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Random vibrations of a damped rotating shaft

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Abstract

Response of a simple Jeffcott rotor to random excitation is considered with both external and internal damping taken into account. Mean square responses are predicted by the method of moments for the cases of transverse and angular (tilting) oscillations. Contrary to unbalance-induced response the random vibrations are shown to depend on the internal or "rotating" damping; in particular, their level increases with approaching threshold speed for dynamic instability. Procedure for estimating this threshold from on-line measurements of the shaft's random vibrations at a constant rotation speed is outlined based on the calculating coherence function of lateral displacements in two perpendicular directions for the case of transverse vibrations and that of tilting angles about two perpendicular axes for the case of tilting oscillations. Dependence of the mean square responses on the rotation speed can also be used for the stability margin evaluation.

1. Introduction

Rotating shafts in certain applications, such as high-power turbines or fans, may experience random excitation of bending vibrations during their operation (say, due to circumferential non-uniformity of a turbulent pressure field in a working fluid flow within a turbine). Whilst the resulting response of the shaft may usually be small compared with its response to unbalance, the measurable random vibration signal (at the shaft's natural frequency) can be detected sometimes if high-precision velocity sensors are used [1]. Thus, analysis of the response may be of interest as long as the available response signal may be used with advantage for on-line diagnostics, or state monitoring of the rotating shaft. Certain potential procedures for such a diagnostics of the shaft's stability margin are considered in the following as based on peak value of a coherence function

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between responses in two perpendicular directions and on the "universal" dependence of the mean square responses on the ratio of rotation speed to its value at the instability threshold.

2. Lateral linear oscillations

Consider a simple Jeffcott rotor with a weightless shaft of stiffness K rotating with angular velocity ν. The horizontal shaft carries a disk of mass m at its midspan and possesses both external or "non-rotating" damping and internal or "rotating" damping with corresponding damping factors c_n and c_r, respectively. Let $X(t)$ and $Y(t)$ be lateral horizontal and vertical displacements, respectively, of the disk's centre in the inertial frame with origin at the undeformed shaft's axis. Then, neglecting gravity force for sufficiently high rotation speeds and adding lateral random excitations one can write equations of lateral motion as [2]

$$\ddot{X} + 2\kappa\dot{X} + \Omega^2 X + 2\beta\nu Y = f_X(t), \qquad \ddot{Y} + 2\kappa\dot{Y} + \Omega^2 Y - 2\beta\nu X = f_Y(t), \tag{1}$$

where $\Omega^2 = K/m$, $\kappa = \alpha + \beta$, $\alpha = c_n/2m$, $\beta = c_r/2m$. The random forces on the RHSs of Eqs. (1) are assumed to be stationary zero-mean uncorrelated Gaussian white noises with the same intensity factor σ^2. As shown in Appendix A, this would be the case if the time-variant part of the pressure field in a flow of working fluid within the machine is delta-correlated both in time and in a circumferential direction.

An analysis of system (1) will be made under the well-known [2] condition for dynamic stability of its LHS $\nu < \nu_*$ where $\nu_* = \kappa\Omega/\beta = \Omega(1 + c_n/c_r)$ is the value of rotation speed at the instability threshold. One of the goals of the analysis is to find a way for evaluating this threshold by appropriate processing of the signals $X(t)$, $Y(t)$ as measured during steady operation at a rotation speed that satisfies the above stability condition.

Mean values, variances and co-variances of the responses as governed by Eqs. (1) can be found by the method of moments [3]. First, direct application of probabilistic averaging, which will be denoted by angular brackets, yields a homogeneous pair of equations for the mean values of $X(t)$ and $Y(t)$; this set of equations has zero steady state solution. To find the second order moments of the response the 4×1 vector $\mathbf{Z}$ of state variables $Z_1 = X$, $Z_2 = Y$, $Z_3 = \dot{X}$, $Z_4 = \dot{Y}$ is then introduced and Eqs. (1) are rewritten in a matrix form as

$$\dot{\mathbf{Z}} = \mathbf{AZ} + \mathbf{Bf}(t), \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\Omega^2 & -2\beta\nu & -2\kappa & 0 \\ 2\beta\nu & -\Omega^2 & 0 & -2\kappa \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

where $\mathbf{f}(t)$ is a 4×1 vector of white noises with

$$\langle f_i(t) f_j(t+\tau) \rangle = W_{ij}\delta(\tau) \quad \text{for } i,j = 1,2,3,4 \text{ and } W_{33} = W_{44} = \sigma^2$$
$$\text{all other } W_{ij} \text{ being zero.} \tag{3}$$

Here $\delta(\tau)$ is the Dirac delta-function.

