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Short Communication

Experimental check on the accuracy of Timoshenko's beam theory

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1. Introduction

Two recent articles [1,2] have dealt with the accuracy of Timoshenko's beam theory (TBT). The flexural normal modes of a long beam with a thin rectangular cross-section are considered. They are calculated in two different ways: using TBT and plane-stress elastodynamic theory. The latter produces solutions which are to be regarded as the exact theoretical benchmark. In these two works, the TBT solutions are compared with the theoretical benchmark. In this paper we make use of a different benchmark, in our case an experimental one.

In Section 2 we indicate how the normal-mode frequencies and amplitudes are measured for flexural waves using the electromagnetic–acoustic transducer (EMAT) we have recently introduced [3,4]. The measurements are performed for an aluminum beam of length L and rectangular cross-section with free ends. In Section 3 we show how the TBT predictions for the normal frequencies are computed. These predictions are then compared with the experimental values in Section 4, providing an additional test of this theory.

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2. Experiment

Our EMAT is very versatile and useful at low frequencies; we have tested it from a few Hertz to 200 kHz. It can excite and detect both longitudinal and transverse waves in a beam or in a rod. The EMAT consists of a coil and a permanent magnet and excites Foucault currents (also called eddy currents), as indicated in Refs. [3,4]. This EMAT is shown in the left lower corner of Fig. 1 and is configured to excite or detect flexural waves without any mechanical contact with the beam.

A block diagram of the experimental setup is given in Fig. 1. The signal of an oscillator (Stanford Research Systems, Function Generator DS345) is sent to a power amplifier (Krhon-Hite 7500) and then to the EMAT exciter. The signal from the transducer scanning detector is amplified by a high-impedance amplifier and then sent to the lock-in amplifier (EG&G PARC 128A). The latter converts the AC signal to a DC voltage, which is digitized by a Computer Automated Measurement And Control (CAMAC), whose output is sent to a PC. The DC voltage is proportional to the wave amplitude when exciting a normal mode of the beam. The reference signal for the lock-in was taken from the oscillator. To count the number of nodes associated with each normal mode, the wave amplitudes were scanned with the detector along the beam. The position z of the scanning detector along the beam axis is measured by mechanically coupling it to a cursor in contact with a nichrome wire, depicted by the bold line in Fig. 1. The signal of the voltage divider is then sent to the CAMAC and finally to the PC. The scanning detector as well as the cursor are moved with a motor (not shown) controlled by the PC and the CAMAC. The normal-mode wave amplitudes are then scanned along the beam.

We should mention that the experimental error in the resonant frequencies due to temperature variations, phase errors and other sources is less than 0.01%. For the first few modes, however, the location of the two nylon threads supporting the beam does matter and could increase the experimental error. To avoid this problem we have located the threads at the nodes of the corresponding wave amplitude.

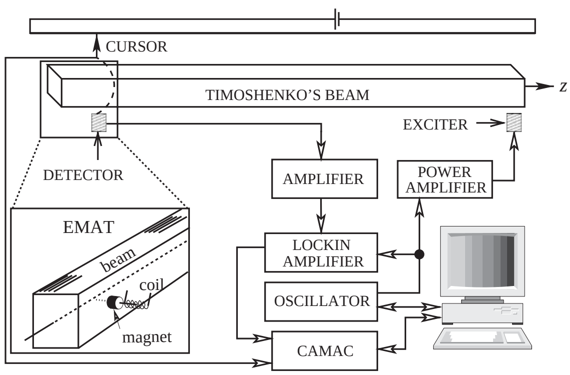

Fig. 1. Block diagram for the experimental setup. The EMATs (exciter and scanning detector) are shown shaded and in a configuration to excite or detect flexural modes. The dashed line indicates mechanical coupling between the detector and the cursor.

3. Solution to Timoshenko beam theory

The fourth-order differential equation for the amplitude ξ in the two-coefficient TBT is [5]

$$\frac{EI}{\rho S}\frac{\partial^4\xi}{\partial z^4}-\frac{I}{S}\left(1+\frac{E}{\kappa_3 G}\right)\frac{\partial^4\xi}{\partial z^2\partial t^2}+\frac{\partial^2\xi}{\partial t^2}+\frac{\rho I}{\kappa_1 GS}\frac{\partial^4\xi}{\partial t^4}=0, \tag{1}$$

which is separable for a normal mode, when $\xi=\Psi(z)\cos(\omega t)$, with $\omega=2\pi f$. Here f is the frequency, G and E the shear and Young modulus, respectively, ρ the density, S the transversal area of the beam and I the moment of inertia. The one-parameter TBT is obtained by setting $\kappa_1=\kappa_3$, equal to κ, the Timoshenko coefficient.

