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Incremental harmonic balance method for predicting amplitudes of a multi-d.o.f. non-linear wheel shimmy system with combined Coulomb and quadratic damping

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Abstract

Incremental harmonic balance (IHB) formulations are derived for general multiple degrees of freedom (d.o.f.) non-linear autonomous systems. These formulations are developed for a concerned four-d.o.f. aircraft wheel shimmy system with combined Coulomb and velocity-squared damping. A multi-harmonic analysis is performed and amplitudes of limit cycles are predicted. Within a large range of parametric variations with respect to aircraft taxi velocity, the IHB method can, at a much cheaper cost, give results with high accuracy as compared with numerical results given by a parametric continuation method. In particular, the IHB method avoids the stiff problems emanating from numerical treatment of aircraft wheel shimmy system equations. The development is applicable to other vibration control systems that include commonly used dry friction devices or velocity-squared hydraulic dampers.

1. Introduction

Aircraft wheel shimmy is a self-excited oscillation caused by coupling of the lateral deflections and torsional oscillations about the gear swivel axis. The basic cause of shimmy is an energy transfer from the moving aircraft to the vibratory modes of the landing-gear system [1]. Increasing emphasis on reducing the weight of landing gear systems in modern aircraft has, in many cases, induced shimmy problems.

A great deal of researches has been devoted to aircraft wheel shimmy problems involving tests, analyses, and remedies to minimize or eliminate shimmy [2–10]. Most of the analytical studies are linear shimmy analyses in nature. Although linear description of the shimmy mechanism will

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usually reveal basic characteristics, it will generally fail to accurately predict the behavior of a landing gear system due to various inherent non-linearities such as non-linear tire/ground interactions, Coulomb friction in the strut oleos, lateral and torsional clearances in the torque links, and the quadratic damping of hydraulic shimmy damper [11].

Few analytical solutions to non-linear aircraft wheel shimmy problems have appeared in the published literature. Gordon Jr. [11] used a perturbation method to predict amplitudes of a two-d.o.f. non-linear shimmy system with velocity-squared damping. The predicted results confine its validity within a very narrow range of taxi velocity, i.e. taxi velocity less than 15 m/s. Burton [12] used the describing function method, which can be considered surrogate single-harmonic analysis technique, to analyze the non-linear motion of landing gear system with non-linear hydraulic steering subsystem. This approach results in a linearization process and the limit cycle amplitudes as well as stability boundaries are predicted using the linear system techniques. A similar approach was also adopted by Grossman [13] to obtain the equivalent damping and stiffness coefficients with respect to clearance, Coulomb friction and velocity-squared damping respectively. It is noteworthy that there exist another type of non-linear shimmy problems which can be classified as parametric shimmy. Parametric shimmy is caused by wheel unbalance and tire imperfections, and has been studied by Zhou and Zhu [14], Ho and Lai [15] and Nybakken [16] using classical perturbation methods.

It is well known that the common weakness of classical perturbation methods and equivalent linearization approaches restricts them to solving problems with weak non-linearities and within a narrow range of parametric variations. Although capable of treating strongly non-linear problems, numerical integration method is, on the other hand, inefficient for performing parametric studies. Furthermore, some special problems such as stiff problems will, with reference to aircraft wheel shimmy system, arise from numerical method [17]. Therefore, special measures and small step size are needed which make parametric studies extremely expensive.

The incremental harmonic balance (IHB) method is a powerful approach with unique advantages: it is capable of dealing with strongly non-linear systems to any desired accuracy, and it is ideally suited to large range parametric studies [18,19]. It was proposed originally by Cheung and Lau [20–22], and has been developed further and successfully applied to various non-linear problems [19]. Among these developments, some developments should be stressed because of their practical significance: Cheung et al. [18] applied the IHB method to cubic non-linear systems; Pierre et al. [23] modified the IHB method to analyze dry friction; Lau and Zhang [24] generalized the method to deal with piecewise-linear systems; Lau and Yuen [25] used the method to perform parametric studies on the Hopf bifurcation and limit cycle problems. Raghothama and Narayanan [26] used IHB method to investigate periodic oscillations and bifurcations of a two-dimensional airfoil in plunge and pitching motions with cubic pitching stiffness in incompressible flow.

In this paper, efforts are devoted to the application of the IHB method to a four-d.o.f. non-linear aircraft wheel shimmy system. Emphases are put on the detailed IHB formulations of velocity- squared damping which are much complicated than the counterpart of Coulomb damping. Limit cycle amplitudes of wheel shimmy are predicted using various harmonic terms. The IHB results are compared, within a large range of aircraft taxi velocity from 10 to 100 m/s, with numerical results given by a parametric continuation method [27,28]. Only four harmonic terms can produce results with high accuracy as compared to numerical results. Great importance lies in the fact that the IHB method is much faster and cheaper than numerical method and avoid stiff issue involved in numerical method.

2. Incremental harmonic balance method for multi-d.o.f. autonomous systems

For a multi-d.o.f. autonomous system, the non-linear equations, in general, have the following form:

$$\omega^2\bar{\mathbf{M}}\ddot{\mathbf{q}} + \omega\bar{\mathbf{C}}\dot{\mathbf{q}} + \bar{\mathbf{K}}\mathbf{q} + \bar{\mathbf{f}}(\mathbf{q}, \dot{\mathbf{q}}, \omega, \lambda) = \mathbf{0}, \tag{1}$$

where the dots denote derivatives with respect to the dimensionless time τ, and ω is the oscillation frequency and λ is the parameter incorporated for the purpose of parametric studies. $\mathbf{q} = \{q_1, q_2, \ldots, q_n\}^{\mathrm{T}}$ is the vector consisting of the unknowns of the system, and $\bar{\mathbf{f}}$ is a non-linear function of ω, λ, the dependent variable $\mathbf{q}$ and its velocity $\dot{\mathbf{q}}$, $\bar{\mathbf{M}}$, $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ are mass, linear viscous damping and linear stiffness matrices respectively.

