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Short Communication

Classical solutions of forced vibration of rod and beam driven by displacement boundary conditions

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1. Introduction

Forced vibration excited by displacement boundary conditions plays an important role in structural dynamics. This type of problem with rod and beam is also used as examples in textbooks of partial differential equations and structural dynamics [1–5]. Usually, the problem is converted, by a transformation, to a forced vibration by body force with homogeneous boundary conditions. Then the solution is obtained by using Fourier series with a Duhamel's integral. For example, see Refs. [4,6] for the rod problems, Refs. [7,8] for the beam problems, and Ref. [9] for extension of approaches of Refs. [7,8] to the Timoshenko beam. However, it is often found that the term-by-term differentiated series, expected to represent the second derivative in time and the second derivative (for rod) or fourth derivative (for beam) in space, does not converge. The solutions are not verified. To the author's knowledge, a rigorous answer in terms of a closed form solution, in the classical sense, is not found in the publications.

An improved approach is developed here to construct the closed form solutions that have continuous derivatives and satisfy the differential equations in a classical sense. The key contribution, improved from the previous approaches, is to let the transformed body force vanish at the end points, in addition to the conditions imposed on the transform function at the boundary by the previous approaches. Then the convergence of the differentiated series is assured.

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The approach is developed in Section 2 for the axial forced vibration of a rod. Then it is extended to the lateral vibration of a beam in Section 3.

The traditional notations for derivatives with respect to time and space variables are adopted, e.g., $\ddot{u}(x,t) = \partial^2 u/\partial t^2$; $u''(x,t) = \partial^2 u/\partial x^2$; $u^{(4)}(x,t) = u_{x^4}(x,t) = \partial^4 u/\partial x^4$.

2. Axial forced vibration of a rod by displacement boundary condition

Consider an axial vibration of a rod, with one end fixed and the other end subject to a prescribed displacement:

$$\rho\ddot{u} - Eu'' = 0, \quad u(0,t) = 0, \quad u(a,t) = g(t), \quad u(x,0) = 0, \quad \dot{u}(x,0) = 0. \tag{1}$$

The problem can be solved with a transformation, widely adopted in publications:

$$u = v + p(x,t), \tag{2}$$

$$p(0,t) = 0, \quad p(a,t) = g(t) \tag{3}$$

It reduces to a problem of forced vibration by a body force with homogeneous boundary conditions:

$$\begin{aligned} &\rho\ddot{v} - Ev'' = q(x,t) = -(\rho\ddot{p} - Ep''), \quad v(0,t) = 0, v(a,t) = 0, \\ &v(x,0) = \varphi(x) = -p(x,0), \quad \dot{v}(x,0) = \psi(x) = -\dot{p}(x,0). \end{aligned} \tag{4}$$

Then the Fourier series with a Duhamel's integral is applied to solve v from Eq. (4):

$$\begin{aligned} &v(x,t) = \sum v_m(t) \sin \alpha_m x, \\ &v_m(t) = \varphi_m \cos \omega_m t + \frac{\psi_m}{\omega_m} \sin \omega_m t + \frac{1}{\rho\omega_m}\int_0^t q_m(\tau) \sin \omega_m (t-\tau)\,\mathrm{d}\tau, \end{aligned} \tag{5}$$

where φ_m, ψ_m and $q_m(x,t)$ are the Fourier coefficients of $\varphi(x)$, $\psi(x)$ and $q(x,t)$ over $\Omega_x = [0,a]$, respectively, corresponding to the basis

$$\{\sin \alpha_m x;\ m = 1,2,\ldots\}$$

with

$$\alpha_m = m\pi/a = O(m), \quad \omega_m = \sqrt{E/\rho}\,\alpha_m = O(m).$$

For example, $\varphi(x) = \sum \varphi_m \sin \alpha_m x$, $\varphi_m = \frac{2}{a}\int_0^a \varphi(x) \sin \alpha_m x\,\mathrm{d}x$.

An example commonly found in the literatures, e.g., Ref. [4], is

$$p(x,t) = g(t)x/a, \tag{6}$$

where $p(x, t)$ meets the requirement (3), and results in $q(x,t) = -\rho\ddot{g}(t)x/a$. It is observed, that the differential equation is not satisfied at $x = a$, if the second derivatives are obtained from term-by-term differentiation. But, this differentiation is built in the solution process and no other definition is available. Furthermore, the Fourier coefficients $q_m(t) = 2\rho\ddot{g}(t)(-1)^{m-1}/m\pi = O(m^{-1})$, so $v_m(t)$ is at most of order $O(m^{-2})$. The twice differentiated series does not converge and cannot represent the second derivative.

