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Semi-active optimal control of linearized systems with multi-degree of freedom and application

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Abstract

A semi-active optimal control method for non-linear multi-degree-of-freedom systems and its application to a building structure for random response reduction are presented in this paper. A structural system with semi-active control devices under random loading is modelled as a controlled, randomly excited and dissipated Hamiltonian system of multi-degree of freedom. The control force produced by a semi-active control device is split into semi-active part and passive part incorporated in the uncontrolled system. Applying the statistical linearization method to the non-linear multi-degree-of-freedom system with passive control force components yields quasi-linear equations of motion, which can tend to corresponding linear ones with system response reduction. By applying the dynamical programming principle to the controlled linearized system, a dynamical programming equation is established and in particular, for a non-filtering white noise excitation, is solved as an optimal regulation problem to determine the quasi-linear quadratic optimal control law and furthermore semi-active optimal control law according to the variational principle. Then the semi-active optimal control of a tall building structure with magnetorheological-tuned liquid column damper (MR-TLCD) under random wind excitation is performed by using the proposed method. The non-linear model of the structural system with semi-active MR-TLCD is formulated in structural mode space and uncoupled between structural and MR fluid accelerations. The quasi-linear equations for system states are derived from the model and the dynamical programming equation for the system is obtained. In the case that the random wind excitation with the Davenport power spectrum cannot be modelled as a linear filtering white noise, the dynamical programming equation is solved as an optimal regulation problem to obtain the semi-active optimal control force, on which the clipping treatment may be performed to ensure the control force implementable actually. Eventually, the response statistics of the semi-actively controlled structure under random wind excitation are evaluated by using the statistical linearization method, and are compared with those of the passively controlled structure to determine the control efficacy.

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Numerical results illustrate the high control effectiveness of the proposed semi-active optimal control method for building structures with MR-TLCDs.

1. Introduction

Semi-active control of structural vibration induced by severe dynamic loading such as strong wind or earthquake ground motion has been an active research subject recently [1,2]. With attractive features such as simplicity, reliability and small power requirement, various semi-active control devices were designed, especially using smart materials, for example, electrorheological (ER) and magnetorheological (MR) dampers. Intensively theoretical and experimental researches were made on the dynamic behavior and potential application of semi-active control devices [2]. The amplitude of semi-active control forces can be adjusted by external small power source and then a number of control methods were presented [3–5]. The control effectiveness of structural systems is highly dependent on the control method used for designing semi-active control law. In general, the optimal control method based on the dynamical programming principle is more reasonable and effective than the others. The optimal control of structural systems can be treated as an optimal regulation problem or optimal track problem [6]. Dynamic loading such as wind or earthquake acting on engineering structures is random in nature. In the case that the random loading cannot be modelled as a filtering white noise, the structural optimal control should be considered as an optimal regulation problem. With a classical explicit solution of control law to the dynamical programming equation, the linear quadratic (LQ) control method [6] is frequently used in structural control.

In dynamic analysis, engineering structures such as tall buildings are usually modelled as multi-degree-of-freedom systems. Structural systems with multi-degree-of-freedom exhibit non-linearity when subjected to strong dynamic loading. In particular, the controlled multi-degree-of-freedom systems are non-linear due to semi-active control devices such as ER or MR dampers [2,3]. Although non-linear control methods, for example, the non-linear stochastic optimal control method based on the stochastic dynamical programming principle and stochastic averaging method [5,7–9] have been proposed, yet their application to non-linear systems with highly multi-degree of freedom is challenging at present since applying the stochastic averaging method [10,11] to the systems is difficult. An alternative method for non-linear multi-degree-of-freedom systems is the statistical linearization method [12–14], by which a good result of random response statistics can be obtained, especially for controlled non-linear systems with response reduction. The linearized result can also converge on corresponding linear one as the non-linearity vanishes. The active optimal control of a non-linear two-degree-of-freedom system under a filtering white noise excitation has been studied based on the statistical linearization method and LQ control method [15]. Therefore, the semi-active optimal control of non-linear multi-degree-of-freedom systems based on the statistical linearization method and dynamical programming principle is a significant research subject.

On the other hand, in structural engineering field, installing supplemental control devices in high-rise building structures is a practical and effective approach to mitigating structural wind or seismic response. The passive control of tall building structures with supplemental control devices such as the tuned mass damper and tuned liquid damper has been researched extensively [16], and the active and semi-active controls of tall building structures under random wind or seismic excitation have been evolved [17–19]. Several semi-active control devices [20,21], especially the magnetorheological-tuned liquid column damper (MR-TLCD) [22] have been designed recently and studied on their dynamic behavior. MR fluids as smart materials possess better essential characteristics such as reversible change between liquid and semi-solid with controllable yield strength in milliseconds when exposed to a magnetic field. TLCDs are a type of U-shape liquid dampers with favorable characteristics such as structural simplicity, convenience of installation and low costs. With the benefits of smart MR fluids and TLCDs, the semi-active MR-TLCD incorporates MR fluids as TLCD-contained liquid to be controlled by applied magnetic field, and has been used as a passive control device for reducing wind response of tall building structures [22]. Consequently, applying the semi-active optimal control method for non-linear multi-degree-of-freedom systems to building structures with MR-TLCDs under random wind excitation is more interesting and would be more effective for wind response mitigation.