The 4×4 matrix of variances $\mathbf{D}$ with elements $D_{ij} = \langle Z_i Z_j \rangle$, $i,j = 1,2,3,4$ satisfy the deterministic matrix ODE [3]

$$\dot{\mathbf{D}} = \mathbf{AD} + \mathbf{DA}^{\mathrm{T}} + \mathbf{BWB}^{\mathrm{T}}, \tag{4}$$

where superscript T denotes transpose of a vector or matrix.

For the present case of matrices $\mathbf{A}, \mathbf{B}, \mathbf{W}$ as defined by expressions (2) and (3), the ODE (4) has the following constant steady-state analytical solution for elements of the matrix $\mathbf{D}$ (as obtained by equating the RHS to zero):

$$D_{11} = D_{22} = \langle X^2 \rangle = \langle Y^2 \rangle = \frac{\sigma^2}{4(\alpha + \beta)\Omega^2[1 - (v/v_*)^2]},$$
$$D_{12} = 0, \quad D_{33} = D_{44} = \Omega^2 D_{11}, \quad D_{13} = D_{24} = D_{34} = 0, \quad D_{14} = -D_{23} = (\beta v/\kappa) D_{11}. \tag{5}$$

The solution shows that the shaft's responses in two fixed perpendicular directions are uncorrelated and have identical mean square values. The mean square response is seen to increase with increasing rotation speed, particularly with approaching instability threshold where it becomes unbounded. In this respect the response to random excitation is different from that to unbalance, the latter being known to be completely insensitive to the influence of "rotating" damping [2]. On the contrary, as can be seen from solution (5) the stability threshold v_* can be estimated from the random vibrations—namely, by calculating the ratio of mean square responses measured at two (or more) different rotation speeds (after the band-pass filter is applied to detect the random component from the overall measured signal). Moreover, as will be shown in the following, using spectral and cross-spectral analysis of the responses $X(t)$ and $Y(t)$, one can obtain the desired estimate from measurements at a *single* rotation speed; that would be truly on-line diagnostics.

Power spectral densities (PSDs) of the responses $X(t)$, $Y(t)$ can be derived by using the following definition of cross-spectral density of any pair of stationary random processes $Z_i(t)$, $Z_j(t)$ [4]:

$$\Phi_{ZiZj}(\omega) = \lim_{T \to \infty} (1/T) \langle \tilde{Z}_i(\omega, T) \tilde{Z}_j^*(\omega, T) \rangle, \qquad \tilde{Z}(\omega, T) = \int_{-T}^{\mathrm{T}} Z(t) \exp(\mathrm{i}\omega t)\, \mathrm{d}t, \quad \mathrm{i} = \sqrt{-1}, \tag{6}$$

where star superscript denotes complex conjugate quantity; this definition covers auto-spectral densities as well if $i = j$. Applying to Eqs. (1) the Fourier transform with finite limits $+T$ and $-T$ as denoted by tilda in Eqs. (6), yields two algebraic equations in the frequency domain:

$$\tilde{X}(-\omega^2 + 2\mathrm{i}\kappa\omega + \Omega^2) + \tilde{Y}(2\beta v) = \tilde{f}_X, \qquad \tilde{X}(-2\beta v) + \tilde{Y}(-\omega^2 + 2\mathrm{i}\kappa\omega + \Omega^2) = \tilde{f}_Y. \tag{7}$$