The first step of the IHB method is a Netwon–Raphson procedure. Letting $\mathbf{q}_0$, ω_0 and λ_0 denote a state of vibration in hand, the neighboring state can be expressed by adding the corresponding increments to them as follows:

$$\mathbf{q} = \mathbf{q}_0 + \Delta\mathbf{q}, \quad \omega = \omega_0 + \Delta\omega, \quad \lambda = \lambda_0 + \Delta\lambda, \tag{2}$$

where

$$\mathbf{q}_0 = \{q_{10}, q_{20}, \ldots, q_{n0}\}^{\mathrm{T}}, \quad \Delta\mathbf{q} = \{\Delta q_1, \Delta q_2, \ldots, \Delta q_n\}^{\mathrm{T}}. \tag{3}$$

Substituting expressions (2) into Eq. (1) and neglecting small terms of higher order, one obtains the following linearized incremental equation:

$$\omega_0^2\bar{\mathbf{M}}\Delta\ddot{\mathbf{q}} + (\omega_0\bar{\mathbf{C}} + \bar{\mathbf{C}}_{\mathbf{N}})\Delta\dot{\mathbf{q}} + (\bar{\mathbf{K}} + \bar{\mathbf{K}}_{\mathbf{N}})\Delta\mathbf{q} = \bar{\mathbf{R}} - (2\omega_0\bar{\mathbf{M}}\ddot{\mathbf{q}}_0 + \bar{\mathbf{C}}\dot{\mathbf{q}}_0 + \bar{\mathbf{Q}})\Delta\omega - \bar{\mathbf{P}}\Delta\lambda, \tag{4}$$

where

$$\bar{\mathbf{K}}_{\mathbf{N}} = \left(\frac{\partial\bar{\mathbf{f}}}{\partial\mathbf{q}}\right)_0, \quad \bar{\mathbf{C}}_N = \left(\frac{\partial\bar{\mathbf{f}}}{\partial\dot{\mathbf{q}}}\right)_0, \quad \bar{\mathbf{Q}} = \left(\frac{\partial\bar{\mathbf{f}}}{\partial\omega}\right)_0, \quad \bar{\mathbf{P}} = \left(\frac{\partial\bar{\mathbf{f}}}{\partial\lambda}\right)_0, \tag{5}$$

$$\bar{\mathbf{R}} = -(\omega_0^2\bar{\mathbf{M}}\ddot{\mathbf{q}}_0 + \omega_0\bar{\mathbf{C}}\dot{\mathbf{q}}_0 + \bar{\mathbf{K}}\mathbf{q}_0 + \bar{\mathbf{f}}_0). \tag{6}$$

If the system concerned is an odd system, the steady state response can be expressed as the sum of N odd harmonic terms as follows [18]:

$$q_{j0} = \sum_{K=1}^{N} [a_{jK}\cos(2K-1)\tau + b_{jK}\sin(2K-1)\tau] = \mathbf{C}_s\mathbf{A}_j, \tag{7}$$

$$\Delta q_{j0} = \sum_{K=1}^{N} [\Delta a_{jK}\cos(2K-1)\tau + \Delta b_{jK}\sin(2K-1)\tau] = \mathbf{C}_s\Delta\mathbf{A}_j, \tag{8}$$

where

$$\mathbf{C}_s = \{\cos\tau, \cos 3\tau, \ldots, \cos(2N-1)\tau, \sin\tau, \sin 3\tau, \ldots, \sin(2N-1)\tau\}, \tag{9}$$

$$\mathbf{A}_j = \{a_{j1}, a_{j2}, \ldots, a_{jN}, b_{j1}, b_{j2}, \ldots, b_{jN}\}^{\mathrm{T}}. \tag{10}$$

The vectors of unknowns and their increments can be expressed as follows:

$$\mathbf{q}_0 = \mathbf{S}\mathbf{A}, \quad \Delta\mathbf{q}_0 = \mathbf{S}\Delta\mathbf{A}, \tag{11}$$

where

$$\mathbf{A} = \{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n\}^{\mathrm{T}}, \qquad \Delta\mathbf{A} = \{\Delta\mathbf{A}_1, \Delta\mathbf{A}_2, \ldots, \Delta\mathbf{A}_n\}^{\mathrm{T}}, \qquad \mathbf{S} = \begin{bmatrix} \mathbf{C}_s & & & \\ & \mathbf{C}_s & & \\ & & \ddots & \\ & & & \mathbf{C}_s \end{bmatrix}. \tag{12}$$

Substituting expression (11) into Eq. (4) and performing the Galerkin procedure give

$$\int_0^{2\pi} \delta(\Delta\mathbf{q})^{\mathrm{T}}[\omega_0^2\bar{\mathbf{M}}\Delta\ddot{\mathbf{q}} + (\omega_0\bar{C} + \bar{\mathbf{C}}_N)\Delta\dot{\mathbf{q}} + (\bar{\mathbf{K}} + \bar{\mathbf{K}}_N)\Delta\mathbf{q}]\,\mathrm{d}\tau = \int_0^{2\pi} \delta(\Delta\mathbf{q})^{\mathrm{T}}[\bar{\mathbf{R}} - (2\omega_0\bar{\mathbf{M}}\ddot{\mathbf{q}}_0 + \bar{\mathbf{C}}\dot{\mathbf{q}}_0 + \bar{\mathbf{Q}})\Delta\omega - \bar{\mathbf{P}}\Delta\lambda]\,\mathrm{d}\tau. \tag{13}$$