The present study is focused firstly on the semi-active optimal control method for a non-linear multi-degree-of-freedom system under random excitation. The control force of a semi-active control device is separated into passive part combined with the uncontrolled system and semi-active part to be determined by an optimal control strategy. The quasi-linear equations of motion for generalized displacements and momenta of the system with passive control force components are derived by using the statistical linearization method. The dynamical programming equation for the linearized system is established based on the stochastic dynamical programming principle. In the case that the random excitation cannot be modelled as a filtering white noise, the optimal control of the system is treated as an optimal regulation problem and the semi-active optimal control law is determined by LQ control. Then the developed control method is applied to a tall building structure for wind response reduction. A multi-degree-of-freedom model of the building structure with an arbitrary number of stories and with a semi-active MR-TLCD at the top floor is formulated and converted into another one by using the modal transformation technique. Since the random wind excitation with the Davenport power spectrum cannot be modelled as a linear filtering white noise, the optimal control of the structure with non-linear MR-TLCD is considered as an optimal regulation problem. The semi-active optimal control force is obtained based on the statistical linearization method and LQ control method, on which the clipping treatment may be performed to ensure the control force implementable actually. Finally, the random wind response of the semi-actively controlled building structure is predicted by using the statistical linearization method and compared with that of the passively controlled structure to evaluate the control efficacy which is illustrated by the numerical results.

2. Linearization and optimal control of multi-degree-of-freedom systems

A non-linear structure with semi-active control devices under random loading can be modelled as a controlled, randomly excited and dissipated Hamiltonian system of multi-degree of freedom, which is governed by n pairs of equations of motion as follows:

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i}, \tag{1a}$$

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i} - c'_{ij}\frac{\partial H'}{\partial P_j} - b_{ir}u_r + f_{ik}\xi_k(t), \tag{1b}$$

$$i,j = 1,2,\ldots,n,\;\; r = 1,2,\ldots,l,\;\; k = 1,2,\ldots,m,$$

where $H' = H'(Q_i, P_i)$ is the Hamiltonian generally representing total energy of the system; Q_i and P_i are generalized displacement and momentum, respectively; $c'_{ij} = c'_{ij}(Q_i, P_i)$ denotes damping coefficient; f_{ik} is the amplitude of random excitation and $\xi_k(t)$ is random process with zero mean and power spectral density $S_{kk'}(\omega)$; u_r represents the control force produced by semi-active control devices and b_{ir} is the control-device placement coefficient.

The random response of the Hamiltonian system in functional form depends on its integrability and resonance which are determined by the structure of its Hamiltonian H' [11,23]. For example, in the case of integrable Hamiltonian system as many engineering structures are modelled generally, there exist n independent integrals of motion, which are in involution and the energy distribution among various degrees of freedom as well as the total system energy is adjustable. The stationary probability density of the system is a functional of independent integrals of motion and thus, the total energy and energy distribution can be controlled by control forces as well as changed by dampings and excitations. The system vibration can be mitigated by the system state control.

In general, the control force u_r produced by semi-active control devices such as ER and MR dampers can be separated into passive part u_{rp} and semi-active part u_{rs} [4,5], that is

$$u_r(Q_i, P_i) = u_{rp}(Q_i, P_i) + u_{rs}(Q_i, P_i), \tag{2}$$

where u_{rp} is the passive control force component of the control devices independent of external voltage and u_{rs} is the semi-active control force component of the control devices dependent on external voltage, which can be adjusted by small power source according to an optimal control strategy. By combining the passive control force component u_{rp} with the uncontrolled system to form a new Hamiltonian H, Eq. (1) is rewritten as

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \tag{3a}$$

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i} - c_{ij}\frac{\partial H}{\partial P_j} - b_{ir}u_{rs} + f_{ik}\xi_k(t), \tag{3b}$$

$$i,j = 1,2,\ldots,n,\;\; r = 1,2,\ldots,l,\;\; k = 1,2,\ldots,m.$$

For non-linear multi-degree-of-freedom system (3) subjected to random excitation, it is difficult to directly use the dynamical programming principle for determining an optimal control law. The statistical linearization method [12–14] can be first applied to system (3) to yield quasi-linear random processes which can converge on corresponding linear ones with non-linearity vanishing, especially for the controlled system with response reduction. The linearized equations for generalized displacements and momenta as state process vector are represented by

$$\dot{\mathbf{Z}} = \mathbf{A}\mathbf{Z} + \mathbf{F}(t) + \mathbf{U}, \tag{4}$$

where the generalized state vector $\mathbf{Z}$, random excitation vector $\mathbf{F}$, semi-active control force vector $\mathbf{U}$ and coefficient matrix $\mathbf{A}$ are, respectively,

$$\mathbf{Z} = \begin{Bmatrix} [Q_i]_{n\times 1} \\ [P_i]_{n\times 1} \end{Bmatrix}, \quad \mathbf{F} = \begin{Bmatrix} 0_{n\times 1} \\ [f_{ik}\xi_k(t)]_{n\times 1} \end{Bmatrix}, \quad \mathbf{U} = \begin{Bmatrix} 0_{n\times 1} \\ -[b_{ir}u_{rs}]_{n\times 1} \end{Bmatrix}, \tag{5a}$$

$$\mathbf{A} = \begin{bmatrix} E\left[\dfrac{\partial^2 H}{\partial Q_i \partial P_i}\right]_{n\times n} & E\left[\dfrac{\partial^2 H}{\partial P_i^2}\right]_{n\times n} \\ -E\left[\dfrac{\partial^2 H}{\partial Q_i^2} + \dfrac{\partial}{\partial Q_i}\left(c_{ij}\dfrac{\partial H}{\partial P_j}\right)\right]_{n\times n} & -E\left[\dfrac{\partial^2 H}{\partial Q_i \partial P_i} + \dfrac{\partial}{\partial P_i}\left(c_{ij}\dfrac{\partial H}{\partial P_j}\right)\right]_{n\times n} \end{bmatrix}, \tag{5b}$$

in which $E[\cdot]$ denotes the expectation operator. Matrix $\mathbf{A}$ can be expressed generally as a function of response statistics of the system so that Eq. (4) is quasi-linear. Such equation is convenient to apply the dynamical programming principle for designing an optimal control law.

