x_3^2 + x_4^2), \tag{65a}$$

$$\dot{x}_2 = -\tfrac{1}{2}cx_2 + \tfrac{1}{2}(\sigma_1 - \alpha_1 F_1)x_1 - \tfrac{1}{32}(2\alpha_2 - 3\alpha_3)x_1(x_1^2 + x_2^2) - \beta_1 x_1(x_3^2 + x_4^2), \tag{65b}$$

$$\dot{x}_3 = -\tfrac{1}{2}cx_3 - \tfrac{1}{2}\sigma_2 x_4 + (\tfrac{1}{2}\alpha_2 - 3\alpha_3)x_4(x_4^2 + x_3^2) + 2\beta_2 x_4(x_2^2 + x_1^2), \tag{65c}$$

$$\dot{x}_4 = -f_2 - \tfrac{1}{2}cx_4 + \tfrac{1}{2}\sigma_2 x_3 - (\tfrac{1}{2}\alpha_2 - 3\alpha_3)x_3(x_3^2 + x_4^2) - 2\beta_2 x_3(x_1^2 + x_2^2), \tag{65d}$$

where $\beta_1 = -\frac{3}{4}\alpha_4 + \frac{9}{4}\alpha_5 - \frac{1}{4}\alpha_3$, $\beta_2 = \frac{3}{4}\alpha_4 + \frac{9}{4}\alpha_5 - \alpha_3$.

Take into account the exciting amplitude f_2 as a perturbation parameter. Amplitude f_2 can be considered as an unfolding parameter when the global bifurcations are investigated. Obviously, when we do not consider the perturbation parameter, Eq. (65) becomes

$$\dot{x}_1 = -\tfrac{1}{2}cx_1 - \tfrac{1}{2}(\sigma_1 + \alpha_1 F_1)x_2 + \tfrac{1}{32}(2\alpha_2 - 3\alpha_3)x_2(x_2^2 + x_1^2) + \beta_1 x_2(x_3^2 + x_4^2), \tag{66a}$$

$$\dot{x}_2 = -\tfrac{1}{2}cx_2 + \tfrac{1}{2}(\sigma_1 - \alpha_1 F_1)x_1 - \tfrac{1}{32}(2\alpha_2 - 3\alpha_3)x_1(x_1^2 + x_2^2) - \beta_1 x_1(x_3^2 + x_4^2), \tag{66b}$$

$$\dot{x}_3 = -\tfrac{1}{2}cx_3 - \tfrac{1}{2}\sigma_2 x_4 + (\tfrac{1}{2}\alpha_2 - 3\alpha_3)x_4(x_4^2 + x_3^2) + 2\beta_2 x_4(x_2^2 + x_1^2), \tag{66c}$$

$$\dot{x}_4 = -\tfrac{1}{2}cx_4 + \tfrac{1}{2}\sigma_2 x_3 - (\tfrac{1}{2}\alpha_2 - 3\alpha_3)x_3(x_3^2 + x_4^2) - 2\beta_2 x_3(x_1^2 + x_2^2). \tag{66d}$$

Eq. (66) has a trivial zero solution $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ at which the Jacobian matrix can be represented as

$$J = D_x X = \begin{bmatrix} -\frac{1}{2}c & -\frac{1}{2}(\sigma_1 + f_0) & 0 & 0 \\ \frac{1}{2}(\sigma_1 - f_0) & -\frac{1}{2}c & 0 & 0 \\ 0 & 0 & -\frac{1}{2}c & -\frac{1}{2}\sigma_2 \\ 0 & 0 & \frac{1}{2}\sigma_2 & -\frac{1}{2}c \end{bmatrix}, \tag{67}$$

where $f_0 = \alpha_1 F_1$.