A set of linear equations in terms of $\Delta\mathbf{A}$, $\Delta\omega$ and $\Delta\lambda$ can be obtained readily

$$K_{mc}\Delta\mathbf{A} = \mathbf{R} - \mathbf{R}_{mc}\Delta\omega - \mathbf{P}\Delta\lambda, \tag{14}$$

where

$$\mathbf{K}_{mc} = \omega_0^2\mathbf{M} + \omega_0\mathbf{C} + \mathbf{C}_N + \mathbf{K} + \mathbf{K}_N, \tag{15}$$

$$\mathbf{R} = -(\omega_0^2\mathbf{M} + \omega_0\mathbf{C} + \mathbf{K})\mathbf{A} - \mathbf{f}_0, \tag{16}$$

$$\mathbf{R}_{mc} = (2\omega_0\mathbf{M} + \mathbf{C})\mathbf{A} + \mathbf{Q}, \tag{17}$$

$$\mathbf{M} = \int_0^{2\pi} \mathbf{S}^{\mathrm{T}}\bar{\mathbf{M}}\ddot{\mathbf{S}}\,\mathrm{d}\tau, \quad \mathbf{C} = \int_0^{2\pi} \mathbf{S}^{\mathrm{T}}\bar{\mathbf{C}}\dot{\mathbf{S}}\,\mathrm{d}\tau, \quad \mathbf{K} = \int_0^{2\pi} \mathbf{S}^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{S}\,\mathrm{d}\tau, \tag{18}$$

$$\mathbf{C}_N = \int_0^{2\pi} \mathbf{S}^{\mathrm{T}}\bar{\mathbf{C}}_N\dot{\mathbf{S}}\,\mathrm{d}\tau, \quad \mathbf{K}_N = \int_0^{2\pi} \mathbf{S}^{\mathrm{T}}\bar{\mathbf{K}}_N\mathbf{S}\,\mathrm{d}\tau, \tag{19}$$

$$\mathbf{f}_0 = \int_0^{2\pi} \mathbf{S}^{\mathrm{T}}\bar{\mathbf{f}}_0\,\mathrm{d}\tau, \quad \mathbf{Q} = \int_0^{2\pi} \mathbf{S}^{\mathrm{T}}\bar{\mathbf{Q}}\,\mathrm{d}\tau, \quad \mathbf{P} = \int_0^{2\pi} \mathbf{S}^{\mathrm{T}}\bar{\mathbf{P}}\,\mathrm{d}\tau. \tag{20}$$

It should be noted that in Eq. (14) the number of incremental unknowns is greater than the number of equations available due to the existence of $\Delta\omega$ and $\Delta\lambda$. A simple approach is adopted herein [25]. As far as only the limit cycles are concerned, λ is taken to be the control parameter (i.e. $\Delta\lambda = 0$) while $\Delta\omega$ should be regarded as an unknown. Therefore, one of the Fourier coefficients has to be fixed (e.g., $a_i = 0$ or $b_j = 0$) to take the number of equations equal to that of the unknowns. As indicated in Ref. [25], this will not cause any effects for autonomous system but it only represents a shift of one of dependent variable q_i on the time axis. Eq. (14) can be solved by a Newton–Raphson iteration scheme for a particular λ, and it is then added by an increment $\Delta\lambda$ and a new solution is sought by iteration. By successive use of augmentation and iteration process, a solution diagram may easily be traced.

It is worth mentioning that the entries of mass, linear damping and linear stiffness matrices given in Eq. (18) can be expressed explicitly for the purpose of simplicity. To do that, the following notations for mass matrix are introduced and the linear damping and stiffness matrices can be treated in a similar manner:

$$\bar{\mathbf{M}} = [\bar{m}_{ij}], \quad \mathbf{M} = [\mathbf{M}_{ij}], \quad \mathbf{M}_{ij} = \int_0^{2\pi} \mathbf{C}_s^{\mathrm{T}} \bar{m}_{ij} \ddot{\mathbf{C}}_s \, \mathrm{d}\tau, \quad [\mathbf{M}_{ij}] = \begin{bmatrix} [\mathbf{M}_{ij,11}] & [\mathbf{M}_{ij,12}] \\ [\mathbf{M}_{ij,21}] & [\mathbf{M}_{ij,22}] \end{bmatrix} \tag{21}$$

in which $i,j = 1, 2, \ldots, n$. From expressions (18), one can obtain the entries corresponding to the kth row and lth column element of each submatrices (e.g., $[\mathbf{M}_{ij,11}]_{kl}$ denotes the kth row and lth column element of submatrix $[\mathbf{M}_{ij,11}]$) as follows:

$$[\mathbf{M}_{ij,11}]_{kl} = -\bar{m}_{ij}\delta_{kl}\pi(2l-1)^2, \quad [\mathbf{M}_{ij,12}]_{kl} = [\mathbf{M}_{ij,21}]_{kl} = \mathbf{0}, \quad [\mathbf{M}_{ij,22}]_{kl} = -\bar{m}_{ij}\delta_{kl}\pi(2l-1)^2, \tag{22}$$

$$[\mathbf{C}_{ij,11}]_{kl} = 0, \quad [\mathbf{C}_{ij,12}]_{kl} = \bar{c}_{ij}\delta_{kl}\pi(2l-1), \quad [\mathbf{C}_{ij,21}]_{kl} = -\bar{c}_{ij}\delta_{kl}\pi(2l-1), \quad [\mathbf{C}_{ij,22}]_{kl} = 0, \tag{23}$$

$$[\mathbf{K}_{ij,11}]_{kl} = \bar{k}_{ij}\delta_{kl}\pi, \quad [\mathbf{K}_{ij,12}]_{kl} = 0, \quad [\mathbf{K}_{ij,21}]_{kl} = 0, \quad [\mathbf{K}_{ij,22}]_{kl} = \bar{k}_{ij}\delta_{kl}\pi, \tag{24}$$

where δ_{kl} is the Kronecker delta function. Expressions given in Eqs. (22)–(24) can simplify the programming and avoid the integration involved in Eq. (18). This rule is complied with in the following, and explicit expressions are given provided the integration can be calculated analytically.

