Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 279 (2005) 1071–1084
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Solution of vibration problems for shallow shells of arbitrary form by the R-function method

L.V. Kurpa[a,*], K.I. Lyubitska[a], A.V. Shmatko[b]

[a] *Department of Applied Mathematics, NTU "Kharkov Polytechnical Institute", 21 Frunze str., 61002 Kharkov, Ukraine*
[b] *Institute for Problems in Machinery (IPMACH) of National Academy of Sciences of Ukraine, 2/10 Pozharsky str., Kharkov 61046, Ukraine*

Received 20 March 2002; accepted 26 November 2003

Abstract

An effective method for the free vibration of arbitrary plan-form shallow shells is proposed. The algorithm is based on the R-function theory and variational Ritz method. The effectiveness of the method offered is illustrated by examples of shallow shells of a complex plan form at different boundary conditions. Three types of curvatures are considered, which are the spherical, circularly cylindrical and of hyperbolic paraboloid shape. A number of the test problems were solved to check the veracity of the proposed method. The comparison between the obtained results and those available in the literature, was carried out.

1. Introduction

The problem of dynamic research into elements of thin-walled designs simulated by shallow shells with the given plan form, is present for many fields of science and engineering. However, in general, the solution is connected with many mathematical difficulties. It is explained by the necessity of solving systems of differential equations with partial derivatives of comparatively high order.

Many studies in the free vibration analysis of shallow shells have been presented in numerous review works. Among them are Leissa's monograph [1], Qatu's review article [2] and a more recent survey by Liew et al. [3], focused on research advances in vibration studies since the 1970s. Detailed analysis of many studies related to the vibration of shallow shells of arbitrary shape leads to the next conclusion. The most widespread numerical methods used for the solution of such

*Corresponding author. Fax: +380-57-70-040-34.
E-mail address: kurpa@kpi.kharkov.ua (L.V. Kurpa).

doi:10.1016/j.jsv.2003.01.002

problems are the finite element method (FEM) [4], finite strip [5] and boundary elements (BEM) [6]. However, more recently many researchers have begun to use the well-known general Rayleigh–Ritz or Ritz method. At this point attention is drawn to papers [7–10], in which the *pb*-2 Ritz method is used. The review of many works [7–13] devoted to the application of the Ritz method to free vibration analysis of shallow shells allows the conclusion to be drawn that the majority of them consider shells with simple enough planform, such as rectangular, circle, annular, rhombic, triangular or trapezoidal, at the best is in the convex domain. From the present point of view it is also explained by those difficulties, which can arise with the construction of the systems of co-ordinate functions for non-convex areas.

The present paper is in agreement with the above mentioned works, but at the same time has principal differences, because the proposed method is based on R-function theory and variational methods [14].

The R-function method (RFM) is less well known than those above-mentioned methods but it is universal and effective [15,16]. One of the main RFM advantages is the possibility of construction the general structure of solution (GSS) in analytical form for plates and shallow shells with any arbitrary planform. These solution's structure exactly satisfy either all boundary conditions or only main (kinematic) boundary conditions and contains indefinite functions, which can be found by means of any variational or other numerical method. The constituent elements of structural formulae, which were constructed, are the functions by means of which either the equation of area's boundary or its separate segment is described. Such equations can be constructed for arbitrary area by means of R-functions theory in the form of a single analytical expression. It is important to note that the solutions structure for different shapes of areas does not vary with fixed type of boundary conditions. It has allowed formalizing a mathematical target setting, to construct algorithms of their solution and to create the computer-aided programming systems fulfilling all the necessary numerical research. Despite wide experience in application of the R-functions theory with respect of plate and shell problems, this method has begun to be applied to dynamic problems of the shallow shells theory only recently. Thus, in Ref. [17] the problem about free oscillations of cantilever shallow shells (cylindrical, spherical and hyperbolic paraboloid) was resolved by the means of a RFM method. The present paper is dedicated to development of the RFM for problem solving about oscillations of shallow shells of the composite shape with different types of boundary conditions.

2. Variational statement of the vibration problem

Consider free vibrations of a shallow shell of uniform thickness h using classical theory based on Kirchhoff–Love hypotheses.