The characteristic equation corresponding to the trivial zero solution is of the form

$$(\lambda^2 + 2c\lambda + c^2 + \sigma_1^2 - f_0^2)(\lambda^2 + 2c\lambda + c^2 + \sigma_2^2) = 0. \tag{68}$$

Let

$$\Delta_1 = c^2 + \sigma_1^2 - f_0^2, \quad \Delta_2 = c^2 + \sigma_2^2. \tag{69}$$

When $c=0$, $\Delta_1=\sigma_1^2-f_0^2=0$ and $\Delta_2=\sigma_2^2>0$ are simultaneously satisfied, system (66) has a double zero and a pair of pure imaginary eigenvalues

$$\lambda_{1,2}=0, \quad \lambda_{3,4}=\pm \mathrm{i}\bar{\omega}_2, \tag{70}$$

where $\bar{\omega}_2^2=\sigma_2^2$.

Letting $\sigma_1=f_0+2\bar{\sigma}_1$ as well as setting $f_0=-1$, the averaged equation (66) without parameter f_2 is changed to

$$\dot{x}_1=x_2+\tfrac{1}{16}\alpha_2(x_2^3+x_1^2x_2)-\tfrac{3}{32}\alpha_3(x_2^3+x_1^2x_2)+\beta_1(x_2x_3^2+x_2x_4^2), \tag{71a}$$

$$\dot{x}_2=-\tfrac{1}{16}\alpha_2(x_1^3+x_1x_2^2)+\tfrac{3}{32}\alpha_3(x_1^3+x_1x_2^2)-\beta_1(x_1x_3^2+x_1x_4^2), \tag{71b}$$

$$\dot{x}_3=-\tfrac{1}{2}\sigma_2x_4+\tfrac{1}{2}\alpha_2(x_4^3+x_3^2x_4)-3\alpha_3(x_4^3+x_3^2x_4)+2\beta_2(x_2^2x_4+x_1^2x_4), \tag{71c}$$

$$\dot{x}_4=\tfrac{1}{2}\sigma_2x_3-\tfrac{1}{2}\alpha_2(x_3^3+x_3x_4^2)+3\alpha_3(x_3^3+x_3x_4^2)-2\beta_2(x_3x_1^2+x_3x_2^2). \tag{71d}$$

In the case considered here, we have

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sigma_2 \\ 0 & 0 & \frac{1}{2}\sigma_2 & 0 \end{pmatrix}, \tag{72}$$

and

$$A^*=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sigma_2 \\ 0 & 0 & -\frac{1}{2}\sigma_2 & 0 \end{pmatrix}. \tag{73}$$

Executing the Maple program given in Appendix A, the 3 order normal form of system (71) is obtained as

$$\dot{y}_1=y_2, \tag{74a}$$

$$\dot{y}_2=(-\tfrac{1}{16}\alpha_2+\tfrac{1}{32}\alpha_3)y_1^3-\beta_1y_1y_3^2-\beta_1y_1y_4^2, \tag{74b}$$

$$\dot{y}_3=-\tfrac{1}{2}\sigma_2y_4+(\tfrac{1}{2}\alpha_2-3\alpha_3)y_4^3+2\beta_2y_1^2y_4+(\tfrac{1}{2}\alpha_2-3\alpha_3)y_3^2y_4, \tag{74c}$$

$$\dot{y}_4=\tfrac{1}{2}\sigma_2y_3+(-\tfrac{1}{2}\alpha_2+3\alpha_3)y_3^3-2\beta_2y_1^2y_3+(-\tfrac{1}{2}\alpha_2+3\alpha_3)y_3y_4^2. \tag{74d}$$

The non-linear transformation used here is given as follows:

$$x_1=y_1+(\tfrac{1}{96}\alpha_2-\tfrac{1}{64}\alpha_3)y_1^3+(\tfrac{1}{16}\alpha_2-\tfrac{1}{32}\alpha_3)y_1y_2^2-\frac{2\beta_1}{\sigma_2}y_2y_3^2+\frac{2\beta_1}{\sigma_2}y_2y_4^2, \tag{75a}$$

$$x_2=y_2+(-\tfrac{1}{32}\alpha_2+\tfrac{3}{64}\alpha_3)y_1y_2^2, \tag{75b}$$

$$x_3=y_3+2\beta_2y_1y_2y_4, \tag{75c}$$

$$x_4 = y_4 - 2\beta_2 y_1 y_2 y_3. \quad (75d)$$

The results obtained above completely agree with that presented by using the method in paper [9].