The auto- and cross-spectral densities of $X(t)$ and $Y(t)$ can now be obtained by solving Eqs. (7) for $\tilde{X}$ and $\tilde{Y}$ and applying the basic definition (6). The result is

$$\Phi_{XX}(\omega) = \Phi_{YY}(\omega) = (\sigma^2/2\pi\Delta\Delta^*)\,[(\omega^2 - \Omega^2)^2 + 4(\kappa^2\omega^2 + \beta^2 v^2)],$$
$$\Phi_{XY}(\omega) = (\sigma^2/2\pi\Delta\Delta^*) 8\mathrm{i}\kappa\omega\beta v, \quad \Delta = (-\omega^2 + 2\mathrm{i}\kappa\omega + \Omega^2)^2 + (2\beta v)^2 \tag{8}$$

(the factor $\sigma^2/2\pi$ appears here as the constant PSD of each of the white noises $f(t)$). This solution can be used to obtain the coherence function [4] of the responses in two perpendicular directions as

$$\gamma_{XY}^2(\omega) = \frac{|\Phi_{XY}^2(\omega)|}{\Phi_{XX}(\omega)\Phi_{YY}(\omega)} = \frac{(8\kappa\omega\beta\nu)^2}{[(\omega^2 - \Omega^2)^2 + 4(\kappa^2\omega^2 + \beta^2\nu^2)]^2}. \tag{9}$$

This function can be seen to have a peak in the immediate vicinity of the rotor's natural frequency Ω, which is known to be also the frequency of forward whirl at the stability boundary, that is of the neutrally stable shaft [2]; shift of the peak is found to be of the order of total damping ratio κ/Ω, and it diminishes also with approaching stability threshold. By introducing the instability threshold rotation speed $\nu_* = \kappa\Omega/\beta = \Omega(1 + c_n/c_r)$ the approximate peak value of the coherence function can be represented, therefore, as

$$\gamma_{XY}^2(\Omega) = \left[\frac{2\nu/\nu_*}{1 + (\nu/\nu_*)^2}\right]^2. \tag{10}$$

Relation (10) provides the desired approximate on-line estimate of the instability threshold ν_* of the rotor from the peak value of the coherence function of the two response signals measured during stable steady state operation of the rotor at a fixed rotation speed ν.

3. Angular oscillations

As a second example consider another simple TDOF rotor, once again with the disk at its midspan but with bearings that allow in this case only angular oscillations, or tilting around the transverse axes (Fig. 1). Let ϕ_X, ϕ_Y be the corresponding rotation angles about axes X and Y,

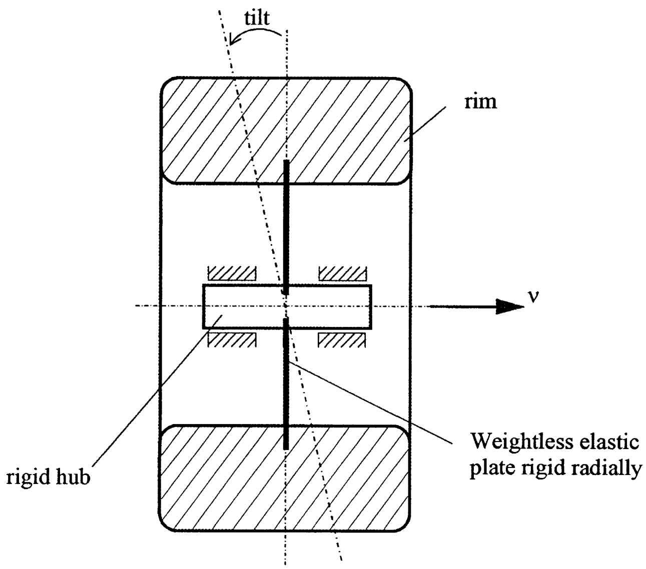

Fig. 1. Rotating disk (rim) with potential for angular (tilting) vibrations.