The optimal control of random process (4) can be performed based on the stochastic dynamical programming principle. The optimal control law depends on the objective of system control, which is expressed in terms of performance index. For $\mathbf{Z}$ control, the performance index in finite time interval is

$$J = E\left[\int_0^{t_f} L(\mathbf{Z}(\tau), \mathbf{u}_s(\tau))\,\mathrm{d}\tau + \Psi(\mathbf{Z}(t_f))\right], \tag{6}$$

where t_f is the terminal time; $L(\mathbf{Z}, \mathbf{u}_s)$ represents a continuous differential convex function; $\mathbf{u}_s = [u_{1s}, u_{2s}, \ldots, u_{ns}]^{\mathrm{T}}$ and $\psi(t_f)$ represents a terminal cost. In infinite time-interval ergodic control, the performance index (6) becomes

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} L(\mathbf{Z}(\tau), \mathbf{u}_s(\tau))\,\mathrm{d}\tau. \tag{7}$$

Obviously, the performance index J depends on the used function L of $\mathbf{Z}$ and $\mathbf{u}_s$. For a convex function L as used in the conventional linear-quadratic-Gaussian control, the random process $\mathbf{Z}$ in entire dynamic process decreases in correspondence with function L and performance index J. Thus the random response can be reduced by minimizing the performance index. In the case of Gaussian white noise excitation $\mathbf{F}(t)$ with intensity $2\mathbf{D}$, applying the stochastic dynamical programming principle [6] yields a dynamical programming equation, for example, to the controlled process (4) with performance index (6) as follows:

$$\frac{\partial V}{\partial t} = -\min_{\mathbf{u}_s}\left\{ L(\mathbf{Z}, \mathbf{u}_s) + (\mathbf{A}\mathbf{Z} + \mathbf{U})^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} + \mathrm{tr}\left[\mathbf{D}\frac{\partial^2 V}{\partial \mathbf{Z}^2}\right]\right\} \tag{8}$$

or to the controlled process (4) with performance index (7) as

$$\lambda = \min_{\mathbf{u}_s}\left\{ L(\mathbf{Z}, \mathbf{u}_s) + (\mathbf{A}\mathbf{Z} + \mathbf{U})^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} + \mathrm{tr}\left[\mathbf{D}\frac{\partial^2 V}{\partial \mathbf{Z}^2}\right]\right\}, \tag{9}$$

where tr[·] denotes the trace operator of square matrix; $V = V(\mathbf{Z}, t)$ is called value function and λ is a constant. If the random excitation $\mathbf{F}(t)$ is modelled as linear-filtering white noises, then Eq. (4) can be rewritten in the augmented form by incorporating the filtering system into the linearized system so that the stochastic dynamical programming principle can be applied similarly.

The optimal control law can be determined by minimizing the right-hand side of Eq. (8) or (9). Its governing equation is

$$\frac{\partial}{\partial \mathbf{u}_s}\left[L(\mathbf{Z}, \mathbf{u}_s) - \mathbf{B}_N^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{P}_N}\right] = 0, \tag{10}$$

where the generalized momentum vector $\mathbf{P}_N = \{P_1, P_2, \ldots, P_n\}^{\mathrm{T}}$ and control-device placement matrix $\mathbf{B}_N = [b_{ir}]_{n\times l}$. In general, let function L be quadratic in control force vector $\mathbf{u}_s$, that is

$$L(\mathbf{Z}, \mathbf{u}_s) = g(\mathbf{Z}) + \mathbf{u}_s^{\mathrm{T}}\mathbf{R}\mathbf{u}_s, \tag{11}$$

where $g(\mathbf{Z}) \geqslant 0$ and $\mathbf{R}$ is a positive-definite symmetric matrix. Then the optimal control force is obtained as follows:

$$\mathbf{u}_s = \tfrac{1}{2}\mathbf{R}^{-1}\mathbf{B}_N^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{P}_N}, \tag{12}$$

which depends on value function V. By substituting the expression of $\mathbf{u}_s$ obtained from Eq. (10) or (12) into Eqs. (8) and (9), the dynamical programming equations become other ones for the value function. If there exists a value function solution quadratic in generalized momenta or velocities, then Eq. (12) implies that the optimal control force $\mathbf{u}_s$ is a quasi-linear damping force due to the coefficients related to the system response statistics.

In the case that the random excitation cannot be modelled as a filtering white noise, the optimal control of system (4) with performance index (6) or (7) is treated as an optimal regulation problem. The optimal control force can be similarly determined by Eq. (10) or obtained as Eq. (12), but the dynamical programming equation for value function is

$$L(\mathbf{Z}, \mathbf{u}_s) + (\mathbf{A}\mathbf{Z} + \mathbf{U})^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} = 0. \tag{13}$$

Based on the LQ control method [6] possessing better characteristics such as simplicity, effectiveness and classical explicit solution of control law, the quasi-linear optimal control can be determined with the following functions:

$$L = \mathbf{Z}^{\mathrm{T}}\mathbf{S}\mathbf{Z} + \mathbf{u}_s^{\mathrm{T}}\mathbf{R}\mathbf{u}_s, \quad V = \mathbf{Z}^{\mathrm{T}}\mathbf{P}\mathbf{Z}, \tag{14}$$

where $\mathbf{S}$ is a positive semi-definite symmetric constant matrix and $\mathbf{P}$ is a symmetric matrix. The matrix $\mathbf{P}$ can be obtained by solving the following equation in the Riccati form:

$$\mathbf{S} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} = 0, \quad \mathbf{B} = [0_{l\times n}\ \mathbf{B}_N^{\mathrm{T}}]^{\mathrm{T}}. \tag{15}$$

