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On the statistics of natural frequency splitting for rings with random mass imperfections

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Abstract

This paper considers the statistical distribution of natural frequency splits for an initially perfect ring with different types of random mass imperfection. The analysis used to derive analytical expressions for the natural frequency splits is based on a Rayleigh–Ritz approach, in which it is assumed that the mode shapes of the imperfect rings are identical to those of a perfect ring. The types of random mass imperfection investigated are: (i) random harmonic variations in the mass per unit length around the circumference of the ring; (ii) the attachment of random point masses at random locations on the ring; and (iii) the attachment of random point masses at uniformly spaced positions on the ring. For case (i) it is found that the frequency splits always have a Rayleigh distribution. For case (ii) an expression for the statistical distribution is deduced which tends to a Rayleigh distribution as the number of attached masses increases. For case (iii) it is found that the frequency split distribution is dependent upon the mode considered and the number of attached masses, and that in some situations the frequency splits have a "half Gaussian" (i.e., non-Rayleigh) distribution.

1. Introduction

It is well known that the vibration modes of a perfectly axi-symmetric ring occur in degenerate pairs, such that the natural frequencies are equal, and the mode shapes are spatially orthogonal having indeterminate angular position. In practice, imperfections due to manufacturing variations and material non-uniformity fix the positions of the modes relative to the ring and yield small frequency splits. In some mechanical systems, such as ring based gyroscopic rate sensors [1,2], the influence of such imperfections is of extreme importance.

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Tobias [3] performed the first studies on the influence of imperfection on the vibration characteristics of axi-symmetric structures. This work gave a good description of the qualitative effects of imperfection, whilst it was not until much later that a quantitative description was provided. Laura et al. [4] and Tonin and Bies [5] have investigated the effect of circumferential variations in wall thickness on a ring and eccentric cylinder, respectively, using the Rayleigh–Ritz approach and the finite element method. More recently, Hwang et al. [6–8] have used the Rayleigh–Ritz analysis to consider general profile variations of a ring, whilst Eley et al. [9] have considered the influence of anisotropy on the vibration characteristics of circular crystalline silicon rings. Whilst the above work considers the influence of imperfection on the vibration characteristics of axi-symmetric structures, some other research relates to the inverse (so-called trimming) problem in which the aim is to modify the structure to eliminate the frequency splits. This problem has been considered by Fox [10,11] and Allaei et al. [12] for a single pair of modes using the Rayleigh–Ritz method and receptance method, respectively. Recent work by Rourke et al. [13,14] has extended the method developed by Fox to the simultaneous trimming of a multiple number of modes of vibration.

All of the above work effectively investigates the influence of particular forms of imperfection on the splitting of the natural frequencies. In practice, the imperfections present will be random, varying from one structure to the next. It is for this reason that the statistical distribution of the frequency splits is of interest. The aim of the work described here is to investigate the influence of different types of random mass imperfection on an initially perfect ring so as to gain a better understanding of the influence of manufacturing uncertainties on the statistical distribution of the frequency splits. Section 2 outlines the results used to calculate the natural frequencies in terms of the initially perfect ring and the attached masses. Section 3 outlines the splitting rules for a perfect ring with uniformly spaced point masses and a perfect ring with random harmonic variations in the mass per unit length. Section 4 investigates the statistical distribution of the frequency splits for a ring with: (i) random harmonic variations in the mass per unit length; (ii) random point masses attached at random locations; and (iii) uniformly spaced random point masses.

2. Natural frequencies of rings with attached masses

Consider an imperfect ring for which the natural frequencies of the pair of orthogonal in-plane modes having n nodal diameters are ω_{n1} and ω_{n2}. Here it is assumed that the radial (w) and tangential displacements (u) of the ring in the nth mode pair are given by

$$w = W\cos n\phi \exp(\mathrm{i}\omega_n t), \quad u = U\sin n\phi \exp(\mathrm{i}\omega_n t), \tag{1,2}$$

where the mode orientations $\psi_{n1} = \psi_{n2} - \pi/2n$. The assumption that the mode shapes are identical to those of a perfect ring adopted in these equations is reasonable provided that the degree of imperfection is sufficiently small. This assumption was tested by Rourke et al. [13] using a 3-term Rayleigh–Ritz procedure.

The effect of attached masses and springs on the natural frequencies and mode positions of a ring has been investigated previously by Fox [10]. In that study, a Rayleigh–Ritz approach was used to determine analytical expressions for the split natural frequencies in terms of the magnitude and location of the added masses, radial springs and torsional springs. Following this approach it can be shown (see Ref. [13] for details) that the natural frequencies ω_{n1} and ω_{n2}, and mode orientation ψ_n of the nth mode for a perfect ring with attached point masses are given by

$$\omega^2_{n1,n2} = \omega^2_{0n}\left(\frac{1+\alpha_n^2}{(1+\alpha_n^2)+\sum_i m_i[(1+\alpha_n^2)\pm(\alpha_n^2-1)\cos 2n(\phi_i-\psi_n)]/M_0}\right), \tag{3}$$

$$\tan 2n\psi_n = \frac{\sum_i m_i \sin 2n\phi_i}{\sum_i m_i \cos 2n\phi_i}, \tag{4}$$

where m_i and ϕ_i denote the magnitude and angular location of the ith added mass ($i = 1, 2, \ldots, N$), α_n is the amplitude ratio W/U, ω_{0n} is the natural frequency of the original perfect ring, and M_0 is the mass of the perfect ring.