respectively, or tilting angles; J_a is the moment of inertia of the (axisymmetric) rotor with respect to these axes; J_p the rotor's moment of inertia with respect to the rotation axis; v the rotation speed (once again), and K_ϕ the angular stiffness of the shaft. Then, instead of Eqs. (1) the equations of angular motion may be written as

$$\begin{aligned} &\ddot{\phi}_X + 2\kappa\dot{\phi}_x + \Omega^2\phi_X + \rho v\dot{\phi}_Y + 2\beta v\phi_Y = g_X(t),\\ &\ddot{\phi}_Y + 2\kappa\dot{\phi}_Y + \Omega^2\phi_Y - \rho v\dot{\phi}_X - 2\beta v\phi_X = g_Y(t), \end{aligned} \tag{11}$$

where $\Omega^2 = K_\phi/J_a$, $\rho = J_p/J_a$. The damping parameters κ and β are defined here similar to the case of lateral linear oscillations: $\kappa = \alpha_\phi + \beta_\phi$, $\alpha_\phi = c_{\phi n}/J_a$, $\beta_\phi = c_{\phi r}/J_a$, where the c's are coefficients of non-rotating and rotating damping in angular rather than translational oscillations as indicated by the additional subscript ϕ. Eqs. (11) and (1) differ in their structure only due to the additional terms with factor ρ in the former which describe the gyroscopic effect in angular oscillations. The random moments in the RHSs of Eqs. (11) are likewise assumed to be stationary zero-mean uncorrelated Gaussian white noises with the same intensity factor σ_ϕ^2.

The condition for dynamic stability of the rotor will be determined first, as governed by LHSs of Eqs. (11). Introducing the complex variable

$$\phi = \phi_X + \mathrm{i}\phi_Y, \tag{12}$$

one can obtain from Eqs. (11) the homogeneous complex differential equation of motion

$$\ddot{\phi} + 2\kappa\dot{\phi} + \Omega^2\phi - \mathrm{i}\rho v\dot{\phi} - 2\mathrm{i}\beta v\phi = 0. \tag{13}$$

Assuming a solution in the form

$$\phi(t) = \phi_0 \exp(\mathrm{i}\omega t) \tag{14}$$

and requiring ω to be real leads to the complex equation for the neutral stability condition in terms of the rotation speed v:

$$\phi_0[-\omega^2 + 2\mathrm{i}\kappa\omega + \Omega^2 + \rho v\omega - 2\mathrm{i}\beta v] = 0. \tag{15}$$

Equating to zero separately the real and imaginary parts of the quantity in brackets yields two relations between frequency of whirl ω and rotation speed v:

$$\omega^2 - \rho v\omega - \Omega^2 = 0, \qquad \beta v = \kappa\omega. \tag{16}$$

The solution to this pair of equations corresponds to the stability boundary and therefore will be denoted by star subscripts. Thus, eliminating ω from relations (16) yields the instability threshold speed $v = v_*$ and the corresponding whirl speed $\omega = \omega_*$ is obtained then as

$$v_* = \frac{(\kappa/\beta)\Omega}{\sqrt{1-\rho\kappa/\beta}}, \qquad \omega_* = \beta v_*/\kappa = \frac{\Omega}{\sqrt{1-\rho\kappa/\beta}}. \tag{17}$$

This stability threshold speed can be reached only if $\rho\kappa/\beta < 1$, or $\rho(1+\alpha/\beta) < 1$. For the case $\rho\kappa/\beta \ll 1$, expressions (17) reduce to those which hold for translational vibrations considered in Section 2.

The mean values, variances, and covariances of the responses are determined once again by the method of moments. Let $\mathbf{Z}$ be now the 4×1 vector of the state variables:

$$Z_1 = \phi_X, \qquad Z_2 = \phi_Y, \qquad Z_3 = \dot{\phi}_X, \qquad Z_4 = \dot{\phi}_Y. \tag{18}$$

Eqs. (11) can then be rewritten in matrix form as

$$\dot{\mathbf{Z}} = \mathbf{AZ} + \mathbf{Bg}(t), \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\Omega^2 & -2\beta v & -2\kappa & -\rho v \\ -2\beta v & -\Omega^2 & \rho v & -2\kappa \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

where $\mathbf{g}(t)$ is a 4×1 vector of white noises with

$$\langle g_i(t) g_j(t+\tau) \rangle = W_{ij}\delta(\tau) \quad \text{for } i,j = 1,2,3,4 \text{ and } W_{33} = W_{44} = \sigma_\phi^2, \\ \text{all other } W_{ij} \text{ being zero.} \tag{20}$$