For non-linear parts of matrices and vectors, analytical expressions are not available and resort must be made to numerical calculation. The same matrix and vector splitting scheme as given in Eq. (21) is used for non-linear damping matrix $\mathbf{C}_N$ and vector $\mathbf{f}_0$, and they can be written as follows:

$$\bar{\mathbf{C}}_N = [\bar{c}_{Nij}], \quad \mathbf{C}_N = [\mathbf{C}_{Nij}], \quad \mathbf{C}_{Nij} = \int_0^{2\pi} \mathbf{C}_s^{\mathrm{T}} \bar{c}_{Nij} \dot{\mathbf{C}}_s \, \mathrm{d}\tau, \quad [\mathbf{C}_{Nij}] = \begin{bmatrix} [\mathbf{C}_{Nij,11}] & [\mathbf{C}_{Nij,12}] \\ [\mathbf{C}_{Nij,21}] & [\mathbf{C}_{Nij,22}] \end{bmatrix}, \tag{25}$$

$$\bar{\mathbf{f}}_0 = \{\bar{f}_{01}, \bar{f}_{02}, \bar{f}_{03}, \ldots, \bar{f}_{0n}\}^{\mathrm{T}}, \quad \mathbf{f}_0 = \{\mathbf{f}_{01}, \mathbf{f}_{02}, \mathbf{f}_{03}, \ldots, \mathbf{f}_{0n}\}^{\mathrm{T}}, \quad \mathbf{f}_{0i} = \int_0^{2\pi} \mathbf{C}_s^{\mathrm{T}} \bar{f}_{0i} \, \mathrm{d}\tau, \quad \mathbf{f}_{0i} = \begin{bmatrix} \mathbf{f}_{0i1} \\ \mathbf{f}_{0i2} \end{bmatrix}. \tag{26}$$

Similar expressions exist for non-linear stiffness matrix $\mathbf{K}_N$ and vector $\mathbf{Q}$.

3. IHB Formulations of an aircraft wheel shimmy system with combined Coulomb and quadratic damping

For a four-d.o.f. non-linear aircraft wheel shimmy system, the analytical model and the non-linearity are presented in Figs. 1 and 2 respectively. The non-linear equations of this autonomous system can be written as follows:

$$I_{LG}\ddot{\theta}_s + mHt\ddot{\theta} + I_{wp}\frac{V}{r}\dot{\theta} - b\sin k/V\dot{y} + K_{11}\theta_s + K_{12}\theta + K_{14}y = 0, \tag{27}$$

$$\begin{aligned} mHt\ddot{\theta}_s + I\ddot{\theta} - I_{wp}\frac{V}{r}\dot{\theta}_s + \frac{b\cos k}{V}\dot{y} + K_{21}\theta_s + K_{22}\theta + K_{24}y \\ + T_{CF}\,\mathrm{sign}(\dot{\theta}) + K_t(\theta - \theta_1) = 0, \end{aligned} \tag{28}$$

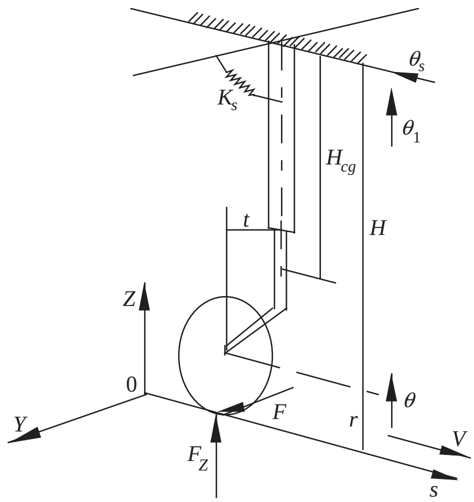

Fig. 1. Aircraft wheel shimmy analytical model.

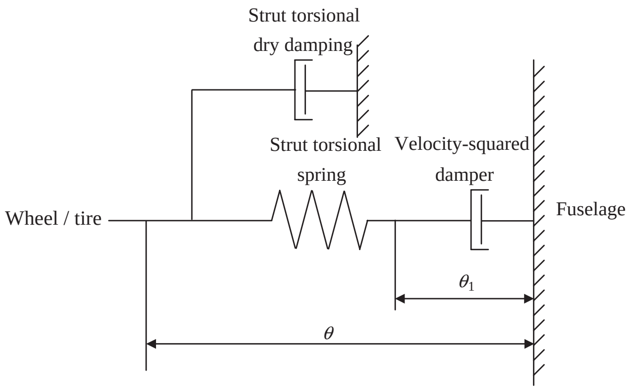

Fig. 2. Non-linearity in aircraft wheel shimmy.

$$I_\beta \ddot{\theta}_1 + C_t \dot{\theta}_1 + C_d \dot{\theta}_1^2 \,\mathrm{sign}(\dot{\theta}_1) - K_t(\theta - \theta_1) = 0, \tag{29}$$

$$\ddot{y} + \beta V \dot{y} + K_{41}\theta_s + K_{42}\theta + K_{44} y = 0, \tag{30}$$

where $\mathrm{sign}(\dot{\theta})$ is the sign function; $\theta_s, \theta_1, \theta$, and y are the lateral tilt angle of strut, the torsional angle of shimmy damper, the swiveling angle of the wheel about pivot and the lateral displacement of tire respectively; V is aircraft taxi velocity; m is the wheel mass, and I_{LG}, I, I_{wp} and I_β are moment of inertia of the landing gear system about trunion, moment of inertia of wheel about strut axle, moment of inertia of wheel about wheel axle, and moment of inertia of shimmy damper respectively; K_s is the lateral tilt stiffness of strut, K_t is the torsional stiffness of damper linkage system; C_t is the linear damping coefficient of shimmy damper, and C_d is the quadratic damping coefficient of damper; T_{CF} is the torque due to Coulomb friction; H and t are the distance from trunion to axle and the trail length and shown in Fig. 1 respectively; β and b

are tire rolling coefficient and tire torsional stiffness coefficient respectively; k is the longitudinal tilt angle of strut and K_{ij} $(i,j = 1,2,3,4)$ are the equivalent stiffness coefficients.