Eqs. (3) and (4) provide a means for determining the mode orientation and frequency splits resulting from the addition of imperfection masses at particular locations to an initially perfect ring, and have been used in previous work by the authors to perform so-called trimming analyses [13,14]. It is worthwhile noting that Eq. (3) is dependent upon knowledge of the mode orientation given by Eq. (4). Given that the orientation varies with the addition of mass, it is necessary to re-express the natural frequencies in a form independent of the orientation. It is shown in Appendix A that this can be achieved by combining Eqs. (3) and (4), such that the natural frequencies can be expressed by

$$\omega^2_{n1,n2} = \frac{\omega^2_{0n}}{1+\sum_i m_i/M_0 \pm \dfrac{(\alpha_n^2-1)}{M_0(\alpha_n^2+1)}\left(\left(\sum_i m_i\cos 2n\phi_i\right)^2+\left(\sum_i m_i \sin 2n\phi_i\right)^2\right)^{1/2}}. \tag{5}$$

This equation provides a means of determining the natural frequencies resulting from the addition of imperfection masses at particular locations to an initially perfect ring. In accordance with the assumption that the mode shapes of the imperfect ring are identical to those of the perfect ring it is justifiable to expand Eq. (5) as a two-term binomial series to determine approximate expressions for the natural frequencies such that

$$\omega_{n1,n2} = \omega_{0n}\left(1+\frac{1}{2}\sum_i m_i/M_0 \pm \frac{(\alpha_n^2-1)}{2M_0(\alpha_n^2+1)}\left(\left(\sum_i m_i\cos 2n\phi_i\right)^2+\left(\sum_i m_i\sin 2n\phi_i\right)^2\right)^{1/2}\right). \tag{6}$$

This equation is well suited to determining the frequency split Δ associated with the attachment of point masses, and yields

$$\Delta = \omega_{n1}-\omega_{n2} = \frac{\omega_{0n}(\alpha_n^2-1)}{M_0(\alpha_n^2+1)}\left(\left(\sum_i m_i\cos 2n\phi_i\right)^2+\left(\sum_i m_i\sin 2n\phi_i\right)^2\right)^{1/2}. \tag{7}$$

For the situation when the mass varies continuously around the circumference of the ring, Eq. (7) is modified as follows:

$$\Delta = \omega_{n1} - \omega_{n2} = \frac{\omega_{0n} R(\alpha_n^2 - 1)}{M_0(\alpha_n^2 + 1)} \left(\left(\int_0^{2\pi} \rho(\phi) \cos 2n\phi \, \mathrm{d}\phi \right)^2 + \left(\int_0^{2\pi} \rho(\phi) \sin 2n\phi \, \mathrm{d}\phi \right)^2 \right)^{1/2}, \quad (8)$$

where $\rho(\phi)$ denotes the mass per unit length at location ϕ around the circumference of the ring, and R is the mean radius of the ring. As a consequence of deriving this expression from a point mass perspective, the influence of rotational inertia is neglected.

In the following, Eqs. (7) and (8) will be used to determine the probability density function of natural frequency splits for rings with different types of mass imperfection. However, before doing this, a review of the splitting rules for rings with mass variations is provided.

3. Frequency splitting rules

In this section a review of the splitting rules is provided for: (i) a perfect ring with identical, uniformly spaced point masses; and (ii) a perfect ring with harmonic variations in the mass per unit length.

3.1. Identical, uniformly spaced point masses

In this case identical masses are distributed uniformly around the circumference of the ring. Setting $m_j = m$ and $\phi_j = 2\pi j/N$, where N is the number of attached point masses, in Eq. (7) (or more generally Eq. (5)) it can be deduced that the frequency split is zero when:

$$\sum_{j=1}^{N} \sin(4n\pi j/N) = 0, \quad (9)$$

$$\sum_{j=1}^{N} \cos(4n\pi j/N) = 0. \quad (10)$$

These equations can be used to deduce the conditions required for the natural frequencies to split. Appendix B proves that Eq. (9) is always satisfied, whilst Appendix C proves that Eq. (10) is only satisfied when $2n/N \neq$ integer. Using these results it can be deduced that the natural frequencies will split only when $2n/N$ is an integer. This rule was discovered first by Charnley and Perrin [15,16] using a combination of perturbation analysis and Group theory. The rule has been used to consider different combinations of n $(= 2, 3, \ldots, 6)$ and N $(= 2, 3, \ldots, 8)$ and the results are shown in Table 1. This table indicates whether a particular combination of n and N yield split (or equal) frequencies.

Table 1
Natural frequency splitting rules for a perfect ring with identical, uniformly spaced masses for various combinations of mode and added masses

	N						
	2	3	4	5	6	7	8
$n=2$	Split	Equal	Split	Equal	Equal	Equal	Equal
$n=3$	Split	Split	Equal	Equal	Split	Equal	Equal
$n=4$	Split	Equal	Split	Equal	Equal	Equal	Split
$n=5$	Split	Equal	Equal	Split	Equal	Equal	Equal
$n=6$	Split	Split	Split	Equal	Split	Equal	Equal

3.2. Harmonic variations in the mass per unit length

In this case the mass per unit length $\rho(\phi)$ is expressed using a Fourier series where

$$\rho(\phi) = a_0 + \sum_{i=1}^{M} (a_i \cos i\phi + b_i \sin i\phi), \tag{11}$$

where a_i and b_i are Fourier coefficients.

Substituting Eq. (11) into Eq. (8) and evaluating the integrals it can be shown easily that the frequency split will be zero only when

$$a_{2n} = b_{2n} = 0, \tag{12}$$

i.e., when there is no "$2n\phi$" variation in the mass per unit length around the circumference of the ring.