The Ritz procedure requires minimizing the functional

$$J = V_{max} - T_{max},$$

where V_{max} and T_{max} are the maximum values of potential energy V and kinetic energy T of the system. The potential energy is

$$V_{max} = \frac{Eh}{2(1-v^2)} \int\int_{\Omega} [(\varepsilon_x + \varepsilon_y)^2 - 2(1-v)(\varepsilon_x \varepsilon_y - \gamma_{xy}^2/4)]\partial\Omega + \frac{Eh^3}{24(1-v^2)} \int\int_{\Omega} [(K_x + K_y)^2 - 2(1-v)(K_x K_y - K_{xy}^2)]\partial\Omega. \quad (1)$$

The kinetic energy is

$$T_{max} = \int\int_{\Omega} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\partial\Omega. \quad (2)$$

The lengthening strain ε_x, ε_y, shift strain γ_{xy} and curvature K_x, K_y, K_{xy} are defined as

$$\varepsilon_x = \partial u/\partial x + w/R_x, \quad \varepsilon_y = \partial v/\partial y + w/R_y, \quad \gamma_{xy} = \partial v/\partial x + \partial u/\partial y, \quad (3)$$

$$K_x = \partial^2 w/\partial x^2, \quad K_y = \partial^2 w/\partial y^2, \quad K_{xy} = \partial^2 w/\partial x \partial y. \quad (4)$$

Assuming harmonic motion for the time response in free vibration in-plane displacements u, v (in directions of Ox- and Oy-axis) and also transverse deflection w (in direction of axis-Oz) may be expressed in the form

$$u(x,y,t) = U(x,y)\sin\lambda t, \quad v(x,y,t) = V(x,y)\sin\lambda t, \quad w(x,y,t) = W(x,y)\sin\lambda t, \quad (5)$$

where λ is a natural frequency.

By considering Eqs. (3)–(5), an initial problem may be reduced to determination of the stationary point of the functional

$$J(U,V,W) = \frac{12D}{h^2} \int\int_{\Omega} \left\{ \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{W}{R_x} + \frac{W}{R_y} \right)^2 - 2(1-v)\left[\left(\frac{\partial U}{\partial x} + \frac{W}{R_x} \right)\left(\frac{\partial V}{\partial y} + \frac{W}{R_y} \right) - \frac{1}{4}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right)^2 \right] \right\} d\Omega + \frac{D}{2} \int\int_{\Omega} \left\{ \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right)^2 - 2(1-v)\left[\left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) - \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \right] \right\} d\Omega - \frac{\lambda^2 \rho h}{2} \int\int_{\Omega} (U^2 + V^2 + W^2)\, d\Omega, \quad (6)$$

where $D = Eh^3/12(1-v^2)$ denotes the flexural rigidity, ρ is the mass density per unit volume, E is Young's modulus, v is the Poisson factor, and U, V, W are the shell's displacements.

According to the Ritz method, unknown displacements functions U, V, W are represented as

$$U = \sum_{i=1}^{N_1} a_i U_i, \quad V = \sum_{i=N_1+1}^{N_2} a_i V_i, \quad W = \sum_{i=N_2+1}^{N_3} a_i W_i, \quad (7)$$

where $\{U_i\}, \{V_i\}, \{W_i\}$ define mathematically complete sets of co-ordinate functions, satisfying, at least, cinematic (main) boundary conditions, a_i is an indefinite factor.

Substituting Eq. (7) into Eq. (6) unknown coefficients can be defined from a functional minimum condition

$$\partial J/\partial a_i = 0, \quad i = 1, 2, 3, \ldots, N_3. \tag{8}$$

The algebraic system of the uniform linear equations (8) may be represented in matrix form as

$$(\mathbf{A} - \lambda^2 \mathbf{B})\mathbf{X} = 0, \tag{9}$$

where $\mathbf{X}^{\mathrm{T}}(a_1, a_2, \ldots, a_n)$ is a vector of unknown factors, $\mathbf{A} = \{a_{ij}\}$, $\mathbf{B} = \{b_{ij}\}$ are Ritz matrices, the elements of which are defined by