In fact, the control force produced by semi-active control devices such as ER and MR dampers [3–5] does not meet Eq. (12) always and thus the required control force (12) needs to be adapted for semi-active control device implementation. The implemented semi-active optimal control force u_{rs}^{*} can be determined by minimizing the difference between implemented control force and required control force, that is

$$u_{rs}^{*} = u_{rs} + \Delta u_{rs}^{m}\,\mathrm{sgn}(u_{rs}^{*} - u_{rs}), \quad \Delta u_{rs}^{m} = \min_{u_{rs}^{*}} |u_{rs}^{*} - u_{rs}|, \tag{16}$$

$$r = 1, 2, \ldots, l,$$

where sgn(·) is the sign operator and sgn(0) = 0. For semi-active ER and MR dampers, the commanded control force (16) is just the clipped optimal control force [3–5] as

$$u_{rs}^{*} = \begin{cases} F_r^{*}\,\mathrm{sgn}(b_{ir}\dot{Q}_i), & F_r^{*} \geqslant 0, \\ 0, & F_r^{*} \leqslant 0, \end{cases} \tag{17a}$$

$$F_r^{*} = [\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{Z}]_r\,\mathrm{sgn}(b_{ir}\dot{Q}_i). \tag{17b}$$

According to the variational principle, the commanded semi-active control force (16) satisfies the dynamical programming equation with the constraint of semi-active control devices and therefore is a semi-active optimal control force implementable actually. The response statistics of the randomly excited controlled system can be first evaluated by using the linearized equation (4). Then the coefficient matrix **A** is calculated as a function of the response statistics and the coefficient matrix **P** of value function is solved by Eq. (15) so that the required control force (12) with (14) can be determined. At last, the semi-active optimal control force $\mathbf{u}_s^{*}$ is obtained from Eq. (16) by iteration.

3. Tall building structures with MR-TLCDs under random wind excitation

To illustrate the application and effectiveness of the proposed semi-active optimal control method for multi-degree-of-freedom systems, consider a high-rise building structure with *n*-storey and a semi-active MR-TLCD (Fig. 1) installed at the top floor. It is assumed that the building structure is subjected to a lateral horizontal wind excitation and the structural response is primarily in the along-wind horizontal direction. In the case of linear elastic shear-type structure,

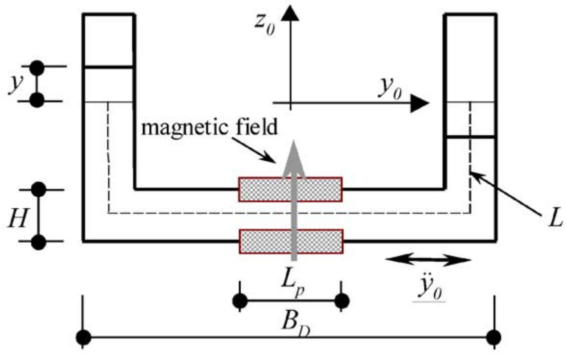

Fig. 1. Semi-active MR-TLCD.

the equations of motion of the structural system are expressed as [22]

$$M\ddot{X} + C\dot{X} + KX = F_W(t) + E_1 f_D, \tag{18a}$$

$$m_D\ddot{y} + u(\dot{y}) + k_D y = -\lambda m_D E_1^{\mathrm{T}}\ddot{X}, \tag{18b}$$

where X denotes the n-dimensional horizontal displacement vector of the structure; M, C and K are the $n \times n$-dimensional symmetric positive-definite mass, damping and stiffness matrices, respectively; $F_W(t)$ is the n-dimensional wind excitation vector; y denotes the relative displacement of liquid to U-shape container of the MR-TLCD; $m_D = \rho A_D L$ is the liquid mass and, ρ, A_D, L are, respectively, the liquid mass density, cross-sectional area and total length; $k_D = 2\rho g A_D$ is the equivalent stiffness and g is the gravity acceleration; $\lambda = B_D/L$ (<1) is the ratio of horizontal and total liquid lengths and B_D is the horizontal liquid length; $E_1 = \{1, 0, \ldots, 0\}^{\mathrm{T}}$ is an n-dimensional identity vector and f_D is the interaction force between the MR-TLCD and top floor of the structure, which is represented by

$$f_D = \frac{1}{\lambda}[(1-\lambda^2)m_D\ddot{y} + u(\dot{y}) + k_D y]. \tag{19}$$

For convenience, matrix and vector symbols are not in boldface here and hereafter. In Eq. (18b), the damping force $u(\dot{y})$ produced by the MR fluids can be separated into two parts as follows:

$$u(\dot{y}) = u_p(\dot{y}) + u_s(\dot{y}) \tag{20}$$

with

$$u_p(\dot{y}) = \tfrac{1}{2}\rho\delta A_D\dot{y}^2\,\mathrm{sgn}(\dot{y}); \quad u_s(\dot{y}) = \tau_y\left(\frac{cA_D L_p}{h}\right)\mathrm{sgn}(\dot{y}), \tag{21}$$

where δ is the overall head loss coefficient; c is a constant; τ_y denotes the yield stress of the MR fluids controlled by applied external voltage; L_p and h are, respectively, the length and depth of the MR fluids; u_p independent of τ_y denotes the uncontrollable part by applied external voltage and is a non-linear passive control damping force which can be incorporated into the uncontrolled system; u_s dependent on τ_y denotes the controllable part by applied external voltage with small power source and then is a semi-active control damping force which can be determined according to an optimal control strategy. It is seen from Eq. (21) that the controllable damping force u_s is positive or negative always corresponding to the relative velocity $\dot{y}$ so that the damping force is in the opposite direction of the relative motion.