5. Conclusions

Comparing the method developed here with other methods given in Refs. [9,17,25], it is observed that we may respectively, obtain the normal forms, the coefficients of the normal forms and the associated near identity non-linear transformations for three cases by using a same main Maple symbolic program. Therefore, it is more convenient to utilize the approach developed here to compute the normal form of the averaged equations for different resonant cases. It is also found that in Refs. [9,17] the researchers only investigate the cases in which the center manifold is four-dimensional. It is known that in paper [25] Leung and Zhang only computed the normal forms of non-linear systems with two-, three- and four-dimensional center manifolds by using the Mathematica language.

A new and efficient method of computing the normal forms for high dimensional non-linear systems is developed based on the adjoint operator method. The newly developed method has the advantage that it is not necessary to find the characteristic system of the sets of partial differential equations, the solutions of which lead to the normal forms. Neither is it necessary to obtain the independent first integrals of the characteristic system. Employing the new method, 3 order polynomial solutions of the sets of partial differential equations are conveniently obtained by using Maple symbolic program. Furthermore, the polynomial solutions can be directly introduced to determine the basis of a vertical complementary subspace. Finally, the method is applied to the averaged equations for non-planar non-linear oscillations of the cantilever beam in two different resonant cases. The normal forms of the averaged equations obtained in this paper can be used to investigate the global bifurcations and chaotic dynamics in non-planar non-linear oscillations of the cantilever beam under combined parametric and forcing excitations. In the next paper [27], the global bifurcations and chaotic dynamics for non-planar non-linear oscillations of the cantilever beam will be investigated.

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Appendix A. Maple programs of computing normal forms

In the following, the Maple source codes for computing the normal forms are presented. Using the Maple programs, we are able to obtain 3 order normal forms of four-dimensional non-linear systems and associated near identity non-linear transformation.