$$\begin{aligned}
a_{ij} = &\frac{12D}{h^2} \int\int_\Omega \Bigg\{ \left(\frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y} + \left(\frac{1}{R_x} + \frac{1}{R_y} \right) W_i \right) \left(\frac{\partial U_j}{\partial x} + \frac{\partial V_j}{\partial y} + \left(\frac{1}{R_x} + \frac{1}{R_y} \right) W_j \right) \\
&- (1 - v) \Bigg[\left(\frac{\partial U_i}{\partial x} + \frac{1}{R_x} W_i \right) \left(\frac{\partial V_j}{\partial y} + \frac{1}{R_y} W_j \right) + \left(\frac{\partial U_j}{\partial x} + \frac{1}{R_x} W_j \right) \left(\frac{\partial V_i}{\partial y} + \frac{1}{R_y} W_i \right) \\
&- \frac{1}{2} \left(\frac{\partial U_j}{\partial y} + \frac{\partial V_i}{\partial x} \right) \left(\frac{\partial U_i}{\partial y} + \frac{\partial V_i}{\partial x} \right) \Bigg] \Bigg\} \, \mathrm{d}\Omega + D \int\int_\Omega \Bigg[\left(\frac{\partial^2 W_i}{\partial x^2} + \frac{\partial^2 W_i}{\partial y^2} \right) \left(\frac{\partial^2 W_j}{\partial x^2} + \frac{\partial^2 W_j}{\partial y^2} \right) \\
&- (1 - v) \left(\frac{\partial^2 W_i}{\partial x^2} \frac{\partial^2 W_j}{\partial y^2} + \frac{\partial^2 W_j}{\partial x^2} \frac{\partial^2 W_i}{\partial y^2} - 2 \frac{\partial^2 W_i}{\partial x \partial y} \frac{\partial^2 W_j}{\partial x \partial y} \right) \Bigg] \, \mathrm{d}\Omega,
\end{aligned}$$

$$b_{ij} = \rho h \int_\Omega (U_i U_j + V_i V_j + W_i W_j) \, \mathrm{d}\Omega. \tag{10}$$

If a shallow shell has complex plan form or boundary conditions then the problem of construction of the sequences of co-ordinate functions is very difficult. In the present work, this problem is solved by a variational–structural method based on the R-functions theory RFM. According to this method the system of co-ordinate functions can be obtained by a solution's structure which can satisfy either all or only main (kinematic) boundary conditions.

3. Structures of the solution for different kinds of boundary conditions

The RFM has been used before for vibration problems of plates of complex form with different boundary conditions [15,16]. The GSS had been constructed for the base types of fastening, clamped edge, simply supported, free edge and mixed boundary conditions.

Table 1
Structural formulas for different boundary conditions

Kind of fastening	Boundary conditions	Structure of the solution
Rigidly clamped	$U = 0, V = 0, W = 0, \partial W/\partial n = 0$	$U = \omega P_1$, $V = \omega P_2$, $W = \omega^2 P_3$
Simply supported	$U = 0,\ V = 0,\ W = 0,\ M_n = 0$	$U = \omega P_1$, $V = \omega P_2$ $W = \omega P_3$
Slip clamped	$W = 0, \partial W/\partial n = 0, \varepsilon_\tau = 0, \sigma_n = 0$	$U = P_1$, $V = \omega P_2$, $W = \omega^2 P_3$
Slip supported	$W = 0,\ M_n = 0,\ \varepsilon_\tau = 0, \sigma_n = 0$	$U = P_1$, $V = \omega P_2$, $W = \omega P_3$

The ways of shells fastening are varied enough. In Table 1 for some types of fastening of shells the examples of the structural formulas, which satisfy only kinematic regional conditions, are given.

In the given formulas, σ_n is the normal stress, ε_τ is the strain along the tangent, M_n is the normal bending moment [18]. The function $\omega(x, y)$ satisfies the conditions

$$\omega(x,y) > 0 \ \ \forall(x,y) \in \Omega, \quad \omega(x,y) = 0 \ \ \forall(x,y) \in \partial\Omega$$

and may be constructed by RFM.

The indefinite components of the structural formulas [15] P_1, P_2, P_3 are represented as expansion of the series on any complete system of functions, for example, power polynomials, spline, trigonometric polynomials or others

$$P_1 = \sum_{i=1}^{N_1} a_i \varphi_i, \quad P_2 = \sum_{i=N_1+1}^{N_2} a_i \psi_i, \quad P_3 = \sum_{i=N_2+1}^{N_3} a_i \chi_i. \tag{11}$$

Substituting P_i $(i = \overline{1,3})$ in the structural formulae, it is possible to receive necessary sequences of co-ordinate functions $\{U_i\}$, $\{V_i\}$, $\{W_i\}$.