In this case the matrix ODE (4) for the corresponding 4×4 matrix $\mathbf{D}$ of covariances has the steady state solution

$$D_{11} = D_{22} = \langle \phi_X^2 \rangle = \langle \phi_Y^2 \rangle = \frac{\sigma_\phi^2}{4(\alpha+\beta)\Omega^2[1-(v/v_*)^2]}, \\ D_{12} = 0, \qquad D_{33} = D_{44} = (\Omega^2 + \rho\beta v^2/\kappa)D_{11}, \qquad D_{13} = D_{24} = D_{34} = 0, \\ D_{14} = -D_{23} = (\beta v/\kappa)D_{11}, \tag{21}$$

where v_* is defined by expression (17). This result is somewhat similar to one obtained in Section 2 and can be interpreted similarly. In particular, the instability threshold speed v_* can be estimated once again from random vibration data, by calculating the ratio of mean square responses as measured at two (or more) different rotation speeds.

Finally, consider the possibility of using spectral and cross-spectral analysis to estimate the stability threshold speed. The coherence function of two tilting angles may be obtained, similar to Section 2, as

$$\gamma^2_{\phi X \phi Y}(\omega) = \left\{ \frac{2\omega v[4\kappa\beta + \rho(\omega^2 - \Omega^2)]}{(\omega^2 - \Omega^2)^2 + 4(\kappa^2\omega^2 + \beta^2 v^2) + \rho^2 v^2 \omega^2} \right\}^2. \tag{22}$$

In case $\rho = 0$, this expression is reduced to expression (9) as obtained for the case of translational vibrations of the shaft.

Value of the coherence function (22) at the frequency $\omega = \omega_*$ can be expressed in terms of the ratio v/v_* by the same relation (10) as had been derived for translational vibrations but with the relevant values of v_* defined by Eq. (17):

$$\gamma^2_{\phi X \phi Y}(\omega_*) = \left[\frac{2v/v_*}{1 + (v/v_*)^2} \right]^2. \tag{23}$$

Besides different definitions of the instability threshold speed v_* in relations (10) and (23), different arguments of the coherence functions are seen in these relations. These arguments are angular frequencies of the shaft's whirl at the corresponding stability boundaries, that is Ω and ω_* for translational and tilting oscillations, respectively. It can also be shown that for the lightly damped shaft the quantity in the LHS of expression (23) is once again (approximately) the peak value of the coherence function—of the two tilting angles in this case.

4. Conclusions

Two cases of random vibrations have been considered for an axisymmetric perfectly balanced TDOF rotating shaft with a disk at its midspan, namely those of translational and of angular or tilting oscillations. A mean square response analysis has been performed for each type of oscillations by the method of moments, as well as an analysis of the response PSDs that lead to analytical expressions for the coherence function of displacements along two mutually perpendicular directions (translational in the first case and angular in the second one).

The mean square response analyses for both cases resulted in analytical expressions that describe a "universal" dependence on rotation speed as $D_{ii} = D_{ii,0}/[1 - (v/v_*)^2]$. Here, $i = 1, 2$ and an additional subscript "zero" for the mean square responses D's that correspond to the shaft at a zero rotation speed, whereas v_* is the relevant rotation speed at the instability threshold. The coherence functions (9) and (22) for the cases of translational and angular (tilting) vibrations, respectively, are also seen to possess certain similarities in spite of a "natural gyroscopic coupling" which is present in the latter case but not in the former one. Namely, peaks of coherence functions of translational displacements in two perpendicular directions and of tilting angles about two perpendicular axes are attained at frequencies of the shaft's whirl at the corresponding stability boundaries—namely at $\omega = \Omega$ for the former of the two functions and at $\omega = \omega_*$ for the latter one. Furthermore, peak values of these two coherence functions exhibit similar dependence on the ratio v/v_* according to relations (10) and (23). In these relations v is the rotation speed at which measurements are made whereas v_* is the relevant instability threshold speed defined by expression (17) for the latter of the two relations and by its special version with $\rho = 0$ for the former one.