Recalling Eq. (1) from Eqs. (27)–(30) one can write

$$q_1 = \theta_s, \quad q_2 = \theta, \quad q_3 = \theta_1, \quad q_4 = y, \quad \lambda = V \tag{31}$$

and

$$\bar{\mathbf{f}} = \{0, T_{CF}\,\mathrm{sign}(\dot{q}_2), C_d\omega^2 q_3^2\,\mathrm{sign}(\dot{q}_3), 0\}^{\mathrm{T}}. \tag{32}$$

Using the definitions given in Eq. (5) gives

$$\bar{c}_{N22} = 2T_{CF}\delta(\dot{q}_{20}), \tag{33}$$

$$\bar{c}_{N33} = 2C_d\omega_0^2[\dot{q}_{30}\,\mathrm{sign}(\dot{q}_{30}) + \dot{q}_{30}^2\delta(\dot{q}_{30})], \quad \bar{c}_{Nij} = 0, \;\; i \neq 3 \text{ or } j \neq 3, \tag{34}$$

$$\bar{\mathbf{K}}_N = \mathbf{0}, \quad \bar{\mathbf{P}} = \mathbf{0}, \quad \bar{\mathbf{Q}} = \{0, 0, 2C_d\omega_0\dot{q}_{30}^2\,\mathrm{sign}(\dot{q}_{30}), 0\}^{\mathrm{T}}, \tag{35}$$

where $\delta(\dot{q}_{20})$ is the Dirac delta function, and Eq. (33) can be obtained, referring to Ref. [23] for details, by taking the derivative of sign function into account.

From Eqs. (25), (26) and (32)–(35), it is clear that the evaluation of the components of $\mathbf{C}_{Nij}$, $\mathbf{f}_{0i}$ and $\mathbf{Q}_i$ requires a knowledge of the zeros of the function $\dot{q}_{20}$ or the function $\dot{q}_{30}$. This is achieved at each iteration using the bisection and secant methods. Let $\tau_1, \tau_2, \ldots, \tau_{M'}$ be the M' zeros of $\dot{q}_{20}$, for example, and $s_1, s_2, \ldots, s_{M'+1}$ be the sign of $\dot{q}_{20}$, respectively, in the intervals $[\tau_0, \tau_1]$, $[\tau_1, \tau_2], \ldots, [\tau_{M'}, \tau_{M'+1}]$, with $\tau_0 = 0$ and $\tau_{M'+1} = 2\pi$. Pierre [23] presented detailed formulations to calculate the non-linear terms stemming from Coulomb damping, and the details are omitted here. The results are summarized as follows:

$$[\mathbf{C}_{N22,11}]_{kl} = 2(2l-1)T_{CF}\sum_{v=1}^{M'}\cos(2k-1)\tau_v\sin(2l-1)\tau_v/\Phi_2(\tau_v), \tag{36}$$

$$[\mathbf{C}_{N22,12}]_{kl} = -2(2l-1)T_{CF}\sum_{v=1}^{M'}\cos(2k-1)\tau_v\cos(2l-1)\tau_v/\Phi_2(\tau_v), \tag{37}$$

$$[\mathbf{C}_{N22,21}]_{kl} = 2(2l-1)T_{CF}\sum_{v=1}^{M'}\sin(2k-1)\tau_v\sin(2l-1)\tau_v/\Phi_2(\tau_v), \tag{38}$$

$$[\mathbf{C}_{N22,22}]_{kl} = -2(2l-1)T_{CF}\sum_{v=1}^{M'}\sin(2k-1)\tau_v\cos(2l-1)\tau_v/\Phi_2(\tau_v), \tag{39}$$

$$[\mathbf{f}_{021}]_k = \frac{T_{CF}}{2k-1}\sum_{v=0}^{M'} s_{v+1}[\sin(2k-1)\tau_{v+1} - \sin(2k-1)\tau_v], \tag{40}$$

$$[\mathbf{f}_{022}]_k = \frac{T_{CF}}{2k-1}\sum_{v=0}^{M'} s_{v+1}[\cos(2k-1)\tau_v - \cos(2k-1)\tau_{v+1}] \tag{41}$$

in which

$$\Phi_i(\tau_v) = \sum_{u=1}^{N} (2u-1)^2 [a_{iu} \cos(2u-1)\tau_v + b_{iu} \sin(2u-1)\tau_v]. \tag{42}$$

In the following, emphasis is placed on the derivation of non-linear terms arising from quadratic damping, since the formulations are much more complicated than that of Coulomb damping. From Eq. (7) one can write

$$\dot{q}_{30} = \sum_{n=1}^{N} (2n-1)[-a_{3n} \sin(2n-1)\tau + b_{3n} \cos(2n-1)\tau], \tag{43}$$

$$\begin{aligned} \dot{q}_{30}^2 = 1/2 \sum_{n=1}^{N} \sum_{m=1}^{N} & (2n-1)(2m-1)[(a_{3n}a_{3m} + b_{3n}b_{3m})\cos(2n-2m)\tau \\ & + (-a_{3n}a_{3m} + b_{3n}b_{3m})\cos(2n+2m-2)\tau + (-a_{3n}b_{3m} + b_{3n}a_{3m})\sin(2n-2m)\tau \\ & + (-a_{3n}b_{3m} - b_{3n}a_{3m})\sin(2n+2m-2)\tau]. \end{aligned} \tag{44}$$