One of the essential practical advantages of the RFM is the possibility of accounting for special behaviour of solutions. Below is presented the example of a structure, which includes a cutout on the free edge of a shallow shell. Suppose that the shell is rigidly clamped on one part of the boundary ($u = v = 0$, $w = \partial w/\partial\eta = 0$, $(x, y) \in \partial\Omega_1$) and free on another ($T_n = 0$, $S = 0$, $Q_n = 0$, $M_n = 0$, $(x, y) \in \partial\Omega_2$). There is a crack/cut on a free side of the boundary (Fig. 1).

First of all it should be noted that the conditions on the free edge are natural, and the conditions on rigidly clamped segment of the boundary are principal. At first sight it seems that the sequence of co-ordinate functions can be obtained by means of structural formulas (types by Kantorovich),

$$W = \omega_1^2 P_1, \quad U = \omega_1 P_2, \quad V = \omega_1 P_3, \tag{12}$$

where $\omega_1 = 0$ is the equation of the rigidly clamped part of the boundary, P_1, P_2, P_3 are indefinite components of the structure of the solution, which are expanded in series on any complete system of functions. The coefficients of this expansion may be found from the condition of a functional minimum (8). However, the edges of shallow shells, which are along a crack/cut, can diverge in the

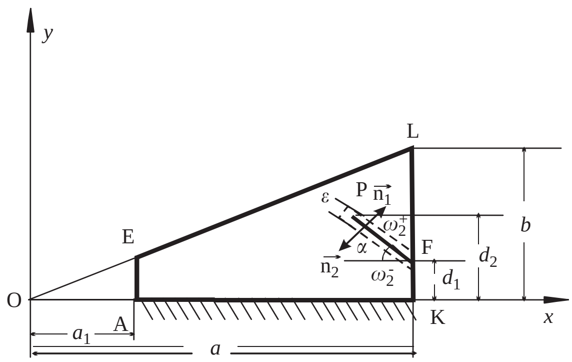

Fig. 1. Plan-form of the console trapezium shallow shell.

direction of a perpendicular median surface during lateral vibrations. Because of the continuity of a function $W(x, y)$ structure (12) fails to take into account such phenomena and consequently the structure must be considered incomplete. It is possible to construct a structure of the solution for the W function, which would be complete, i.e., one would provide the "jump" for the deflection $W(x, y)$. Using the R-functions theory, the normalized equations of a crack/cut as $\omega_2 = \sqrt{l^2 \vee_\alpha \bar{\varphi}}$ will be constructed, where $l = 0$ is the normalized (sign l changes at PF line crossing) equation of a line including the crack/cut. The function $\varphi(x, y) \geqslant 0$ describes the area, which separates the crack/cut from the PF line, $\vee_\alpha$ is the sign of the R-operation, namely R-disjunction [15]

$$x \vee_\alpha y = x + y + \sqrt{x^2 + y^2}. \tag{13}$$

By designating the parties of the crack/cut $\omega_2 = 0$ as ω_2^+ and ω_2^-, then the approach to the crack/cut from the different parties, the function [15]

$$D_1^{(l)}\omega_2 = (\partial l/\partial x)\partial\omega_2/\partial x + (\partial l/\partial y)\partial\omega_2/\partial y \tag{14}$$

behaves as

$$D_1^{(l)}\omega_2 \begin{Bmatrix} -1 \text{ along } \omega_2^- \\ +1 \text{ along } \omega_2^+ \end{Bmatrix}.$$