The wind excitation F_W can be modelled as a random process vector with the Davenport power spectrum [24]. According to this model, the cross power spectral density of wind forces is

$$S_{Wij}(\omega) = \rho_a^2 C_D^2 V_{10}^2 A_i A_j \left(\frac{h_i h_j}{100}\right)^{\alpha} \mathrm{coh}(h_i, h_j, \omega)\frac{S_0(\omega)}{2\pi}, \tag{22}$$

$$i, j = 1, 2, \ldots, n,$$

where $S_{Wij}(\omega)$ denotes the cross spectrum of wind forces at the ith floor and jth floor; ρ_a is the air mass density; C_D is the drag coefficient; V_{10} is the mean wind velocity at 10 m height; A_i and A_j are the equivalent projection areas about the ith and jth floors, respectively; h_i and h_j are heights of the ith and jth floors; and α is a constant. $\mathrm{coh}(h_i, h_j, \omega)$ is the coherence function of wind forces, describing the spanwise correlation features of fluctuating wind forces; and $S_0(\omega)$ is the power spectral density of wind velocity. They are

$$\mathrm{coh}(h_i, h_j, \omega) = \exp\left\{-\frac{C_h|\omega||h_i - h_j|}{2\pi V_{10}}\right\}, \tag{23}$$

$$S_0(\omega) = 8\pi K_D V_{10}^2 \frac{\eta_0^2}{|\omega|(1+\eta_0^2)^{4/3}}, \quad \eta_0 = \frac{600\omega}{\pi V_{10}}, \tag{24}$$

where C_h is a decay constant and K_D is the ground coarse coefficient. The random wind excitation cannot be modelled as a filtering white noise.

The random wind response of the building structure can be expressed accurately using first several structural modes [22]. For the structural response, the first m ($\leqslant n$) modes are aimed at and taken to assemble into an $n \times m$-dimensional mode matrix Φ normalized with respect to the mass matrix M. By using the modal transformation technique, the equations of motion of the structural system (18) are converted into

$$\ddot{Y} + \Xi\dot{Y} + \Omega Y = \Phi^{\mathrm{T}} F_W(t) + \Phi^{\mathrm{T}} E_1 f_D, \tag{25a}$$

$$\ddot{y} + \frac{1}{m_D} u(\dot{y}) + \frac{k_D}{m_D} y = -\lambda E_1^{\mathrm{T}} \Phi \ddot{Y}, \tag{25b}$$

where $Y = \{y_1, y_2, \ldots, y_m\}^{\mathrm{T}}$ denotes the m-dimensional modal displacement vector; the $m \times m$-dimensional diagonal matrices $\Omega = \Phi^{\mathrm{T}} K \Phi = [\omega_i^2]$ and $\Xi = \Phi^{\mathrm{T}} C \Phi = [2\zeta_i\omega_i]$ under the assumption of damping matrix C diagonalizable by the mode matrix Φ, in which ω_i and ζ_i are, respectively, structural natural frequency and damping ratio of the ith mode. Combining Eqs. (25a) and (25b) yields the following augmented matrix equation for the structural system with MR-TLCD:

$$M_A \ddot{\bar{Y}} + C_A \dot{\bar{Y}} + K_A \bar{Y} + F_{Ap}(\bar{Y}) = F_{AW}(t) + F_{As}, \tag{26}$$

where the $(m+1)$-dimensional generalized displacement vector $\bar{Y}$, the $(m+1) \times (m+1)$-dimensional generalized mass matrix M_A, damping matrix C_A and stiffness matrix K_A, the $(m+1)$-dimensional generalized external force vector F_{AW}, passive control force vector F_{Ap} and semi-active control force vector F_{As} are, respectively,

$$\bar{Y} = \begin{Bmatrix} Y \\ y \end{Bmatrix}, \quad M_A = \begin{bmatrix} I_m & -(1-\lambda^2) m_D \Phi^{\mathrm{T}} E_1/\lambda \\ \lambda E_1^{\mathrm{T}} \Phi & 1 \end{bmatrix}, \tag{27a}$$

$$C_A = \begin{bmatrix} \Xi & 0_m \\ 0_m & 0_m \end{bmatrix}, \quad K_A = \begin{bmatrix} \Omega & -k_D \Phi^{\mathrm{T}} E_1/\lambda \\ 0 & k_D/m_D \end{bmatrix}, \tag{27b}$$

$$F_{AW} = \begin{Bmatrix} \Phi^{\mathrm{T}} F_W \\ 0 \end{Bmatrix}, \quad F_{Ap} = B_p u_p(\dot{y}), \tag{27c}$$

$$F_{As} = -B_p u_s(\dot{y}), \quad B_p = \begin{Bmatrix} -\Phi^{\mathrm{T}} E_1/\lambda \\ 1/m_D \end{Bmatrix} \tag{27d}$$

and I_m is the $m \times m$-dimensional identity matrix. The passive control force component u_p of the MR-TLCD is incorporated into the uncontrolled system. The generalized mass matrix M_A is non-singular since the determinant $|M_A| = 1 + (1-\lambda^2)m_D\sum_{i=1}^m \phi_{i1}^2 > 0$, in which ϕ_{i1} is the first element of the ith mode vector in matrix Φ. Pre-multiplying Eq. (26) by the inverse matrix M_A^{-1} and rewriting it in the form of state equation yield