An alternative approach exists, which can improve this situation and derive a closed form solution in the classical sense. In addition to Eq. (3), let the transform satisfy

$$q(0,t) = q(a,t) = 0. \tag{7}$$

There are four conditions in Eqs. (3) and (7). A four-term polynomial blending is suggested

$$p(x,t) = C_0(t) + C_1(t)x + C_2(t)x^2 + C_3(t)x^3,$$

$$q(x,t) = -\rho(\ddot{C}_0 + \ddot{C}_1 x + \ddot{C}_2 x^2 + \ddot{C}_3 x^3) + E(2C_2 + 6C_3 x).$$

The answer is straightforward, with an additional term to Eq. (6):

$$p(x,t) = g(t)x/a + E^{-1}\rho\gamma(x)\ddot{g}(t), \quad q(x,t) = -E^{-1}\rho^2\gamma(x)g^{(4)}(t), \tag{8}$$

where

$$\gamma(x) = \frac{a^2}{6}\left(-\frac{x}{a} + \left(\frac{x}{a}\right)^3\right). \tag{9}$$

In fact, for u, as well as v, to have continuous second derivatives it is necessary that $\ddot{g}(t)$ is continuous and certain consistency conditions are satisfied:

$$g(0) = \dot{g}(0) = \ddot{g}(0) = 0. \tag{10}$$

The first two equations are required by the third, fourth and fifth equations of Eq. (1) at $x = a$ and $t = 0$. The last one is required by the first, third and fourth equations of Eq. (1). The initial conditions in the auxiliary system (4) are then reduced to

$$\varphi(x) = -p(x,0) = -g(0)x/a - \ddot{g}(0)\gamma(x)\rho/E = 0,$$
$$\psi(x) = -\dot{p}(x,0) = -\dddot{g}(0)\gamma(x)\rho/E.$$

The function $\gamma(x)$ is involved in Eq. (4) now. With $\gamma(0) = \gamma(a) = 0$, its Fourier coefficients are

$$\gamma_m = (-1)^m 2a^{-1}\alpha_m^{-3}. \tag{11}$$

Thus,

$$v_m(t) = \frac{\rho\gamma_m}{E\omega_m}\left(-\dddot{g}(0)\sin\omega_m t - \int_0^t g^{(4)}(\tau)\sin\omega_m(t-\tau)\,\mathrm{d}\tau\right). \tag{12}$$

Since $\alpha_m = O(m)$ and $\omega_m = O(m)$, therefore $\gamma_m = O(m^{-3})$ and $v_m(t) = O(m^{-4})$. It is straightforward to verify that the twice differentiated series, with respect to x or t, have coefficients of order $O(m^{-2})$. Hence, the series converge uniformly and thus represent the second derivatives. Then the Eq. (1) is satisfied in the classical sense.

In summary, for Eq. (1) to have a classical solution with continuous second derivatives, it is necessary that $\ddot{g}(t)$ is continuous and the consistency conditions (10) are satisfied. When $g^{(4)}(t)$ is continuous, the classical solution can be constructed, in a closed form, with the transform (2), define by Eqs. (8) and (9), and the Fourier series (5); $v_m(t)$ of Eq. (5) is defined in Eq. (12) and γ_m in Eq. (12) is defined in Eq. (11).

3. Lateral forced vibration of beam by displacement boundary condition

The approach developed in Section 2 can be extended to beam bending problems. As a demonstration, consider the following case:

$$\rho A\ddot{w} + EIw^{(4)} = 0, \quad w(0,t) = 0, \quad w''(0,t) = 0, \quad w(a,t) = g(t),$$
$$w''(a,t) = 0, \quad w(x,0) = 0, \quad \dot{w}(x,0) = 0. \tag{13}$$

Similarly, a transform is introduced:

$$w = v + p(x,t), \tag{14}$$

$$p(0,t) = 0, \quad p''(0,t) = 0, \quad p(a,t) = g(t), \quad p''(a,t) = 0. \tag{15}$$

The forced vibration by body force with homogeneous boundary conditions is reduced to

$$\rho A\ddot{v} + EIv^{(4)} = q(x,t) = -(\rho A\ddot{p} + EIp^{(4)}), \quad v(0,t) = 0, \quad v''(0,t) = 0,$$
$$v(a,t) = 0, \quad v''(a,t) = 0, \quad v(x,0) = \varphi(x) = -p(x,0), \quad \dot{v}(x,0) = \psi(x) = -\dot{p}(x,0). \tag{16}$$