4. Statistical distribution of natural frequency splits

It is convenient at this stage to express Eqs. (7) and (8) as follows:

$$\Delta = \omega_{n1} - \omega_{n2} = \frac{\omega_{0n}(\alpha_n^2 - 1)}{M_0(\alpha_n^2 + 1)} (z_1^2 + z_2^2)^{1/2}, \tag{13}$$

where z_1, z_2 are random variables defined by

$$z_1 = \sum_{i=1}^{N} m_i \cos 2n\phi_i, \quad z_2 = \sum_{i=1}^{N} m_i \sin 2n\phi_i, \tag{14, 15}$$

when N point masses are attached, and

$$z_1 = R \int_0^{2\pi} \rho(\phi) \cos 2n\phi \, \mathrm{d}\phi, \quad z_2 = R \int_0^{2\pi} \rho(\phi) \sin 2n\phi \, \mathrm{d}\phi, \tag{16, 17}$$

when the mass per unit length varies continuously around the circumference of the ring.

The random nature of z_1, z_2 will directly influence the probability density function of the frequency split given in Eq. (13). In the following the probability density function of the natural frequency split is investigated for a number of different situations. These include: (i) random harmonic variations of the mass per unit length around the circumference of ring; (ii) attachment of random point masses at random locations; and (iii) attachment of random point masses at uniformly spaced locations. Each of these will be considered in turn.

4.1. Random harmonic variations in the mass per unit length

Harmonic variations in the mass per unit length are expected to arise as a result of material processing and/or the manufacturing process. For the purposes of analysis, the mass per unit length is defined by Eq. (11) and it is assumed that the Fourier coefficients are statistically independent, zero mean Gaussian random variables, possessing the following statistical properties:

$$E[a_i^2] = E[b_i^2] = \sigma_i^2, \tag{18}$$

$$E[a_i a_j] = E[b_i b_j] = 0, \quad i \neq j, \tag{19}$$

$$E[a_i b_j] = 0 \quad \text{for all } i, j, \tag{20}$$

where $E[\ldots]$ is the expectation operator.

Substituting Eq. (11) into Eqs. (16) and (17) and using Eqs. (18)–(20), it can be shown easily that z_1 and z_2 are statistically independent, zero-mean Gaussian random variables, such that

$$\sigma_{z1} = \sigma_{z2} = \sigma_{2n}, \tag{21}$$

where σ_{zi} is the standard deviation of z_i. Using these results in conjunction with Eq. (13) and a transformation of variables it can be shown easily that the frequency split will have a Rayleigh distribution such that

$$p(\Delta) = \frac{\Delta}{\sigma^2} \exp\left(-\frac{1}{2}\left(\frac{\Delta^2}{\sigma^2}\right)\right), \tag{22}$$

where σ is given by

$$\sigma = \frac{\omega_{0n} R \sigma_{2n} (\alpha_n^2 - 1)}{M_0 (\alpha_n^2 + 1)}. \tag{23}$$

Fig. 1 shows a Rayleigh distribution plotted on normalized axes.

In summary, the presence of independent random harmonic variations in the mass per unit length yield frequency splits that possess a Rayleigh distribution.

4.2. Random point masses at random locations

Small discrete changes in the mass properties of the ring can arise due to the presence of manufacturing imperfections, damage to the structure, and/or material non-uniformities. For the purposes of analysis these changes are modelled as random point masses, in which the point masses are assumed to be zero-mean Gaussian random variables with the following statistical

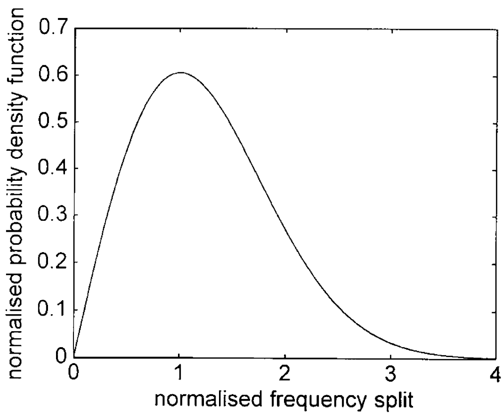

Fig. 1. Rayleigh distribution.

properties:

$$E[m_i^2] = \sigma_m^2, \tag{24}$$

$$E[m_i m_j] = 0, \quad i \neq j, \tag{25}$$

where σ_m is the standard deviation of each attached mass. Furthermore, it is assumed that the point masses are distributed around the circumference of the ring with a uniform distribution, ensuring that no location around the circumference of the ring is preferred.

To determine the statistical distribution of the frequency split it is necessary to consider the joint statistical distribution of z_1 and z_2, and to achieve this it is necessary to consider the joint characteristic function $M(\theta_1, \theta_2)$ where

$$M(\theta_1, \theta_2) = E[\exp(\mathrm{i}\theta_1 z_1 + \mathrm{i}\theta_2 z_2)], \tag{26}$$

and the joint probability density function of z_1 and z_2 is given by

$$p(z_1, z_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M(\theta_1, \theta_2) \exp(-\mathrm{i}\theta_1 z_1 - \mathrm{i}\theta_2 z_2)\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2. \tag{27}$$