1. The main Maple symbolic program is as follows:

```
nm1 :=proc(b::array,L1::array)
local F, C, H, H1, i, j;
F :=evalm(b&*A);
C :=array(1..4,1..4);
for i from 1 to 4 do
  for j from 1 to 4 do
    C[i,j] :=diff(F[i],x[j]);
  od;
od;
H :=evalm(C&*L1&*x-L1&*F);
H1 :=array(1..4);
for i from 1 to 4 do
  H1[i] :=simplify(H[i]);
od;
H1;
end:

nm2 :=proc(T1::array)
local a, eq1, eq2, eq3, i, j, k, m, n, n1;
a :=array(1..4,1..20);
for i from 1 to 4 do
  for j from 1 to 4 do
    a[i,j] :=coeff(T1[i],x[j]^3);
  od;
  n :=0;
  for j from 1 to 4 do
    eq1 :=coeff(T1[i],x[j]^2);
    for k from 1 to 4 do
      if k<>j then
        n :=n+1;
        a[i,4+n] :=coeff(eq1,x[k]);
      fi;
    od;
  od;
  n1 :=4+n;
  n :=0;
  for j from 1 to 4 do
    eq2 :=coeff(T1[i],x[j]);
    if j+1<=4 then
  for k from j+1 to 4 do
    eq3 :=coeff(eq2,x[k]);
```

```
        if k+1<=4 then
          for m from k+1 to 4 do
            n :=n+1;
            a[i,n1+n] :=coeff(eq3,x[m]);
          od;
        fi;
      od;
    fi;
  od;
od;
a;
end:

nm3 :=proc(C2::array,c::array)
global NN, b3;
local eqb0, eqb1, eqs, b1, b2, AA, BB, CC, i, j, n, k, M;
eqb0 :=array(1..80);
eqb1 :=array(1..80);
n :=0;
for i from 1 to 4 do
  for j from 1 to 20 do
    n:=n+1;
    eqb0[n] :=C2[i,j]=0;
    eqb1[n] :=c[i,j];
  od;
od;
eqs :=convert(eqb0,list);
b1 :=convert(eqb1,list);
AA :=genmatrix(eqs,b1,'v');
b2 :=linsolve(AA,v,'r',eqb1);
b3 :=array(1..4,1..20);
for i from 1 to 4 do
  for j from 1 to 20 do
    b3[i,j] :=b2[j+20*(i-1)];
  od;
od;
n :=0;
BB :=convert(b2,set);
CC :=convert(BB,list);
M :=nops(CC);
for i from 1 to 4 do
```

```
  for j from 1 to 20 do
    k :=1;
    while coeff(CC[k],c[i,j])=0 and k<M do
      k :=k+1;
    od;
    if k<M then
      n :=n+1;
    elif coeff(CC[M],c[i,j])<>0 then
      n :=n+1;
    fi;
  od;
od;
NN :=n;
end:

nm4 :=proc(G21::array,g::array,b::array)
local eqb0, eqb1, eqb2, eqs, b1, b2, b4, AA, i, j, k, n;
eqb0 :=array(1..80);
eqb1 :=array(1..80+NN);
eqb2 :=array(1..80);
b4 :=array(1..4,1..20);
n :=0;
for i from 1 to 4 do
  for j from 1 to 20 do
    n:=n+1;
    eqb0[n] :=G21[i,j]=h[i,j];
    eqb1[n] :=g[i,j];
    eqb2[n] :=b[i,j];
  od;
od;
for j from 1 to NN do
  eqb1[80+j] :=eqb2[j];
od;
eqs :=convert(eqb0,list);
b1 :=convert(eqb1,list);
AA :=genmatrix(eqs,b1,'v');
b2 :=linsolve(AA,v,'r',eqb1);
for k from 1 to 80+NN do
 for i from 1 to 4 do
  for j from 1 to 20 do
    b2[k] :=subs(g[i,j]=0,b2[k]);
```

```
   od;
  od;
od;

for j from 1 to NN do
  eqb2[j] :=b2[80+j];
od;
for i from 1 to 4 do
  for j from 1 to 20 do
    b[i,j] :=eqb2[j+20*(i-1)];
    b4[i,j] :=b2[j+20*(i-1)];
  od;
  od;
  b4;
  end:

  with(linalg):
  read input1;
  x :=vector(4):
  A :=vector(20,[x[1]^3, x[2]^3, x[3]^3, x[4]^3, x[1]^2*x[2],
      x[1]^2*x[3], x[1]^2*x[4], x[1]*x[2]^2, x[2]^2*x[3],
          x[2]^2*x[4], x[1]*x[3]^2, x[2]*x[3]^2, x[3]^2*x[4],
          x[1]*x[4]^2, x[2]*x[4]^2, x[3]*x[4]^2, x[1]*x[2]*x[3],
          x[1]*x[2]*x[4], x[1]*x[3]*x[4], x[2]*x[3]*x[4]]):
  b :=array(1..4,1..20):
  g :=array(1..4,1..20):
  h :=array(1..4,1..20):

  L :=array(1..4,1..4):
  for i from 1 to 4 do
    for j from 1 to 4 do
      L[i,j] :=w[i,j];
    od;
  od;
  print(L);
  for i from 1 to 4 do
    for j from 1 to 20 do
      h[i,j] :=u[i,j];
    od;
  od:
  L1 :=transpose(L):
```

```
B1 :=nm1(b,L1):
B2 :=nm2(B1):
nm3(B2,b):
G1 :=nm1(g,L):
G2 :=nm2(G1):
G21 :=evalm(G2+b3):
G3 :=nm4(G21,g,b):
G4 :=evalm(b3&*A):
G5 :=evalm(G3&*A):
for i from 1 to 4 do
  G4[i] :=convert(G4[i],rational):
od;
for i from 1 to 4 do
  G5[i] :=convert(G5[i],rational):
od;
```