For example, in the case of a plan-form represented in Fig. 1, one has $l = x\sin\alpha + y\cos\alpha - d_1\cos\alpha + a\sin\alpha$. $D_1^{(l)}\omega_2|_{\omega_2^+=0} = \partial\omega_2/\partial n = 1$, $D_1^{(l)}\omega_2|_{\omega_2^-=0} = -\partial\omega_2/\partial n = -1$. Thus the function $D_1^{(l)}\omega_2$ is determined everywhere on the Ω, and it is equal to $D_1^{(l)}\omega_2 = \pm 1 + O(\omega_2)$. Using a function $D_1^{(l)}\omega_2$ it is possible to construct functions q_1 and q_2, which ones, approaching to a crack/cut on a normal to the one side, tend to 0 and at the approaching to the other side, tend to 1, namely

$$q_1 = \tfrac{1}{2}(1 + D_1^{(l)}\omega_2) = \begin{Bmatrix} 0 \text{ when } l \to -0 \\ K \text{ when } l \to +0 \end{Bmatrix}, \quad q_2 = \tfrac{1}{2}(1 - D_1^{(l)}\omega_2) = \begin{Bmatrix} 1 \text{ when } l \to -0 \\ 0 \text{ when } l \to +0 \end{Bmatrix}. \tag{15}$$

Assuming that a deflection function on a line of a crack/cut looks like $W_2 = P_{12}q_1 + P_{22}q_2$, where P_{12} and P_{22} are indefinite components, freedom of the behaviour of the W function will be provided on each of the parties of a crack/cut. Thus, a structure of the solution for W function in a considered case may be presented as

$$W = \begin{Bmatrix} \omega_1^2 P_1 \text{ in the vicinity of } \omega_1 = 0 \\ W_2 + P_1 \text{ in the vicinity of } \omega_3 = 0 \end{Bmatrix}. \tag{16}$$

where $\omega_3 = 0$ is the equation of the free part including a crack/cut. Using the interlocational Lagrange formula [15,16], the final structure of the solution for $W(x, y)$ looks like

$$W = (\omega_1^2/(\omega_3 + \omega_1^2))[W_2 + (\omega_3 + 1)P_1]. \tag{17}$$

4. Numerical examples

The specialized problem-oriented system POLE-SHELL [19] was created for numerical implementation of the given algorithm. This system has the necessary computational base, which

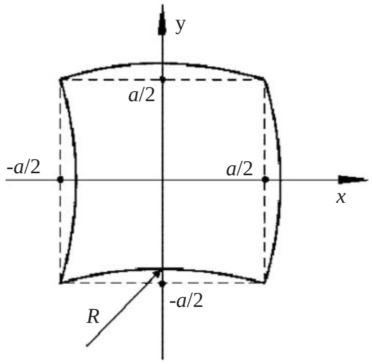

Fig. 2. Shallow spherical shell with square planform.

Table 2
Vibration of a slip supported and slip clamped shallow shell with square planform

	Λ, Slip supported				Λ, Slip clamped			
K	$ncf = 30$	$ncf = 35$	$ncf = 45$	[5]	$ncf = 30$	$ncf = 35$	$ncf = 45$	[5]
0.0	19.810	19.786	19.785	19.785	36.068	36.068	36.066	36.002
3.0	20.036	20.012	20.012	20.011	36.193	36.193	36.191	36.125
5.0	20.431	20.407	20.407	20.405	36.413	36.412	36.412	36.344
10	22.190	22.168	22.168	22.160	37.431	37.426	37.426	37.351
20	28.142	28.125	28.125	28.104	41.250	41.232	41.231	41.134
50	53.690	53.681	53.681	53.678	61.636	61.563	61.559	61.404
100	101.267	101.262	101.262	101.738	105.844	105.693	105.683	105.797

allows fulfilling a broad numerical experiment. The source programs RL [20] were compounded and their testing was executed to solve the problem of free oscillations of shallow shells within the limits of this system.

Example 1. Consider a shallow spherical shell with a square planform with the parameters: $a/h = 100$; $\nu = 0.3$ (Fig. 2). Boundary conditions: (a) slip supported; (b) slip clamped. Table 2 compares the fundamental frequency parameter $\Lambda = \lambda a^2\sqrt{(\rho h/D)}$ given by Bucco and Mazumdar [21] who used the FEM with the present method. The parameter K^* has the same sense, as in Ref. [21] and is defined as $K^* = [12(1-\nu^2]^{1/2}a^2/Rh$, where R is the radius of curvature of the spherical surface of the shell. To determine convergence of the obtained results calculations were carried out at different numbers of co-ordinate functions (ncf). The *ncf* was determined by the degree of approximating polynomials. From Table 2 it can be seen that the stabilization of results occurs at 35 co-ordinate functions. It corresponds the eighth degrees of approximating polynomials for functions $U(x,y)$, $V(x,y)$, $W(x,y)$. The consequent increasing of number *ncf* changes the results in the fourth order character, it means that the error is about 0.5%.