Substituting Eqs. (43) and (44) into Eq. (34) and considering Eq. (25) gives

$$[\mathbf{C}_{N33,11}]_{kl} = -2(2l-1)C_d\omega_0^2 \int_0^{2\pi} [\dot{q}_{30} \operatorname{sign}(\dot{q}_{30}) + \dot{q}_{30}^2 \delta(\dot{q}_{30})] \cos(2k-1)\tau \sin(2l-1)\tau \, d\tau, \tag{45}$$

$$[\mathbf{C}_{N33,12}]_{kl} = 2(2l-1)C_d\omega_0^2 \int_0^{2\pi} [\dot{q}_{30} \operatorname{sign}(\dot{q}_{30}) + \dot{q}_{30}^2 \delta(\dot{q}_{30})] \cos(2k-1)\tau \cos(2l-1)\tau \, d\tau, \tag{46}$$

$$[\mathbf{C}_{N33,21}]_{kl} = -2(2l-1)C_d\omega_0^2 \int_0^{2\pi} [\dot{q}_{30} \operatorname{sign}(\dot{q}_{30}) + \dot{q}_{30}^2 \delta(\dot{q}_{30})] \sin(2k-1)\tau \sin(2l-1)\tau \, d\tau, \tag{47}$$

$$[\mathbf{C}_{N33,22}]_{kl} = 2(2l-1)C_d\omega_0^2 \int_0^{2\pi} [\dot{q}_{30} \operatorname{sign}(\dot{q}_{30}) + \dot{q}_{30}^2 \delta(\dot{q}_{30})] \sin(2k-1)\tau \cos(2l-1)\tau \, d\tau. \tag{48}$$

Efforts are devoted here to express Eqs. (45)–(48) to facilitate computer programming. To do that, we introduce

$$\Gamma(i, a_{3n}, b_{3n}) = a_{3n}(\sin i\tau_{v+1} - \sin i\tau_v)/i - b_{3n}(\cos i\tau_{v+1} - \cos i\tau_v)/i. \tag{49}$$

$$\Pi(i, a_{3n}, b_{3n}) = a_{3n}(\cos i\tau_{v+1} - \cos i\tau_v)/i + b_{3n}(\sin i\tau_{v+1} - \sin i\tau_v)/i. \tag{50}$$

Here $\tau_1, \tau_2, \ldots, \tau_M$ be the M zeros of $\dot{q}_{30}$, for example, and $s_1, s_2, \ldots, s_{M+1}$ be the sign of $\dot{q}_{30}$ following the fashion of $\dot{q}_{20}$. The first part in the integrals in Eqs. (45)–(48) then can be calculated from the following expressions:

$$\begin{aligned}&\int_0^{2\pi} \dot{q}_{30}\,\mathrm{sign}(\dot{q}_{30})\cos(2k-1)\tau\,\sin(2l-1)\tau\,\mathrm{d}\tau\\&\quad=\frac{1}{4}\sum_{v=0}^{M} s_{v+1}\left\{\sum_{n=1}^{N}(2n-1)\{\Gamma(2n+2k+2l-3,a_{3n},b_{3n})-\Gamma(2n-2k-2l+1,a_{3n},b_{3n})\right.\\&\qquad\left.-\Gamma(2n+2k-2l-1,a_{3n},b_{3n})+\Gamma(2n-2k+2l-1,a_{3n},b_{3n})\}\right\},\end{aligned}\tag{51}$$

$$\begin{aligned}&\int_0^{2\pi} \dot{q}_{30}\,\mathrm{sign}(\dot{q}_{30})\cos(2k-1)\tau\,\cos(2l-1)\tau\,\mathrm{d}\tau\\&\quad=\frac{1}{4}\sum_{v=0}^{M} s_{v+1}\left\{\sum_{n=1}^{N}(2n-1)\{\Pi(2n+2k+2l-3,a_{3n},b_{3n})+\Pi(2n-2k-2l+1,a_{3n},b_{3n})\right.\\&\qquad\left.+\Pi(2n+2k-2l-1,a_{3n},b_{3n})+\Pi(2n-2k+2l-1,a_{3n},b_{3n})\}\right\},\end{aligned}\tag{52}$$

$$\begin{aligned}&\int_0^{2\pi} \dot{q}_{30}\,\mathrm{sign}(\dot{q}_{30})\sin(2k-1)\tau\,\sin(2l-1)\tau\,\mathrm{d}\tau\\&\quad=\frac{1}{4}\sum_{v=0}^{M} s_{v+1}\left\{\sum_{n=1}^{N}(2n-1)\{-\Pi(2n+2k+2l-3,a_{3n},b_{3n})-\Pi(2n-2k-2l+1,a_{3n},b_{3n})\right.\\&\qquad\left.+\Pi(2n+2k-2l-1,a_{3n},b_{3n})+\Pi(2n-2k+2l-1,a_{3n},b_{3n})\}\right\},\end{aligned}\tag{53}$$

$$\begin{aligned}&\int_0^{2\pi} \dot{q}_{30}\,\mathrm{sign}(\dot{q}_{30})\cos(2k-1)\tau\,\sin(2l-1)\tau\,\mathrm{d}\tau\\&\quad=\frac{1}{4}\sum_{v=0}^{M} s_{v+1}\left\{\sum_{n=1}^{N}(2n-1)\{\Gamma(2n+2k+2l-3,a_{3n},b_{3n})-\Gamma(2n-2k-2l+1,a_{3n},b_{3n})\right.\\&\qquad\left.+\Gamma(2n+2k-2l-1,a_{3n},b_{3n})-\Gamma(2n-2k+2l-1,a_{3n},b_{3n})\}\right\}.\end{aligned}\tag{54}$$