2. The input program for two pairs of pure imaginary eigenvalues is listed in the following.

```
with(linalg):
w:=array(1..4, 1..4, [[0, -(1/2)*sigma[1], 0, 0], [(1/2)*sigma[1], 0, 0, 0],
        [0, 0, 0, -(1/2)*sigma[2]], [0, 0, (1/2)*sigma[2], 0]]):
u:=array(1..4, 1..20, [[0, (1/8)*(2*alpha[2]-3*alpha[3]), 0, 0, (1/8)*(2*alpha[2]-3*alpha[3]),
        0, 0, 0, 0, 0, 0, -(1/8)*(2*alpha[2]+alpha[3]), 0, 0,
        (1/8)*(2*alpha[2]-3*alpha[3]), 0, 0, 0, (1/4)*(2*alpha[2]-alpha[3]), 0],
        [-(1/8)*(2*alpha[2]-3*alpha[3]), 0, 0, 0, 0, 0, 0, -(1/8)*(2*alpha[2]-3*alpha[3]),
        0, 0, -(1/8)*(2*alpha[2]-3*alpha[3]), 0, 0, (1/8)*(2*alpha[2]+alpha[3]),
        0, 0, 0, 0,0,-(1/4)*(2*alpha[2]-alpha[3])],
        [0, 0, 0, (1/8)*(2*alpha[2]-3*alpha[3]), 0, 0, -(1/8)*(2*alpha[2]+alpha[3]),
        0, 0, (1/8)*(2*alpha[2]-3*alpha[3]), 0, 0, (1/8)*(2*alpha[2]-3*alpha[3]),
        0, 0, 0, (1/4)*(2*alpha[2]-alpha[3]), 0, 0, 0],
        [0, 0, -(1/8)*(2*alpha[2]-3*alpha[3]), 0, 0, -(1/8)*(2*alpha[2]-3*alpha[3]),
        0, 0, (1/8)*(2*alpha[2]+alpha[3]), 0, 0, 0, 0, 0, 0, -(1/8)*(2*alpha[2]-3*alpha[3]),
        0, -(1/4)*(2*alpha[2]-alpha[3]), 0, 0]]):
```

3. The input program for a double zero and a pair of pure imaginary eigenvalues is listed in the following.

```
with(linalg):
w:=array(1..4, 1..4, [[0, 1, 0, 0], [0, 0, 0, 0], [0, 0, 0, -(1/2)*sigma[2]],
         [0, 0, (1/2)*sigma[2], 0]]):
u:=array(1..4, 1..20, [[0, ((1/16)*alpha[2]-(3/32)*alpha[3]), 0, 0,
```

```
((1/16)*alpha[2]-(3/32)*alpha[3]), 0, 0, 0, 0, 0, 0, beta[1], 0, 0, beta[1], 0, 0, 0, 0, 0],
[(-(1/16)*alpha[2]+(3/32)*alpha[3]),  0, 0, 0, 0, 0, 0,
(-(1/16)*alpha[2]+(3/32)*alpha[3]), 0, 0, -beta[1], 0, 0, -beta[1], 0, 0, 0, 0, 0, 0],
[0, 0, 0, (0.5*alpha[2]-3*alpha[3]), 0, 0, 2*beta[2], 0, 0, 2*beta[2],
0, 0, (0.5*alpha[2]-3*alpha[3]), 0, 0, 0, 0, 0, 0, 0],
[0, 0, (-0.5*alpha[2]+3*alpha[3]), 0, 0, -2*beta[2], 0, 0, -2*beta[2],
0, 0, 0, 0, 0, 0, (-0.5*alpha[2]+3*alpha[3]), 0, 0, 0, 0]]):
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