It is shown in Appendix D that the joint characteristic function is given by

$$M(\theta_1, \theta_2) = \exp\left(-\frac{N(\theta_1^2 + \theta_2^2)\sigma_m^2}{4}\right) \left[\mathrm{I}_0\left(\frac{(\theta_1^2 + \theta_2^2)\sigma_m^2}{4}\right)\right]^N, \tag{28}$$

where I_0 is the zeroth order modified Bessel function of the first kind. This equation indicates that the joint characteristic function is axially symmetric, and that a more convenient representation is afforded by using polar co-ordinates. Substituting Eq. (28) into Eq. (27), and using the transformations of variables: $\theta_1 = s\cos\xi$, $\theta_2 = s\sin\xi$, $z_1 = (\Delta/\lambda)\cos\psi$, and $z_2 = (\Delta/\lambda)\sin\psi$, where $\lambda = \omega_{0n}(\alpha_n^2 - 1)/(M_0(\alpha_n^2 + 1))$ it can be shown that the joint pdf $p(\Delta, \psi)$ is given by

$$p(\Delta, \psi) = \frac{\Delta}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} sM(s, \xi) \exp(-\mathrm{i}s\Delta\cos(\xi - \psi))\, \mathrm{d}s\, \mathrm{d}\xi, \tag{29}$$

where

$$M(s,\xi) = \exp\left(-\frac{Ns^2\sigma_m^2}{4}\right)\left[\mathrm{I}_0\left(\frac{s^2\sigma_m^2}{4}\right)\right]^N. \tag{30}$$

Performing the integration over ξ and then integrating the resulting expression over ψ from 0 to 2π, it can be shown that the probability density function of the frequency split is given by

$$p(\Delta) = \Delta\int_0^\infty s\exp\left(-\frac{Ns^2\sigma_m^2}{4}\right)\mathrm{I}_0\left(\frac{s^2\sigma_m^2}{4}\right)^N \mathrm{J}_0(s\lambda\Delta)\,\mathrm{d}s, \tag{31}$$

where J_0 is the zeroth order Bessel function of the first kind. Eq. (29) is an analytical expression for the statistical distribution of the frequency split for a perfect ring with random (zero mean) point masses attached at random locations. It is worthwhile noting that as the number of added masses N increases, the central limit theorem [17] ensures that the probability density function Δ approaches the Rayleigh distribution (see Eq. (22)) with

$$\sigma = \frac{\omega_{0n}\sqrt{N}\sigma_m(\alpha_n^2-1)}{M_0(\alpha_n^2+1)}. \tag{32}$$

The Rayleigh character of the distribution can be proved easily by noting that the joint characteristic function (Eq. (28)) tends to

$$M(\theta_1,\theta_2) = \exp\left(-\frac{N(\theta_1^2+\theta_2^2)\sigma_m^2}{4}\right) \quad \text{for } N \text{ large} \tag{33}$$

as N becomes large. Eq. (33) ensures that $p(z_1, z_2)$ is Gaussian, and that z_1 and z_2 are statistically independent. As a consequence of this, the frequency split tends to a Rayleigh distribution as N increases.

Figs. 2(a)–(f) show some numerical results for the cases $N(=1,2,3,5,10,20)$ attached masses, where the distribution has been plotted on normalized axes in each case. For the purposes of validation, the plots are compared with the results from numerical simulation (40,000 samples). In addition, a Rayleigh distribution is shown for comparison. In each case it can be seen that the developed results give excellent agreement with numerical simulation. For the case when $N=1$ the distribution is highly non-Rayleigh—this case is considered in the following Section where it will be shown that it is in fact a "half-Gaussian" distribution. For $N=2,3,\ldots$ the statistical distribution takes a form that is much more representative of a Rayleigh distribution, and as N increases the frequency split distribution quickly approaches a Rayleigh distribution.

4.3. Uniformly spaced, random point masses

This situation models the manufacturing imperfections present due to the presence of a symmetric supporting structure for the ring. For example, they could represent the point of attachment for spoked supports. For the purposes of analysis, the magnitude of the attached point masses is expressed as the sum of a mean component, representing the mean added mass from the support (say), and a zero-mean Gaussian random variable, such that

$$m_i = \mu_m + \sigma_m\,\mathrm{d}m_i, \tag{34}$$

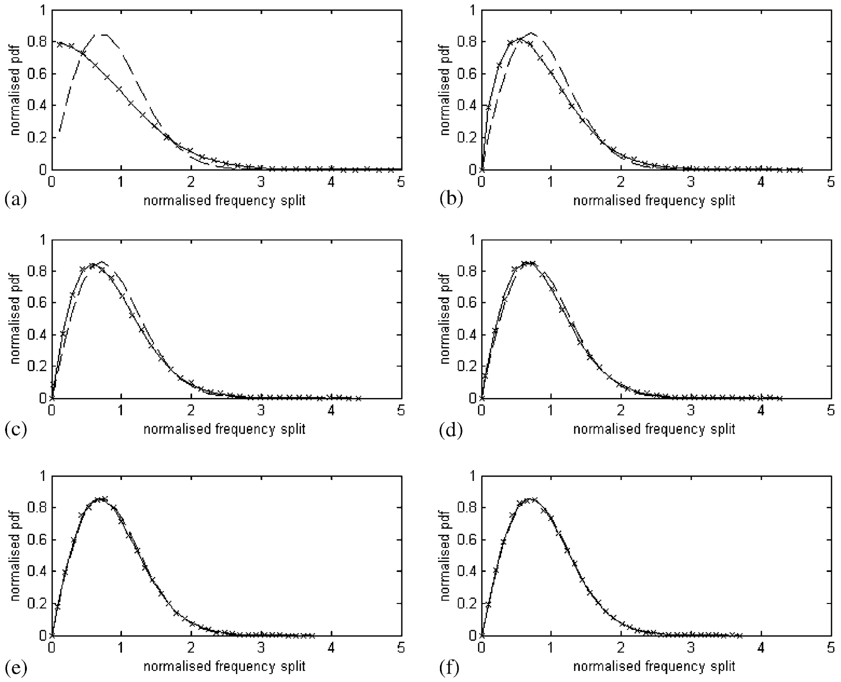

Fig. 2. Frequency split distribution for a perfect ring with N randomly spaced, random masses: (a) $N = 1$; (b) $N = 2$; (c) $N = 3$; (d) $N = 5$; (e) $N = 10$; (f) $N = 20$—"solid line", Eq. (31); "dashed line", Rayleigh (Eq. (22)); '×', simulation of Eq. (13).

where μ_m and σ_m are the mean and standard deviation of the magnitude of each mass, and $\mathrm{d}m_i$ are zero-mean, statistically independent Gaussian random variables possessing the following statistical properties:

$$E[\mathrm{d}m_i\,\mathrm{d}m_j] = \delta_{ij}, \tag{35}$$

where δ_{ij} is zero when $i \neq j$ and unity otherwise.