Let

$$\begin{aligned}\Lambda(\tau_v)=&\left\{\sum_{n=1}^{N}\sum_{m=1}^{N}(2n-1)(2m-1)\{(a_{3n}a_{3m}+b_{3n}b_{3m})\cos(2n-2m)\tau_v\right.\\&+(-a_{3n}a_{3m}+b_{3n}b_{3m})\cos(2n+2m-2)\tau_v+(-a_{3n}b_{3m}+b_{3n}a_{3m})\sin(2n-2m)\tau_v\\&\left.-(a_{3n}b_{3m}+b_{3n}a_{3m})\sin(2n+2m-2)\tau_v\}\right\}/\Phi_3(\tau_v),\end{aligned}\tag{55}$$

where $\Phi_3(\tau_v)$ is given in Eq. (42), then the second part in the integrals in Eqs. (45)–(48) can be calculated from the following expressions:

$$\int_0^{2\pi} \dot{q}_{30}^2 \delta(\dot{q}_{30}) \cos(2k-1)\tau \sin(2l-1)\tau \, d\tau = -\frac{1}{2} \sum_{v=1}^{M} \Lambda(\tau_v) \cos(2k-1)\tau_v \sin(2l-1)\tau_v, \quad (56)$$

$$\int_0^{2\pi} \dot{q}_{30}^2 \delta(\dot{q}_{30}) \cos(2k-1)\tau \cos(2l-1)\tau \, d\tau = -\frac{1}{2} \sum_{v=1}^{M} \Lambda(\tau_v) \cos(2k-1)\tau_v \cos(2l-1)\tau_v, \quad (57)$$

$$\int_0^{2\pi} \dot{q}_{30}^2 \delta(\dot{q}_{30}) \sin(2k-1)\tau \sin(2l-1)\tau \, d\tau = -\frac{1}{2} \sum_{v=1}^{M} \Lambda(\tau_v) \sin(2k-1)\tau_v \sin(2l-1)\tau_v, \quad (58)$$

$$\int_0^{2\pi} \dot{q}_{30}^2 \delta(\dot{q}_{30}) \sin(2k-1)\tau \cos(2l-1)\tau \, d\tau = -\frac{1}{2} \sum_{v=1}^{M} \Lambda(\tau_v) \sin(2k-1)\tau_v \cos(2l-1)\tau_v. \quad (59)$$

In a similar manner, if one designates

$$\Theta_s(i) = [\sin(i\tau_{v+1}) - \sin(i\tau_v)]/i, \qquad \Theta_c(i) = [\cos(i\tau_{v+1}) - \cos(i\tau_v)]/i, \quad (60)$$

$$\begin{aligned}\Omega_k = \sum_{n=1}^{N} \sum_{m=1}^{N} & (2n-1)(2m-1)\{(a_{3n}a_{3m} + b_{3n}b_{3m})[\Theta_s(2n-2m+2k-1) \\ & + \Theta_s(2n-2m-2k+1)] \\ & + (-a_{3n}a_{3m} + b_{3n}b_{3m})[\Theta_s(2n+2m+2k-3) + \Theta_s(2n+2m-2k-1)] \\ & + (a_{3n}b_{3m} - b_{3n}a_{3m})[\Theta_c(2n-2m+2k-1) + \Theta_c(2n-2m-2k+1)] \\ & + (a_{3n}b_{3m} + b_{3n}a_{3m})[\Theta_c(2n+2m+2k-3) + \Theta_c(2n+2m-2k-1)]\} \end{aligned} \quad (61)$$

and

$$\begin{aligned}\Psi_k = \sum_{n=1}^{N} \sum_{m=1}^{N} & (2n-1)(2m-1)\{(a_{3n}a_{3m} + b_{3n}b_{3m})[-\Theta_c(2n-2m+2k-1) \\ & + \Theta_c(2n-2m-2k+1)] \\ & + (-a_{3n}a_{3m} + b_{3n}b_{3m})[-\Theta_c(2n+2m+2k-3) + \Theta_s(2n+2m-2k-1)] \\ & + (a_{3n}b_{3m} - b_{3n}a_{3m})[\Theta_s(2n-2m+2k-1) - \Theta_s(2n-2m-2k+1)] \\ & + (a_{3n}b_{3m} + b_{3n}a_{3m})[\Theta_s(2n+2m+2k-3) - \Theta_s(2n+2m-2k-1)]\} \end{aligned} \quad (62)$$

one can obtain

$$[\mathbf{f}_{031}]_k = \int_0^{2\pi} \bar{f}_{03} \cos(2k-1)\tau \, d\tau = \tfrac{1}{4} C_d \omega_0^2 \sum_{v=0}^{M} s_{v+1} \Omega_k, \quad (63)$$

$$[\mathbf{f}_{032}]_k = \int_0^{2\pi} \bar{f}_{03} \sin(2k-1)\tau \, d\tau = \tfrac{1}{4} C_d \omega_0^2 \sum_{v=0}^{M} s_{v+1} \Psi_k, \quad (64)$$

and

$$[\mathbf{Q}_{31}]_k = \frac{2}{\omega_0}[\mathbf{f}_{031}]_k, \quad [\mathbf{Q}_{32}]_k = \frac{2}{\omega_0}[\mathbf{f}_{032}]_k. \tag{65}$$

4. Numerical results

With all the above IHB formulations in hand, the implementation procedure is straightforward. A computer code is written to analyze the wheel shimmy of a fighter using IHB method. Four different cases are studied: system with friction torque $T_{CF} = 5.0$ N m and quadratic damping coefficient $C_d = 5.0$ N m s^2; system with friction torque $T_{CF} = 20.0$ N m and quadratic damping coefficient $C_d = 10.0$ N m s^2; system with friction torque $T_{CF} = 15.0$ N m and quadratic damping coefficient $C_d = 20.0$ N m s^2; and system with friction torque $T_{CF} = 10.0$ N m and quadratic damping coefficient $C_d = 50.0$ N m s^2. Taking one harmonic term and four harmonic terms into consideration, the IHB results for limit cycle amplitudes which correspond to these four cases are presented and compared with numerical results. The numerical results are obtained by a parametric continuation method which combines shooting technique, arc-length technique and continuation method to form a versatile tool for various non-linear vibration problems [27,28]. Figs. 3–6 indicate that, within a large range of aircraft taxi velocity from 10 to 100 m/s, good agreements are obtained between IHB results and numerical results, which demonstrate the correctness and effectiveness of the proposed method. Furthermore, four harmonic terms can give IHB results nearly identical to numerical results with regard to limit cycle amplitudes. This proves the method has a good convergent property.