It may be speculated that strong coupling between vibrations in two mutually perpendicular directions, as resulting in high peak values of the coherence function, may also be used as an indicator of closeness to the instability threshold for rotating shafts for other sources of instability than "rotating" internal damping—for example, for the cases of potential instability due to thin fluid films in plain journal bearings, labyrinth seals, etc. Increasing the mean square level of the random response component with increasing rotation speed seems also to be generally useful for detecting an approach to the stability threshold for such cases.

Finally, some comments on potential applications to real rotating machinery seem appropriate. The proposed algorithms for condition monitoring, as derived from analytical solutions for random vibrations of two versions of the basic Jeffcott rotor model, may not be directly applicable to complicated rotor systems that are described only approximately by these models. Therefore, these algorithms may be in need of verification of their "robustness" with respect to potential deviations from the basic model—such as small anisotropy in stiffness of the shaft and/or its supports, additional degrees of freedom, etc.; some adjustment(s) of the algorithms may be required. The verification may be implemented by a numerical solution of the equations for moments and/or by direct numerical Monte Carlo simulation of the random vibration problem for a given system. For example, a four-degrees-of-freedom shaft with a single disk that is offset from the shaft's midspan may experience both translational and tilting oscillations that would be coupled. Whilst applicability of the coherence approach may still be expected in this case, the ultimate verification would be necessary, which can be obtained by a numerical solution of the equations for the response moments. Numerical studies may also be applied for cases where random excitations in two perpendicular directions are correlated—for example if concentrated random force(s) of fixed direction is (are) applied to the shaft from the vibrating support(s). Thus, the analytically derived methods may also be viewed as benchmarks that may prove to be helpful for the numerical studies of more complicated models of random vibrations, including those based on direct Monte Carlo simulations.

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Appendix A. Excitation model

Consider a solid circular disk exposed to an external stationary homogeneous random pressure field along its boundary. Let the time-invariant part of the field be constant along circumference whereas the time-variant part $p(\theta, t)$ is delta-correlated both in time and in angular position. Thus

$$\langle p(\theta, t) \rangle = 0, \quad \langle p(\theta, t) p(\theta', t+\tau) \rangle = \sigma_0^2 \delta(\theta - \theta') \delta(\tau). \tag{A.1}$$

Let $f_X(t)$ and $f_Y(t)$ be horizontal and vertical forces, respectively, as applied to the disk, so that

$$f_X(t) = \int_0^{2\pi} p(\theta, t) \cos\theta \, \mathrm{d}\theta, \qquad f_Y(t) = \int_0^{2\pi} p(\theta, t) \sin\theta \, \mathrm{d}\theta. \tag{A.2}$$

Then, using definitions (A.2) and properties (A.1) of the pressure field one can see that the first order moments of both forces are zero, whereas for the second order moments we obtain

$$\begin{aligned}
\langle f_X(t) f_X(t+\tau) \rangle &= \sigma_0^2 \delta(\tau) \int_0^{2\pi} \cos\theta \, \mathrm{d}\theta \int_0^{2\pi} \cos\theta' \cdot \delta(\theta - \theta') \, \mathrm{d}\theta' \\
&= \sigma_0^2 \delta(\tau) \int_0^{2\pi} \cos^2\theta \, \mathrm{d}\theta = \sigma^2 \delta(\tau), \\
\langle f_Y(t) f_Y(t+\tau) \rangle &= \sigma_0^2 \delta(\tau) \int_0^{2\pi} \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \sin\theta' \cdot \delta(\theta - \theta') \, \mathrm{d}\theta' \\
&= \sigma_0^2 \delta(\tau) \int_0^{2\pi} \sin^2\theta \, \mathrm{d}\theta = \sigma^2 \delta(\tau), \\
\langle f_X(t) f_Y(t+\tau) \rangle &= \sigma_0^2 \delta(\tau) \int_0^{2\pi} \cos\theta \, \mathrm{d}\theta \int_0^{2\pi} \sin\theta' \cdot \delta(\theta - \theta') \, \mathrm{d}\theta' \\
&= \sigma_0^2 \delta(\tau) \int_0^{2\pi} \sin 2\theta \, \mathrm{d}\theta = 0,
\end{aligned} \tag{A.3}$$

where $\sigma^2 = \pi \sigma_0^2$. Thus we arrived at the excitation model used in Eqs. (1).