Substituting Eq. (34) into Eqs. (14) and (15), and taking the expected value, it can be shown that z_1, z_2 are Gaussian random variables such that

$$\begin{aligned}
&\mu_{z1} = \mu_m \sum_i \cos\frac{4n\pi i}{N}, \quad \mu_{z2} = \mu_m \sum_i \sin\frac{4n\pi i}{N},\\
&\sigma_{z1}^2 = \sigma_m^2 \sum_i \cos^2\frac{4n\pi i}{N}, \quad \sigma_{z2}^2 = \sigma_m^2 \sum_i \sin^2\frac{4n\pi i}{N},\\
&E[(z_1-\mu_{z1})(z_2-\mu_{z2})] = \sigma_m^2 \sum_i \sin\frac{8n\pi i}{N},
\end{aligned} \tag{36–40}$$

where μ_{zi} and σ_{zi} are the mean and standard deviation of z_i.

Using the results proved in Appendices B and C in Eqs. (36)–(40) it can be shown that

$$\mu_{z1} = \begin{cases} 0 & \text{if } 2n/N \neq \text{ integer,} \\ N\mu_m & \text{otherwise,} \end{cases} \qquad \mu_{z2} = 0,$$
$$\sigma_{z1}^2 = \begin{cases} N\sigma_m^2/2 & \text{if } 4n/N \neq \text{ integer,} \\ N\sigma_m^2 & \text{otherwise,} \end{cases} \qquad \sigma_{z2}^2 = \begin{cases} N\sigma_m^2/2 & \text{if } 4n/N \neq \text{ integer,} \\ 0 & \text{otherwise} \end{cases} \tag{41–44}$$

and that z_1 and z_2 are always uncorrelated (i.e., statistically independent). Given the conditions associated with Eqs. (41)–(44), it is convenient in what follows to consider the cases when $4n/N =$ integer separately from the situation when $4n/N \neq$ integer. These are considered next.

(i) $4n/N =$ *integer*: Noting that for this situation $\sigma_{z2} = 0$ and using Eq. (13) it can be shown easily that the frequency split is given by

$$\Delta = \frac{\omega_{0n}(\alpha_n^2 - 1)}{M_0(\alpha_n^2 + 1)}|z_1|, \tag{45}$$

where z_1 is a Gaussian random variable. Using a transformation of variable it can be shown that the probability density function of the frequency split is given by

$$p(\Delta) = \sqrt{\frac{2}{\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\Delta^2 + \mu^2}{\sigma^2}\right)\right)\cosh\left(\frac{\mu\Delta}{\sigma^2}\right), \tag{46}$$

where

$$\mu = \begin{cases} 0 & \text{if } 2n/N \neq \text{integer,} \\ \dfrac{\omega_{0n}\mu_m N(\alpha_n^2 - 1)}{M_0(\alpha_n^2 + 1)} & \text{otherwise,} \end{cases} \tag{47}$$

$$\sigma = \frac{\omega_{0n}\sigma_m\sqrt{N}(\alpha_n^2 - 1)}{M_0(\alpha_n^2 + 1)}. \tag{48}$$

It should be noted that it is not surprising that $\mu = 0$ when $2n/N \neq$ integer, since it was shown in Section 3.1 that no splits occur for the "nominal" case. In this case Eq. (46) reduces to a "half Gaussian" distribution, so that

$$p(\Delta) = \sqrt{\frac{2}{\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\Delta^2}{\sigma^2}\right)\right). \tag{49}$$

A further situation when a "half Gaussian" distribution occurs is when $N = 1$ and $\mu_m = 0$; this special case is the limiting case for the case of randomly positioned random point masses considered in Section 4.2 when $N = 1$.

Figs. 3(a)–(d) shows a plot of the frequency split distribution given by Eq. (46) for different values of the coefficient of variation ($c = \mu/\sigma$), where $c = 0, 1, 2, 3$. In this plot normalized axes are used throughout, and the "half Gaussian" case is shown in Fig. 3(a).

In summary, for the case when $4n/N$ is an integer the natural frequency splits have the distribution given by Eq. (46) which reduces to a "half Gaussian" distribution for the special case when $2n/N \neq$ integer.