Example 2. The free vibration of the shallow spherical shell with planform submitted in Fig. 3 is considered. The edges of the shell are assumed to be slip supported. Investigation of the

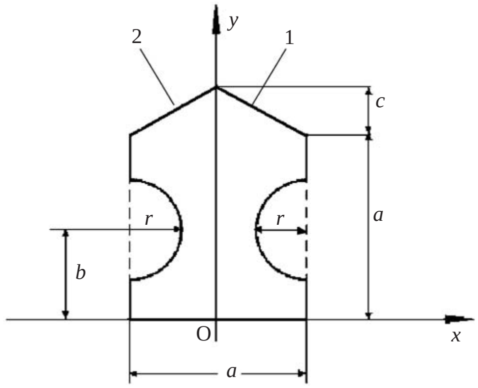

Fig. 3. Plan-form of the shallow spherical shell.

Table 3
Convergence of frequencies of vibration for a shallow shell (Example 2, $K^* = 50$)

r/a	ncf	Λ_1	Λ_2	Λ_3	r/a	ncf	Λ_1	Λ_2	Λ_3
0	44	52.69	62.77	70.48	0.2	44	56.69	70.97	83.38
	56	52.67	62.67	69.21		56	56.48	70.44	82.64
	70	52.66	62.65	69.17		70	56.36	70.32	81.84
0.1	44	55.66	65.88	80.52	0.3	44	60.76	88.68	94.35
	56	55.35	65.61	78.72		56	60.46	87.66	92.99
	70	55.20	65.57	77.89		70	60.34	87.49	92.26

dependence of vibration frequencies from curvature and the depth of cut (r/a) in the shell were made. The values of parameters of the shell were selected as: $a/h = 100$; $v = 0.3$, $b = 0.4a$, $c = 0.3a$, $r/a = 0, 0.1, 0.2, 0.3$. Convergence of the frequencies for such a shell is shown in Table 3. It is interesting to note that for higher numbers of mode causes the convergence is slower. It also depends on shell geometry. For example, for $r/a = 0$ convergence is faster than for $r/a = 0.3$. The further increasing a degree of approximating polynomials does not influence the obtained results essentially. Error makes no more than 0.3%.

The numerical values of the frequency parameter for various values $K^* = [12(1 - v^2)]^{1/2}a^2/Rh$ are presented in Table 4. By setting $r/a = 0$ and $b = 0$ the results (not shown here) coincide with the given test, confirming the validity of the proposed algorithm. From Table 4 it can be seen, that at the increasing depth of a crack/cut parameter of frequency is increasing, especially in case of plates that corresponds to the physical sense of a problem. With increasing curvature of the shell this influence appreciably decreases. Also it may be interpreted from Table 4 that the frequency will decrease with increases in shallowness for a given geometric parameter r/a.

Example 3. Calculate the frequency of free shallow shell oscillation, which is considered at Example 2. Suppose that the boundary conditions are mixed and the depth cut is fixed i.e.,

Table 4
Frequencies of vibration for a slip supported shallow shell (Fig. 3), $a/h = 100$, $\nu = 0.3$, $b = 0.4a$, $c = 0.3a$, $r/a = 0, 0.1, 0.2, 0.3$

r/a	K^*	Λ_1	Λ_2	Λ_3	r/a	K^*	Λ_1	Λ_2	Λ_3
0.0	0	16.80	37.85	47.14	0.2	0	25.93	49.30	63.92
	10	19.55	39.15	48.22		10	27.80	50.31	64.74
	20	26.11	42.80	51.32		20	32.77	53.23	67.12
	30	34.36	48.28	56.10		30	39.69	57.76	70.91
	40	43.34	55.04	62.16		40	47.71	63.56	75.89
	50	52.66	62.65	69.17		50	56.36	70.32	81.84
	60	62.16	70.85	76.86		60	65.38	77.77	88.56
	70	71.76	79.44	85.03		70	74.63	85.74	95.87
	80	81.41	88.31	93.55		80	84.01	94.08	103.66
	90	91.08	97.36	102.32		90	93.49	102.70	111.80
	100	100.75	106.56	111.26		100	103.02	111.54	120.22
0.1	0	23.42	42.14	58.88	0.3	0	33.64	71.56	76.72
	10	25.47	43.32	59.76		10	35.10	72.26	77.40
	20	30.81	46.69	62.32		20	39.16	74.34	79.41
	30	38.08	51.82	66.37		30	45.12	77.68	82.66
	40	46.36	58.24	71.64		40	52.33	82.12	86.99
	50	55.20	65.57	77.89		50	60.34	87.49	92.26
	60	64.35	73.53	84.89		60	68.86	93.63	98.30
	70	73.70	81.93	92.46		70	77.72	100.40	104.98
	80	83.15	90.65	100.46		80	86.80	107.68	112.18
	90	92.67	99.60	108.80		90	96.03	115.36	119.80
	100	102.23	108.71	117.38		100	105.37	123.37	127.75