A similar model may be adopted for the case of angular oscillations of the disk, as governed by Eqs. (11). Assume for simplicity that the width h of the fluid-filled annulus around the disk (say, height of turbine blades) is small compared with the disk's radius R. Then the moments $g_X(t)$, $g_Y(t)$ as produced by distributed axial forces would be defined by Eqs. (A.2) with each integral multiplied by factor $R + h/2$—and of course with f's replaced by g's with the same subscripts.

Appendix B. Nomenclature

c_n	coefficient of "non-rotating" damping in translational motion, that is, proportionality factor between velocity and corresponding damping force in non-rotating frame
$c_{\phi n}$	coefficient of "non-rotating" damping in tilting motion, that is, proportionality factor between angular velocity and moment of the corresponding damping force in non-rotating frame
c_r	coefficient of "rotating" damping in translational motion, that is, proportionality factor between velocity and corresponding damping force in rotating frame
$c_{\phi r}$	coefficient of "rotating" damping in tilting motion, that is, proportionality factor between angular velocity and moment of the corresponding damping force in rotating frame
D_{ij}, $i,j = 1,2,3,4$	variances and covariances of the state variables—that is, of translational displacements and velocities in Section 2 and of angular displacements and velocities in Section 3
$f_X(t), f_Y(t)$	random forces applied to the rotor in directions X and Y, respectively
$g_X(t)$, $g_Y(t)$	random moments about axes X and Y, respectively, applied to the rotor
J_a	moments of inertia of the (axisymmetric) rotor with respect to axes X and Y
J_p	rotor's moment of inertia with respect to its rotation axis
K	rotor's stiffness with respect to translational displacements
K_ϕ	rotor's angular stiffness with respect to tilt
W_{ij}	intensities of white-noise excitations—forces in Section 2 (see relations (3)) and moments in Section 3 (see relations (20))
X, Y	horizontal and vertical axes, respectively, of a non-rotating Cartesian frame with origin at the intersection point of disk with the undeformed shaft's axis
Z	vectors of state variables—translational displacements and velocities in Section 2 and angular displacements and velocities in Section 3
α	$c_n/2m$
β	$c_r/2m$
α_ϕ	$c_{\phi n}/2J_a$
β_ϕ	$c_{\phi r}/2J_a$
$\gamma^2_{XY}(\omega)$	coherence function of translational displacements $X(t)$, $Y(t)$
$\gamma^2_{\phi X \phi Y}(\omega)$	coherence function between angular displacements $\phi_X(t)$, $\phi_Y(t)$
$\phi_X(t)$, $\phi_Y(t)$	angular displacements about axes X, Y, respectively, or tilt angles
$\Phi_{ZiZj}(\omega)$	auto- and cross-spectral densities of arbitrary stationary processes $Z_i(t)$, $Z_j(t)$
κ	$\alpha + \beta$ for translational oscillations—Eqs. (1) in Section 2
κ	$\alpha_\phi + \beta_\phi$ for tilting oscillations—Eqs. (11) in Section 3

ρ	J_p/J_a
ν	angular frequency or speed of shaft's rotation
ν_*	speed of shaft's rotation at the instability threshold; defined by Eq. (17) for tilting oscillations and by its special case for $\rho = 0$ for translational oscillations
σ^2, σ_ϕ^2	intensities, respectively, of white-noise forces $f(t)$ (Section 2) and moments $g(t)$ (Section 3)
ω	frequency
ω_*	frequency of the shaft's forward whirl at the neutral stability boundary; defined by Eq. (17) for tilting oscillations and by its special case for $\rho = 0$ for translational oscillations
Ω	$\sqrt{K/m}$ and $\sqrt{K_\phi/J_a}$ in Sections 2 and 3, respectively—natural frequency of a non-rotating shaft

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