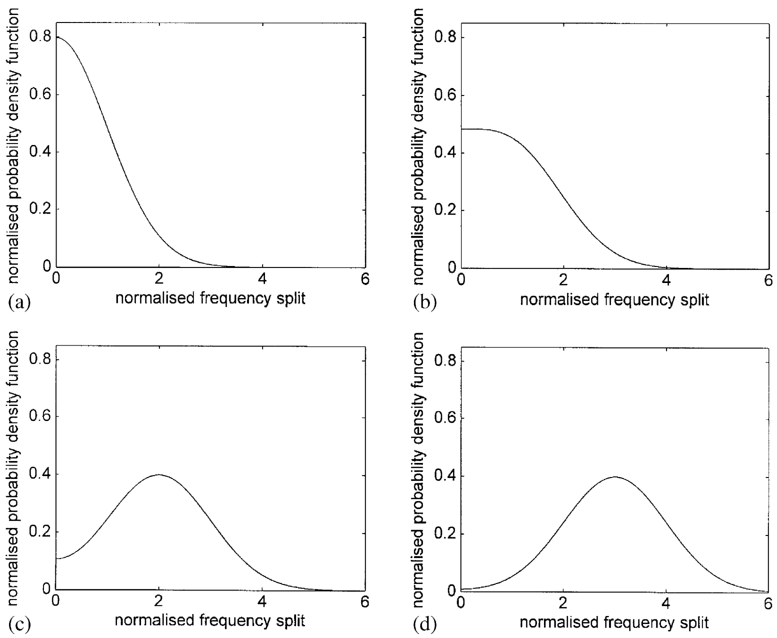

Fig. 3. Frequency split distribution for a perfect ring with N uniformly spaced random masses ($4n/N = \text{integer}$): (a) $c = 0$; (b) $c = 1$; (c) $c = 2$; (d) $c = 3$.

(ii) $4n/N \neq$ *integer*: Using Eq. (13), and Eqs. (41)–(44) and noting that when $4n/N \neq$ integer it is not possible for $2n/N$ to be an integer, it can be shown that the natural frequency split is given by

$$\Delta = \frac{\omega_{0n}(\alpha_n^2 - 1)}{M_0(\alpha_n^2 + 1)}\sqrt{z_1^2 + z_2^2}, \tag{50}$$

where z_1 and z_2 are uncorrelated, zero-mean Gaussian random variables.

Using a transformation of variables in an identical manner to that briefly described in Section 4.1 it can be shown easily that the frequency split has a Rayleigh distribution (see Eq. (22)) where

$$\sigma = \frac{\omega_{0n} N \sigma_m(\alpha_n^2 - 1)}{M_0(\alpha_n^2 + 1)}. \tag{51}$$

In summary, for the case when $4n/N$ is not an integer the natural frequency splits always have a Rayleigh distribution (see Fig. 1).

(iii) *Summary*: The above rules (for $4n/N = \text{integer}$ and $4n/N \neq \text{integer}$) have been used to deduce the statistical distribution of the frequency split for different combinations of n ($= 2, 3, \ldots, 6$) and N ($= 2, 3, \ldots, 8$) and the results are summarized in Table 2. This table indicates whether a particular combination of n and N yield frequency split distributions that are Rayleigh (R), "half Gaussian" (HG) or given by Eq. (46) (S).

Table 2
Natural frequency statistics rules for a perfect ring with uniformly spaced masses for various combinations of mode and added masses

	N						
	2	3	4	5	6	7	8
$n=2$	S	R	S	R	R	R	HG
$n=3$	S	S	HG	R	S	R	R
$n=4$	S	R	S	R	R	R	S
$n=5$	S	R	HG	S	R	R	R
$n=6$	S	S	S	R	S	R	HG

S—Eq. (46); HG (half-Gaussian)—Eq. (49); R (Rayleigh)—Eq. (22).

5. Conclusions

The statistical distribution of the frequency splits arising for in-plane modes of vibration of circular rings with different types of mass imperfection have been investigated. In particular the distribution of the frequency split was sought for: (i) random harmonic variations in the mass per unit length around the circumference of the ring; (ii) the attachment of random point masses at random locations on the ring; and (iii) the attachment of random point masses at uniformly spaced positions on the ring. For the purposes of analysis the imperfections considered were composed of independent random quantities, and expressions for the statistical distributions were determined. For case (i) it was found that the frequency splits had a Rayleigh distribution. For case (ii) an analytical representation of the distribution was found which was shown to tend to Rayleigh distribution as the number of point masses increased. For case (iii) it was found that the frequency split distribution was dependent upon the mode considered and the number of attached point masses, and that in some situations the frequency splits had a "half Gaussian" (i.e. non-Rayleigh) distribution.

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Appendix A. Derivation of Eq. (5)

With the purpose of deriving an expression for the natural frequencies that are independent of the orientation of the model, it is convenient to re-express Eq. (4) as follows:

$$\sum_i m_i \sin 2n(\phi_i - \psi_n) = 0, \tag{A.1}$$

and re-arrange Eq. (3) as follows:

$$\omega_{n1,n2}^2 = \omega_{0n}^2 \left(\frac{M_0(\alpha_n^2+1)}{(M_0+\sum_i m_i)(1+\alpha_n^2) \pm (\alpha_n^2-1)\sum_i m_i \cos 2n(\phi_i-\psi_n)} \right). \tag{A.2}$$

Without loss of generality, it can be deduced that Eq. (A.2) can be expressed as

$$\omega_{n1,n2}^2 = \omega_{0n}^2 \left(\frac{M_0(\alpha_n^2+1)}{(M_0+\sum_i m_i)(1+\alpha_n^2) \pm (\alpha_n^2-1)(\sum_i\sum_j m_i m_j \cos 2n(\phi_i-\psi_n)\cos 2n(\phi_j-\psi_n))^{1/2}} \right). \tag{A.3}$$

Using standard trigonometric identities it can be shown that

$$\begin{aligned}\sum_i\sum_j m_i m_j \cos 2n(\phi_i-\psi_n)\cos 2n(\phi_j-\psi_n) \\ = \sum_i\sum_j \tfrac{1}{2} m_i m_j (\cos 2n(\phi_i-\phi_j) + \cos 2n(\phi_i-\phi_j+2\psi_n)).\end{aligned} \tag{A.4}$$