Table 5
Convergence of frequencies of vibration for a shallow shell (Example 3, $K^* = 50$)

ncf	C–F				C–SS				C–C			
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_1	Λ_2	Λ_3	Λ_4	Λ_1	Λ_2	Λ_3	Λ_4
44	4.79	16.07	23.42	33.05	78.71	83.57	104.23	109.96	82.11	95.77	118.32	124.00
56	4.68	14.92	22.50	32.37	78.68	83.47	103.24	107.58	82.07	95.44	117.73	123.03
70	4.61	14.59	22.24	32.04	78.64	83.39	103.00	107.29	81.99	95.36	117.29	122.13

$r/a = 0.2$. Consider that parts of boundary 1, 2 (Fig. 3) are rigidly clamped (C), other parts of the boundary are: (a) free (F); (b) simply supported (SS); (c) clamped (C). The convergence of the frequencies and its numerical results are respectively shown in Tables 5 and 6.

From the characteristics given in Table 5 it may be concluded that Λ converges faster for a shell with more boundary constraints. It may also be noted that convergence is slower with increasing mode order (as in the previous example). The analysis of the received results shows that increases

Table 6
Frequencies of vibration for shallow shell (Fig. 3) with mixed boundary conditions, $a/h = 100$, $\nu = 0.3$, $b = 0.4a$, $c = 0.3a$, $r/a = 0.2$

K^*	C–F				C–SS				C–C			
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_1	Λ_2	Λ_3	Λ_4	Λ_1	Λ_2	Λ_3	Λ_4
0	3	8.88	17.15	29.98	32.41	54.57	73.65	90.47	44.96	74.76	94.39	107.99
10	3.10	9.24	17.45	30.09	35.56	56.02	75.06	91.18	47.06	75.69	95.42	108.58
50	4.61	14.59	22.24	32.04	78.64	83.39	103.00	107.29	81.99	95.36	117.29	122.13
100	6.18	21.12	28.84	35.02	133.26	135.14	152.90	159.14	136.58	139.06	160.74	166.69

in the boundary constraints always lead to increases in frequency parameter. For example in the case of a combination rigidly clamped–free edge (C–F) values of the frequency parameter are much less than with conditions rigidly clamped or combinations rigidly clamped–simply supported. Thus increasing curvature does not exert strong influence on values of frequencies. In the cases C–SS and C–C curvature of the shell puts the main influence on the frequency parameter. Thus the values of the frequency parameter differ a little among themselves at high enough curvature. On the whole it can be seen in Table 6 that an increase in curvature of the shell (see K^* values) causes an increase in its frequency.

Example 4. Consider the free vibration problem of the shallow shell upon the base represented in Fig. 1, and with the crack/cut PF on a free side. Assume, that the shell can be: (a) cylindrical ($1/R_x = 0$); (b) spherical ($1/R_x = 1/R_y = 1/R$); (c) doubly curved ($1/R_x = -1/R_y$). Suppose $\alpha = 0$, i.e., line of crack/cut is parallel to clamped edge. The numerical values of the frequency parameter $\Lambda_i = \lambda_i a^2 \sqrt{\rho h/D}$ are presented in Table 7. From Table 7 it follows that spherical and doubly curved shallow shells are most rigid. For these shells the frequencies are higher than for cylindrical shallow shells and the presence of the crack/cut has no essential influence on their frequencies. But the cylindrical shell is less rigid. Its frequencies are less. However, the presence of a crack has more essential influence on frequency of cylindrical shell oscillations.