$$\dot{Z} = A_L Z + F_N(Z) + F(t) + U, \tag{28}$$

where the $(2m+2)$-dimensional generalized state vector Z, the $(2m+2)\times(2m+2)$-dimensional coefficient matrix A_L, the $(2m+2)$-dimensional non-linear force vector F_N, external force vector F and control force vector U are

$$Z = \begin{Bmatrix} \bar{Y} \\ \dot{\bar{Y}} \end{Bmatrix}, \quad A_L = \begin{bmatrix} 0_{m+1} & I_{m+1} \\ -M_A^{-1}K_A & -M_A^{-1}C_A \end{bmatrix}, \tag{29a}$$

$$F_N = \begin{Bmatrix} 0_{(m+1)\times 1} \\ -M_A^{-1}B_p \end{Bmatrix} u_p(\dot{y}), \quad F = \begin{Bmatrix} 0_{(m+1)\times 1} \\ M_A^{-1}F_{AW} \end{Bmatrix}, \tag{29b}$$

$$U = \begin{Bmatrix} 0_{(m+1)\times 1} \\ -M_A^{-1}B_p \end{Bmatrix} u_s(\dot{y}). \tag{29c}$$

By applying the statistical linearization method [12–14] to non-linear state equation (28), the linearized equation is obtained as follows:

$$\dot{Z} = AZ + F(t) + U, \tag{30}$$

where the coefficient matrix $A = A_L + A_N$ and

$$A_N = \begin{bmatrix} 0_{(m+1)\times(2m+1)} & 0_{(m+1)\times 1} \\ 0_{(m+1)\times(2m+1)} & -M_A^{-1}B_p c_{eq} \end{bmatrix}, \quad c_{eq} = \rho\delta A_D\sqrt{\frac{2E[\dot{y}^2]}{\pi}}, \tag{31a}$$

$$U = -Bu_s(\dot{y}), \quad B = \begin{Bmatrix} 0_{(m+1)\times 1} \\ M_A^{-1}B_p \end{Bmatrix}, \tag{31b}$$

in which c_{eq} is an equivalent damping coefficient of the passive control force component u_p of the MR-TLCD. The semi-active control damping force component u_s of the MR-TLCD is expressed in the form of separation such that it is convenient to applying an optimal control strategy.

4. Semi-active optimal control law and response prediction

The response control of the structural system (18a) with the semi-active MR-TLCD (18b) under random wind excitation (22) can be achieved by the response control of the linearized system (30). Since the random wind excitation cannot be modelled as a filtering white noise, the optimal control of system (30) is treated as an optimal regulation problem. It is assumed that the system states such as displacements and velocities can be determined exactly by measurement. Then the optimal control problem is independent of the state observation problem and the optimal control law can be determined directly based on the dynamical programming principle [6]. For the system response control, performance indexes can be expressed as Eqs. (6) and (7). The performance index in infinite time interval is of the form

$$J = \lim_{T \to \infty} \frac{1}{T} \int_0^{\mathrm{T}} L(Z(t), u_s(t))\, \mathrm{d}t. \tag{32}$$

Based on the dynamical programming principle [6], the dynamical programming equation for system (30) with performance index (32) as an optimal regulation problem is established as follows:

$$\min_{u_s} \left\{ L + (AZ + U)^{\mathrm{T}} \frac{\partial V}{\partial Z} \right\} = 0. \tag{33}$$

With the similarity to Eqs. (10)–(15), the quasi-linear optimal control law can be determined by minimizing the left side of Eq. (33) and using Eq. (31b). For function L and value function V (14) quadratic in control force and system state, the optimal control force is

$$u_s = R^{-1} B^{\mathrm{T}} P Z, \tag{34}$$

where R is a positive constant and the coefficient matrix P can be obtained by solving algebraic matrix equation (15). Since matrix P is related to the system response statistics due to matrix A (31a), the control force u_s (34) is quasi-linear. Note that the control force required by (34) is a function of the state vector Z (29a) and the damping force produced by the MR-TLCD (21) is just a function of the state variable $\dot{y}$ (27a), so that they are not in agreement always. When the required control force is positive (or negative) while the producible damping force is negative (or positive), the clipping treatment to the required control force needs to be performed. The clipped optimal control force is

$$u_s^* = \begin{cases} F^* \operatorname{sgn}(\dot{y}), & F^* \geqslant 0, \\ 0, & F^* \leqslant 0, \end{cases} \tag{35a}$$

$$F^* = R^{-1} B^{\mathrm{T}} P Z \operatorname{sgn}(\dot{y}), \tag{35b}$$

which satisfies the dynamical programming equation (33) with the damping force constraint (21) according to the variational principle. The optimal control force (35) is implementable by the semi-active MR-TLCD in terms of Eq. (21) and then is the semi-active optimal control force. By denoting matrices

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_3 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & P_2 \\ P_2^{\mathrm{T}} & P_3 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \tag{36}$$

where S_i, P_i and A_i are the $(m+1) \times (m+1)$-dimensional sub-matrices, and by letting $S_1 = 0$, Eqs. (35b) and (15) are further expressed as

$$F^* = R^{-1} B_p^{\mathrm{T}} Q_3^{\mathrm{T}} \dot{Y} \operatorname{sgn}(\dot{y}), \tag{37}$$

$$S_3 - \bar{C}_A^{\mathrm{T}} Q_3^{\mathrm{T}} - Q_3 \bar{C}_A - R^{-1} Q_3 B_p B_p^{\mathrm{T}} Q_3^{\mathrm{T}} = 0, \tag{38}$$

where $Q_3 = P_3 M_A^{-1}$, $\bar{C}_A = C_A + C'_A$ and $C'_A = [0_{(m+1)\times m}, B_p c_{eq}]$.