Then, the solution in the form of Fourier series with a Duhamel's integral can be obtained:

$$v(x,t) = \sum v_m(t)\,\sin\,\alpha_m x,$$
$$v_m(t) = \varphi_m\,\cos\,\omega_m t + \frac{\psi_m}{\omega_m}\,\sin\,\omega_m t + \frac{1}{\rho A\omega_m}\int_0^t q_m(\tau)\,\sin\,\omega_n(t-\tau)\,\mathrm{d}\tau. \tag{17}$$

Here, φ_m, ψ_m and $q_m(x,\ t)$ are the Fourier coefficients of $\varphi(x)$, $\psi(x)$ and $q(x,\ t)$ over $\Omega_x = [0,a]$, respectively, corresponding to the basis $\{\sin(m\pi x/a);\ m = 1,2,\ldots\}$, with $\alpha_m = m\pi/a = O(m)$, $\omega_m = \sqrt{EI/\rho A}\,\alpha_m^2 = O(m^2)$.

For $\ddot{v}$ and $v^{(4)}$ to be continuous and to be obtained from differentiating the Fourier series (17), it is required that

$$q(0,t) = q(a,t) = 0. \tag{18}$$

For the six conditions (15) and (18), a six-term polynomial (fifth-degree in x) is suggested and solved:

$$p(x,t) = C_0(t) + C_1(t)x + C_2(t)x^2 + C_3(t)x^3 + C_4(t)x^4 + C_5(t)x^5,$$

$$p(x,t) = \frac{x}{a}\,g(t) + \frac{\rho A}{EI}\,\zeta(x,a)\ddot{g}(t), \tag{19}$$

where

$$\zeta(x,a) = \frac{a^4}{360}\left(-7\frac{x}{a} + 10\left(\frac{x}{a}\right)^3 - 3\left(\frac{x}{a}\right)^5\right). \tag{20}$$

Thus,

$$q(x,t) = -(\rho^2A^2/EI)\zeta(x,a)g^{(4)}(t).$$

Now the solution needs to be verified. From Eq. (13), consistency requires

$$g(0) = \dot{g}(0) = \ddot{g}(0) = 0. \tag{21}$$

Therefore,

$$p(x,0) = g(0)x/a + (\rho A/EI)\ddot{g}(0)\zeta(x,a) = 0, \quad \dot{p}(x,0) = (\rho A/EI)\dddot{g}(0)\zeta(x,a).$$

Note that the Fourier coefficients of $\zeta(x,a)$ are

$$\zeta_m = (-1)^m 2a^{-1}\alpha_m^{-5}. \tag{22}$$

Then

$$\psi_m = -\dddot{g}(0)\,\frac{\rho A\zeta_m}{EI} = O(m^{-5}),$$

$$q_m(t) = -\frac{\rho^2 A^2 \zeta_m}{EI}\int_0^t g^{(4)}(\tau)\sin\,\omega_m(t-\tau)\,\mathrm{d}\tau = O(m^{-5}),$$

$$v_m(t) = \frac{\rho A\zeta_m}{EI\omega_m}\left(-\dddot{g}(0)\,\sin\,\omega_m t - \int_0^t g^{(4)}(\tau)\,\sin\,\omega_m(t-\tau)\,\mathrm{d}\tau\right). \tag{23}$$

It is straightforward to verify that the differentiation of series (17) twice with respect to t or four times with respect to x results in a series whose coefficients are of order $O(m^{-3})$. Therefore, the differentiated series converge uniformly, the derivatives are valid, and the solution is verified in the classical sense.

As a summary, for Eq. (13) to have a classical solution, with continuous second t-derivative and fourth x-derivative, it is necessary that $\ddot{g}(t)$ is continuous and the consistency conditions (21) are satisfied. When $g^{(4)}(t)$ is continuous, the classical solution can be constructed with the transform (14), defined in Eqs. (19) and (20), and the Fourier series (17); $v_m(t)$ of Eq. (17) is defined in Eq. (23) and ζ_m in Eq. (23) is defined in Eq. (22).

4. Conclusions

Closed form solutions, in the classical sense, were obtained constructively for the forced vibrations of a rod and beam excited by displacement boundary conditions. The deficiency in a previous approach solving the rod problem, commonly seen in textbooks, was discussed. The new approach had additional requirements for the transformed body force to vanish at the end points. The uniform convergence of the differentiated series was illustrated and the solutions were verified in the classical sense.

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