Furthermore, squaring Eq. (A.1) and using standard trigonometric identities it can be shown that

$$\begin{aligned}\sum_i\sum_j m_i m_j \sin 2n(\phi_i-\psi_n)\sin 2n(\phi_j-\psi_n) \\ = \sum_i\sum_j \tfrac{1}{2} m_i m_j (\cos 2n(\phi_i-\phi_j) - \cos 2n(\phi_i-\phi_j+2\psi_n)) = 0.\end{aligned} \tag{A.5}$$

Adding Eqs. (A.4) and (A.5) gives

$$\begin{aligned}\sum_i\sum_j m_i m_j \cos 2n(\phi_i-\psi_n)\cos 2n(\phi_j-\psi_n) \\ = \sum_i\sum_j m_i m_j \cos 2n(\phi_i-\phi_j).\end{aligned} \tag{A.6}$$

Expanding the right hand side of this equation using standard trigonometric identities it can be shown that

$$\begin{aligned}&\sum_i\sum_j m_i m_j \cos 2n(\phi_i-\psi_n)\cos 2n(\phi_j-\psi_n) \\ &= \sum_i\sum_j m_i m_j (\cos 2n\phi_i \cos 2n\phi_j + \sin 2n\phi_i \sin 2n\phi_j) \\ &= \left(\sum_i m_i \cos 2n\phi_i\right)^2 + \left(\sum_i m_i \sin 2n\phi_i\right)^2.\end{aligned} \tag{A.7}$$

Substituting Eq. (A.7) into Eq. (A.2) and re-arranging gives

$$\omega_{n1,n2}^2 = \frac{\omega_{0n}^2}{1+\sum_i m_i/M_0 \pm \frac{(\alpha_n^2-1)}{M_0(\alpha_n^2+1)}\left((\sum_i m_i \cos 2n\phi_i)^2 + (\sum_i m_i \sin 2n\phi_i)^2\right)^{1/2}}. \tag{A.8}$$

Appendix B. Proof that Eq. (9) is always satisfied

This appendix proves that

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = 0, \tag{B.1}$$

where n and N are integers.

Consider the situation when N is even such that $N = 2M$, where M is an integer. In this case the left hand side of Eq. (B.1) can be expressed as

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = \sum_{j=1}^{M} \sin\frac{4\pi nj}{N} + \sum_{j=M+1}^{2M} \sin\frac{4\pi nj}{N}. \tag{B.2}$$

Noting that $\sin 2n\pi = \sin 4n\pi = 0$ for all n, it can be deduced that

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = \sum_{j=1}^{M-1} \sin\frac{4\pi nj}{N} + \sum_{j=M+1}^{2M-1} \sin\frac{4\pi nj}{N}. \tag{B.3}$$

Re-ordering the terms in the second term on the right hand side of Eq. (B.3) gives

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = \sum_{j=1}^{M-1} \sin\frac{4\pi nj}{N} + \sum_{k=1}^{M-1} \sin\frac{4\pi n(2M-k)}{N}. \tag{B.4}$$

Expanding the second term on the right hand side of Eq. (B.4) using standard trigonometric equations gives

$$\sin\frac{4\pi n(2M-k)}{N} = \sin\frac{8\pi nM}{N}\cos\frac{4\pi nk}{N} - \cos\frac{8\pi nM}{N}\sin\frac{4\pi nk}{N}. \tag{B.5}$$

Recalling that $N = 2M$, ensures that $\sin(8\pi nM/N) = 0$ and $\cos(8\pi nM/N) = 1$. Using these results in Eqs. (B.5) and the resulting equation in Eq. (B.4) gives

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = \sum_{j=1}^{M-1} \sin\frac{4\pi nj}{N} - \sum_{k=1}^{M-1} \sin\frac{4\pi nk}{N}. \tag{B.6}$$

Noting that the two terms on the right hand side of this equation are identical, it can be deduced that when N is even:

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = 0. \tag{B.7}$$

Consider next the situation when N is odd such that $N = 2M + 1$, where M is an integer. In this case, the left hand side of Eq. (B.1) can be expressed as

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = \sum_{j=1}^{M} \sin\frac{4\pi nj}{N} + \sum_{j=M+1}^{2M+1} \sin\frac{4\pi nj}{N}. \tag{B.8}$$

Noting that $\sin 4n\pi = 0$ for all n, it can be deduced that

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = \sum_{j=1}^{M} \sin\frac{4\pi nj}{N} + \sum_{j=M+1}^{2M} \sin\frac{4\pi nj}{N}. \tag{B.9}$$

Re-ordering the terms in the second term on the right hand side of Eq. (B.9) gives

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = \sum_{j=1}^{M} \sin\frac{4\pi nj}{N} + \sum_{k=1}^{M} \sin\frac{4\pi n(2M+1-k)}{N}. \tag{B.10}$$

Expanding the second term on the right hand side of Eq. (B.10) using standard trigonometric identities gives

$$\sin\frac{4\pi n(2M+1-k)}{N} = \sin\frac{4\pi n(2M+1)}{N}\cos\frac{4\pi nk}{N} - \cos\frac{4\pi n(2M+1)}{N}\sin\frac{4\pi nk}{N}. \tag{B.11}$$

Recalling that $N = 2M+1$, ensures that $\sin[4\pi n(2M+1)/N] = 0$ and $\cos[4\pi n(2M+1)/N] = 1$. Using these results in Eq. (B.11) and the resulting equation in Eq. (B.10) gives

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = \sum_{j=1}^{M} \sin\frac{4\pi nj}{N} - \sum_{k=1}^{M} \sin\frac{4\pi nk}{N}. \tag{B.12}$$

Noting that the two terms on the right hand side of this equation are identical, it can be deduced that when N is odd:

$$\sum_{j=1}^{N} \sin\frac{4\pi nj}{N} = 0. \tag{B.13}$$

Thus, it has been proved that Eq. (B.1) is always satisfied.