To evaluate the control efficacy of the proposed non-linear stochastic optimal control method, the random response of the semi-actively controlled structure under wind excitation is then predicted and compared with that of the passively controlled structure in terms of performance criteria. By substituting the optimal control force (35) into Eq. (26) and applying the statistical linearization method to it, the following matrix equation for the semi-actively controlled structural system with MR-TLCD is obtained:

$$M_A \ddot{\bar{Y}} + \tilde{C}_A \dot{\bar{Y}} + K_A \bar{Y} = F_{AW}(t), \tag{39}$$

where $\tilde{C}_A = \bar{C}_A + C''_A$, $C''_A = B_p C_{seq}^{\mathrm{T}}$ and the equivalent coefficient vector

$$C_{seq} = \frac{1}{2R}\left\{ Q_3 B_p + E\left[\frac{\partial}{\partial \dot{\bar{Y}}} |B_p^{\mathrm{T}} Q_3^{\mathrm{T}} \dot{\bar{Y}}| \,\mathrm{sgn}(\dot{y})\right]\right\}. \tag{40}$$

The cross-power spectrum matrix of random responses of system (39) is represented by

$$S_{\bar{Y}\bar{Y}}(\omega) = H(\mathrm{j}\omega) S_F(\omega) H^{\mathrm{T}}(-\mathrm{j}\omega), \tag{41}$$

where the frequency-response function matrix $H(\mathrm{j}\omega)$ and the power spectrum matrix of random excitation $S_F(\omega)$ are

$$H(\mathrm{j}\omega) = (K_A - \omega^2 M_A + \mathrm{j}\omega \tilde{C}_A)^{-1}, \quad \mathrm{j} = \sqrt{-1}, \tag{42a}$$

$$S_F(\omega) = \begin{bmatrix} \Phi^{\mathrm{T}} S_W(\omega) \Phi & 0_{m\times 1} \\ 0_{1\times m} & 0 \end{bmatrix}, \tag{42b}$$

in which $S_W(\omega)$ is the wind power spectrum matrix with elements given by Eq. (22). The mean square response of system (30) can be evaluated by using the cross-power spectrum (41) as follows:

$$E[\bar{Y}_i^2] = \int_{-\infty}^{+\infty} S_{\bar{Y}_i\bar{Y}_i}(\omega)\,\mathrm{d}\omega, \quad E[\dot{\bar{Y}}_i^2] = \int_{-\infty}^{+\infty} \omega^2 S_{\bar{Y}_i\bar{Y}_i}(\omega)\,\mathrm{d}\omega. \tag{43}$$

Then the mean square displacement, acceleration and optimal control force of the semi-actively controlled structure (18a) are represented based on the modal transformation technique by

$$E[X_i^2] = \int_{-\infty}^{+\infty} S_{X_iX_i}(\omega)\,\mathrm{d}\omega, \quad E[\ddot{X}_i^2] = \int_{-\infty}^{+\infty} \omega^4 S_{X_iX_i}(\omega)\,\mathrm{d}\omega, \tag{44a}$$

$$E[u_s^{*2}] = C_{seq}^{\mathrm{T}} E[\dot{\bar{Y}} \dot{\bar{Y}}^{\mathrm{T}}] C_{seq}, \tag{44b}$$

where $S_{XX}(\omega) = \Phi S_{YY}(\omega) \Phi^{\mathrm{T}}$ is the cross-power spectrum matrix of the structural displacements and $S_{YY}(\omega)$ is the cross-power spectrum matrix of the modal displacements, which is a sub-matrix of the power spectrum matrix $S_{\bar{Y}\bar{Y}}(\omega)$.

The mean square response of the passively controlled structure corresponding to (18a) under random wind excitation can be obtained in the same way by eliminating the semi-active optimal control force. At last, the following performance criteria [7–9] are used for evaluating the control efficacy:

$$K_{response} = \frac{RMS(response_p) - RMS(response_s)}{RMS(response_p)} \times 100\%, \tag{45a}$$

$$K_{u_s} = \frac{\sqrt{E[u_s^{*2}]}}{\mathrm{tr}[M]g} \times 100\%, \tag{45b}$$

where $RMS(\cdot)$ denotes the root-mean-square operator; and subscripts p and s denote the passive and semi-active control, respectively. The ratio $K_{response}$ measures the percentage response reduction of the semi-actively and passively controlled structures or the control effectiveness. The ratio K_{u_s} measures the percentage value of the semi-active optimal control force relative to the total structural weight. The higher $K_{response}$ and smaller K_{u_s} indicate the control method with more response reduction capabilities.

5. Numerical results

A numerical study is conducted on the semi-active optimal control of a 51-storey building structure [22] subjected to wind loading and with an MR-TLCD at the top floor. The height of the building structure is 161.65 m, the total structural mass is 2.774×10^7 kg and the modal damping ratio is 0.03. The first five natural frequencies are 0.216, 0.940, 2.278, 3.941 and 5.932 Hz obtained from the three-dimensional finite element model of the structure. The parameter values are $\rho_a = 1.28\ \mathrm{kg/m^3}$, $C_D = 1.2$, $V_{10} = 45.3\ \mathrm{m/s}$, $A_i = 1$ (that is, the following numerical results for unit equivalent projection area of the wind loading), $\alpha = 0.19$, $C_h = 10$, $K_D = 0.02$ for the wind loading; and $m_D = 2.774 \times 10^5$ kg, $\omega_D = \sqrt{k_D/m_D} = 1.2195$ rad/s, $\lambda = 0.8, \delta = 30$ for the MR-TLCD. The wind power spectral density (22) for different floor height is given in Fig. 2. The weighting coefficients of control force and system state are $R = 10^6/\mathrm{tr}^2[M]$ and $S_3 = diag\{0.7, 0.8, 1.2, 1.5, 1.5, 0.3\}$. Some numerical results are displayed in Figs. 3–5.