If the normal mode condition is imposed, Eq. (1) reduces to the following fourth-order differential equation for $\Psi(z)$:

$$\frac{d^4\Psi}{dz^4}-\alpha\frac{d^2\Psi}{dz^2}-\beta\Psi=0 \tag{2}$$

with

$$\alpha=-\frac{\rho\omega^2}{E}-\frac{\rho\omega^2}{\kappa_3 G}\quad\text{and}\quad\beta=\frac{\rho S\omega^2}{EI}-\frac{\rho^2\omega^4}{E\kappa_1 G}. \tag{3}$$

Eq. (2) can be solved by making it equivalent to four first-order differential equations. To implement this procedure, we set $g_1=\Psi$, $g_2=\Psi'$, $g_3=\Psi''$ and $g_4=\Psi'''$, where the prime indicates differentiation with respect to z. Eq. (2) is then equivalent to the following set of equations:

$$\begin{pmatrix}g_1'\\g_2'\\g_3'\\g_4'\end{pmatrix}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\\beta&0&\alpha&0\end{pmatrix}\begin{pmatrix}g_1\\g_2\\g_3\\g_4\end{pmatrix}. \tag{4}$$

The solution to this system is given by

$$\Psi(z)=A\lambda_1^{-3}e^{\lambda_1 z}+B\lambda_2^{-3}e^{\lambda_2 z}+C\lambda_3^{-3}e^{\lambda_3 z}+D\lambda_4^{-3}e^{\lambda_4 z}, \tag{5}$$

where

$$\lambda_i=(-1)^{[i/2]}\sqrt{\frac{\alpha+(-1)^i\sqrt{\alpha^2+4\beta}}{2}},\quad\text{for } i=1,2,3,4, \tag{6}$$

where $[i/2]$ is the integer less than $i/2$. We impose free-end boundary conditions at $z=0$ and L. These conditions are fulfilled only when

$$\det\begin{pmatrix}1&1&1&1\\\lambda_1^{-1}&\lambda_2^{-1}&\lambda_3^{-1}&\lambda_4^{-1}\\e^{\lambda_1 L}&e^{\lambda_2 L}&e^{\lambda_3 L}&e^{\lambda_4 L}\\\lambda_1^{-1}e^{\lambda_1 L}&\lambda_2^{-1}e^{\lambda_2 L}&\lambda_3^{-1}e^{\lambda_3 L}&\lambda_4^{-1}e^{\lambda_4 L}\end{pmatrix}=0. \tag{7}$$

The eigenfrequencies ω_i are then obtained numerically by scanning this determinant as a function of ω. When a change of sign is found, a routine for root finding is called.

4. Comparison between theory and experiment

We have computed the predictions of TBT for various values of κ in the one-parameter theory and one set of values of κ_1 and κ_3 in the two-parameter version of TBT. The values used for κ, κ_1 and κ_3 are given in Table 1 [2], where ν is the Poisson ratio. We then plot in Fig. 2 the normal-mode frequency percentage error for each of these calculations as a function of the modal number, using the experimental benchmark instead of the theoretical one.

From Fig. 2 we see that for low frequencies, that is, large wavelengths, all theories agree with the experimental values. At high frequencies, the relative error grows and theory E produces the best results. However, using the one-parameter TBT and a least-squares fit to the experimental

Table 1
Different values of the Timoshenko coefficients used to test TBT

Theory	Timoshenko coefficient(s)
A	$\kappa = 5/6$
B	$\kappa = 0.83945$
C	$\kappa = 10(1+\nu)/(12+11\nu)$
D	$\kappa = 5(1+\nu)/(6+5\nu)$
E	$\kappa_1 = 10(1+\nu)/(12+11\nu)$, $\kappa_3 = 5(1+\nu)/(6+5\nu)$

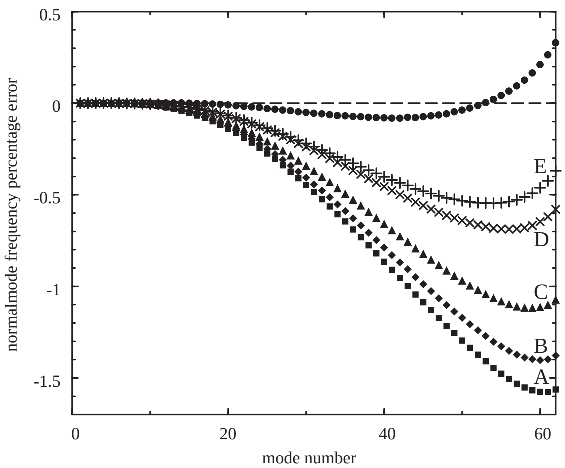

Fig. 2. Percentage error in the frequency prediction with respect to the experimental benchmark. Theory A (squares), B (diamonds), C (triangles), D (crosses) and theory E (plus signs). The circles correspond to a least-squares fit. The value $\nu = 0.3$ was used and $L = 1.000 \pm 0.0005\,\text{m}$. The length of the sides of the cross-section was $w = 1.9088 \pm 0.0004 \times 10^{-2}\,\text{m}$ and $h = 1.9063 \pm 0.0004 \times 10^{-2}\,\text{m}$.

measurements, we can obtain a value of κ that, for a given number N of normal-mode frequencies, produces an even better fit, as shown by the dots in Fig. 2. These dots correspond to the somewhat larger value of $\kappa = 0.8987$ when $N = 60$.

One should mention here that a somewhat less detailed experimental analysis is given in Ref. [6], but the experimental method followed in that work is much less precise than the one using the EMAT we have presented here, so the authors could not conclude which value of the Timoshenko coefficient fits best.

5. Conclusions

Comparing with experimental values, we have shown that TBT predictions are rather accurate for not too high frequencies. In any case, the two-parameter TBT relative error is always less than 0.6%.

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