It should be noted that in Eqs. (27)–(30) $I_\beta \ll I$, this causes stiff problems using numerical methods to solve Eqs. (27)–(30) [17]. Ill-conditions of the equations may cause unstable solutions and loss of convergence in parametric continuation method. Therefore, small step sizes are used to guarantee obtaining stable convergent numerical solutions. This causes the parametric studies of aircraft wheel shimmy analysis to be extremely expensive. On the contrary, the IHB method, at

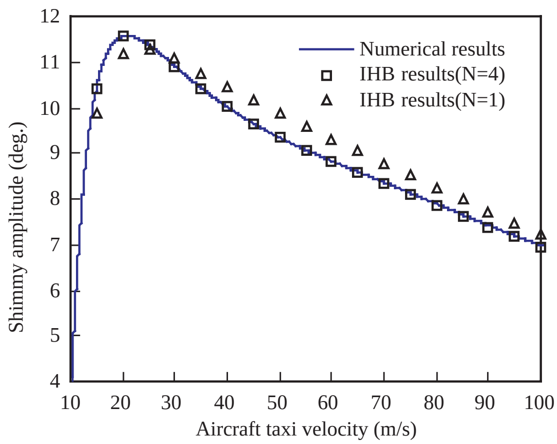

Fig. 3. Shimmy amplitude given by IHB and numerical methods ($C_d = 5, T_{CF} = 5$).

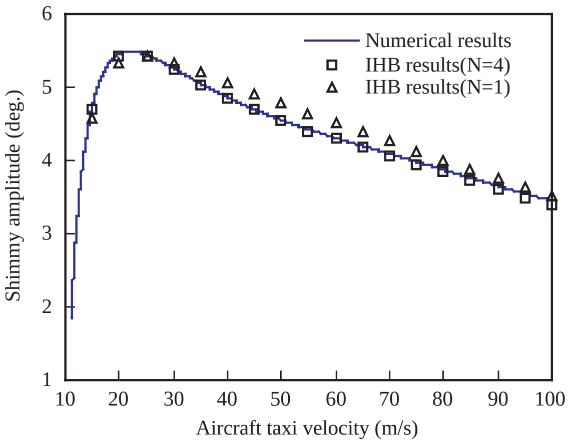

Fig. 4. Shimmy amplitude given by IHB and numerical methods ($C_d = 10, T_{CF} = 20$).

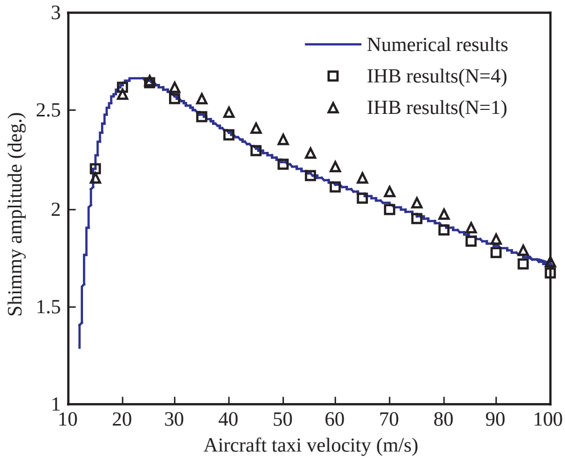

Fig. 5. Shimmy amplitude given by IHB and numerical methods ($C_d = 20, T_{CF} = 15$).

the price of tedious but once-done IHB derivations, bears the merits of efficiency, good convergence and particular avoidance of stiff problems for aircraft wheel shimmy system.

5. Conclusions

IHB formulations for a four-d.o.f. autonomous aircraft wheel shimmy system are presented with emphases put on the detailed derivations of non-linear terms arising from quadratic velocity-squared damping. The development presented in this paper, we believe, is of great importance due to common applications of hydraulic dampers, which in general exhibit a non-linear velocity-squared damping characteristics. Furthermore, it is also of practical significance since numerical

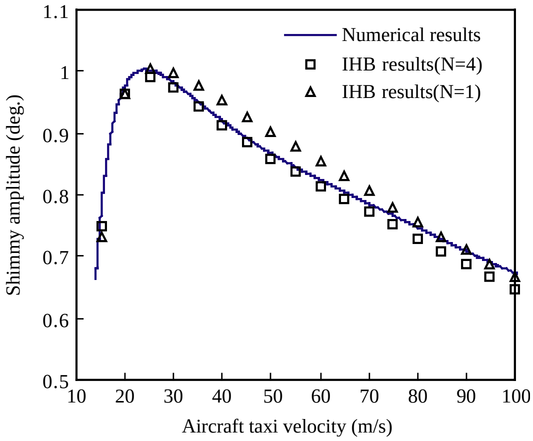

Fig. 6. Shimmy amplitude given by IHB and numerical methods ($C_d = 50, T_{CF} = 10$).

methods may suffer from stiff problems for particular aircraft wheel shimmy system, and the IHB method constitutes great advantages over numerical methods for large range parametric studies. The IHB method is not limited to weak non-linearity and can provide results of good accuracy within a large range of parametric variations. Four different cases are studied by IHB method and the IHB results are compared with numerical results for limit cycle amplitudes. Good agreements are obtained within a large range of aircraft taxi velocity from 10 to 100 m/s, and four harmonic terms are found to be adequate for good accuracy at a much cheaper cost as compared with numerical methods. The numerical results demonstrate the correctness and effectiveness of present development, and it can be used for non-linear dynamic analysis of other systems with dry friction devices or hydraulic dampers.

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