Figs. 3 and 4 show, respectively, the displacement and acceleration responses of the semi-actively controlled structure by using the proposed method and the passively controlled structure

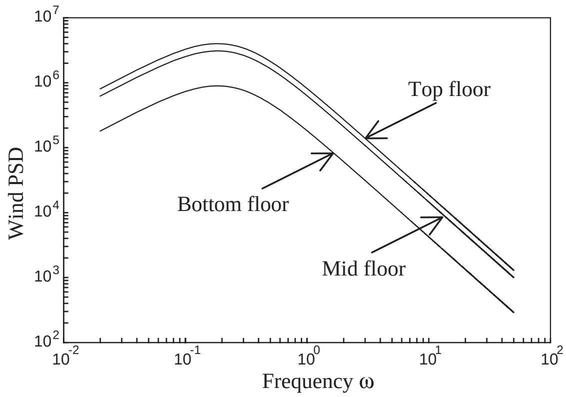

Fig. 2. Wind power spectral density (PSD) for different floor height.

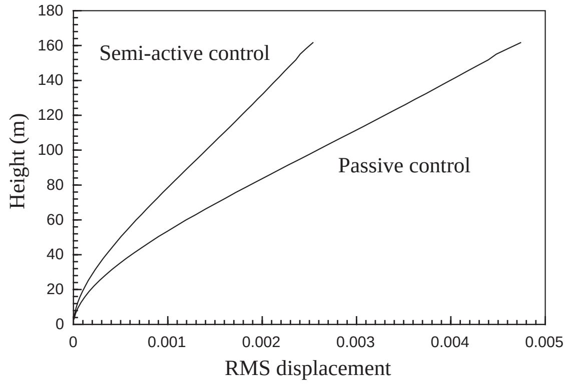

Fig. 3. Root-mean-square (RMS) displacements of the semi-actively and passively controlled structures versus floor height.

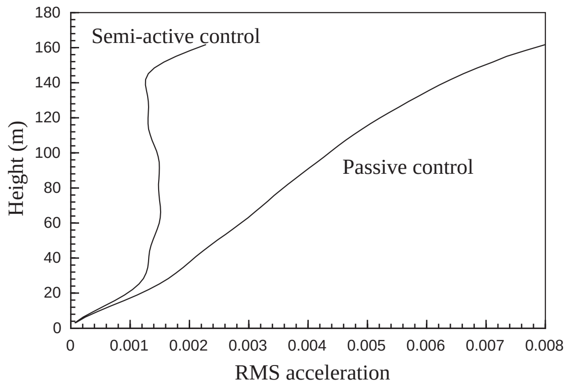

Fig. 4. Root-mean-square (RMS) accelerations of the semi-actively and passively controlled structures versus floor height.

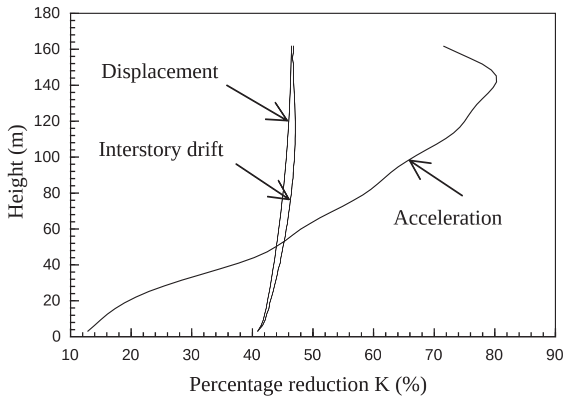

Fig. 5. Percentage relative reductions of the structural displacements, interstory drifts and accelerations versus floor height.

for various floor heights. It is seen that the random response reduction increases with the floor height. Fig. 5 illustrates the percentage relative reductions of structural displacements (K_X), interstory drifts ($K_{\Delta X}$) and accelerations ($K_{\ddot{X}}$) with the percentage relative optimal control force (K_{u_s}) equal to 0.0088%. About 46% displacement reduction and 72% acceleration reduction at the top floor are achieved. The percentage acceleration reduction at higher floor is more than at lower floor while the percentage displacement reduction varies slightly.

6. Conclusions

A semi-active optimal control method for non-linear multi-degree-of-freedom systems has been developed based on the dynamical programming principle, statistical linearization method and variational principle, and has been applied to a tall building structure with MR-TLCD for random wind response reduction. The developed control method has the following advantages: (a) it is applicable to non-linear and multi-degree-of-freedom systems under random excitation and with non-linear semi-active control devices; (b) it is simple, effective and has a classical explicit solution of control law to the dynamical programming equation; (c) it is uniform and available as the non-linear controlled system tends to corresponding linear one; (d) it combines the benefits of active and passive control methods, as illustrated by the random wind response control of the building structure with semi-active MR-TLCD. The semi-active optimal control force for MR-TLCDs is obtained in the form of a quasi-linear dissipative damping force, which does not have the potential to destabilize the structure. Numerical results for the structural system with an MR-TLCD at the top show that more random response reduction can be achieved by using the developed control method. In consequence, the developed semi-active optimal control method is potentially promising for structural control applications.

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