Appendix C. Proof that Eq. (10) is satisfied only when $2n/N \neq$ integer

This appendix proves that

$$\sum_{j=1}^{N} \cos\frac{4\pi nj}{N} = 0 \tag{C.1}$$

only when $2n/N \neq$ integer, where n and N are integers.

The basis of the proof presented here is that the left hand side of Eq. (C.1) is independent of the order of the summation. A consequence of this is that

$$\sum_{j=1}^{N} \cos\frac{4\pi n(j-k)}{N} = \sum_{j=1}^{N} \cos\frac{4\pi nj}{N} \tag{C.2}$$

must be satisfied for all integer values of k.

Expanding the left hand side of Eq. (C.2) using standard trigonometric identities it can be shown that

$$\sum_{j=1}^{N}\left(\cos\frac{4\pi nj}{N}\cos\frac{4\pi nk}{N}+\sin\frac{4\pi nj}{N}\sin\frac{4\pi nk}{N}\right)=\sum_{j=1}^{N}\cos\frac{4\pi nj}{N}. \tag{C.3}$$

It was shown in Appendix B that

$$\sum_{j=1}^{N}\sin\frac{4\pi nj}{N}=0, \tag{C.4}$$

where n and N are integers. Using this result in Eq. (C.3) and rearranging it can be shown that the following equation must be satisfied:

$$\left(1-\cos\frac{4\pi nk}{N}\right)\sum_{j=1}^{N}\cos\frac{4\pi nj}{N}=0. \tag{C.5}$$

The possible solutions to Eq. (B.5) are that

$$\cos\frac{4n\pi k}{N}=1 \quad \text{for all } k, \tag{C.6}$$

and/or

$$\sum_{j=1}^{N}\cos\frac{4n\pi j}{N}=0. \tag{C.7}$$

For Eq. (C.6) to be valid it is necessary that $2n/N$ is an integer. Under these conditions, it can be shown easily that $\sum_{j=1}^{N}\cos(4n\pi j/N)=N$, indicating that Eqs. (C.6) and (C.7) cannot be satisfied simultaneously. Thus for those situations when $2n/N$ is not an integer, it can be deduced that Eq. (C.7) must be satisfied.

Thus, it has been proved that Eq. (C.1) is only satisfied under the stated conditions.

Appendix D. Derivation of Eq. (28)

Consider the random variables z_1 and z_2 defined by Eqs. (14) and (15), where the independent random variables m_i and ϕ_i have the following statistical distributions:

$$p(m_i)=\frac{1}{\sqrt{2\pi\sigma_m^2}}\exp\left(-\frac{1}{2}\frac{m_i^2}{\sigma_m^2}\right), \quad p(\phi_i)=\frac{1}{2\pi}. \tag{D.1, D.2}$$

To evaluate the joint probability density function of z_1, z_2 it is convenient to note that the probability density function of $x_i=\cos(\phi_i-\zeta)$ is given by [17]

$$p(x_i)=\frac{1}{\pi\sqrt{1-x_i^2}}, \tag{D.3}$$

where $-1\leqslant x_i\leqslant 1$ and ζ is a constant phase angle. This equation is valid for $x_i=\sin\phi_i$ and $\cos\phi_i$ and ensures that the probability density functions for z_1 and z_2 are identical.

Using the definition of the joint characteristic function given by Eq. (29) and noting that the random variables are statistically independent, it follows that the joint characteristic function is given by

$$M(\theta_1, \theta_2) = \prod_{j=1}^{N} E\left[\exp\left(\mathrm{i}m_j\sqrt{\theta_1^2+\theta_2^2}\cos(\phi_j-\zeta)\right)\right]$$
$$= \prod_{j=1}^{N}\left(\int_{-1}^{1}\int_{-\infty}^{\infty}\exp\left(\mathrm{i}m_j\sqrt{\theta_1^2+\theta_2^2}x_j\right)p(x_j)p(m_j)\,\mathrm{d}m_j\,\mathrm{d}x_j\right). \tag{D.4}$$

Substituting Eq. (D.3) into Eq. (D.4) and evaluating the integral wrt x_j it can be shown that

$$M(\theta_1, \theta_2) = \prod_{j=1}^{N}\left(\int_{-\infty}^{\infty}\mathrm{J}_0\left(m_j\sqrt{\theta_1^2+\theta_2^2}\right)p(m_j)\,\mathrm{d}m_j\right), \tag{D.5}$$

where $\mathrm{J}_0(\ldots)$ is the zeroth order Bessel function of the first kind.

Substituting Eq. (D.1) in Eq. (D.5) and evaluating the integral wrt m_i it can be shown that

$$M(\theta_1, \theta_2) = \prod_{i=1}^{N}\exp\left(-\frac{(\theta_1^2+\theta_2^2)\sigma_m^2}{4}\right)\mathrm{I}_0\left(\frac{(\theta_1^2+\theta_2^2)\sigma_m^2}{4}\right)$$
$$= \exp\left(-\frac{N(\theta_1^2+\theta_2^2)\sigma_m^2}{4}\right)\left[\mathrm{I}_0\left(\frac{(\theta_1^2+\theta_2^2)\sigma_m^2}{4}\right)\right]^N, \tag{D.6}$$

where $\mathrm{I}_0(\ldots)$ is the zeroth order modified Bessel function of the first kind.

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