Presented data were obtained by approximation of indefinite components in structural formulas, 66 co-ordinate functions for U and V and 78 for W. Integration was executed with the help of 16-pointed Gauss's formula. If the crack's length tends to zero the obtained results may be compared with the ones, which are known in Ref. [22]. Comparing the results obtained by the offered method with results in Ref. [22] (marked *) is submitted in the Table 7. It can be seen that they are in the good agreement.

Example 5. Consider a shallow shell with plan-form shown in Fig. 4. In this case function ω_1, which corresponds to a clamped edge, should be constructed by means of an R-functions i.e., $\omega_1 = F_1 \vee_0 F_2$, where $\vee_0$ is a sign of R-disjunction, which is determined by formula (13) at $\alpha = 0$,

$$F_1 = (y \geqslant 0);\ \ F_2 = ((R^2 - (x - (a + a_1)/2)^2 - y^2)/(2R) \geqslant 0).$$

Table 7
Influence of the curvature and length of the crack/cut PF on frequency parameter Λ for the console shallow shell (Fig. 1), $a/R_y = 0.5$, $b/a = 1$, $a/h = 100$, $d/a = 0.25$, $a_1/a = 0.5$, $r/a = 0.125$, $\nu = 0.3$, $\alpha = 0$

		Number of the mode					
R_y/R_x	PF/a	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6
+1	0	7.64*	22.27*	34.71*	64.67*	87.26*	115.3*
	0	7.66	22.5	34.9	65.0	87.7	117.0
	0.05	7.63	22.5	34.2	64.7	85.5	116.0
	0.2	7.54	22.2	34.4	64.5	85.2	115.0
	0.3	7.71	22.5	34.5	65.1	86.9	116.0
	0.4	7.58	22.4	34.4	64.8	85.4	115.0
0	0	5.452*	20.69*	30.78*	60.97*	84.55*	112.1*
	0	5.47	20.9	30.9	61.3	84.7	113.0
	0.05	5.18	20.8	30.2	60.2	82.2	112.0
	0.2	5.01	20.7	29.8	58.5	81.7	110.0
	0.3	5.16	20.8	30.4	59.7	83.3	112.0
	0.4	5.03	20.7	30.0	58.9	81.5	110.0
−1	0	8.138*	22.54*	37.56*	74.15*	89.78*	130.8*
	0	8.16	22.7	37.6	74.3	90.2	132.0
	0.05	7.90	22.6	37.1	73.6	88.8	130.0
	0.2	7.97	22.4	37.0	72.3	88.4	127.0
	0.3	8.00	22.6	37.4	73.5	90.2	130.0
	0.4	7.89	22.4	37.1	72.4	88.6	129.0

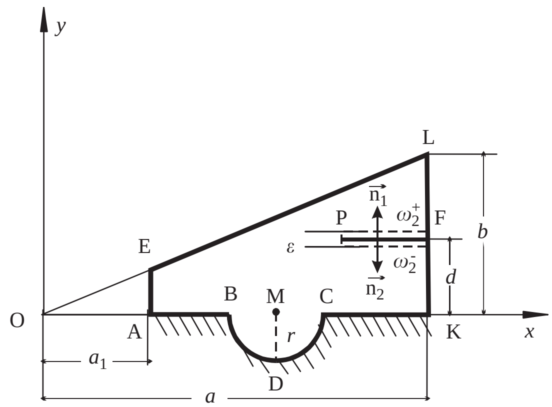

Fig. 4. Plan-form of the console shallow shell with a crack/cut PF on a free site.

The sequence of co-ordinate functions has been constructed by means of the formula (17) as well as in Example 4. The function ω_3 looks like

$$\omega_3 = F_3 \wedge_0 F_4 \wedge_0 \omega_2,$$

where $\wedge_0$ is sign R-conjunction at $\alpha = 0$ and

$$F_3 \wedge_0 F_4 = F_3 + F_4 - \sqrt{F_3^2 + F_4^2}.$$

The functions F_3, F_4 and ω_2 are determined as

$$F_3 = \left(\frac{b}{x}x - y \geqslant 0\right), \quad F_4 = \left(\left(\left(\frac{a-a_1}{2}\right)^2 - \left(x - \frac{a+a_1}{2}\right)^2\right) \Big/ (a - a_1) \geqslant 0\right).$$

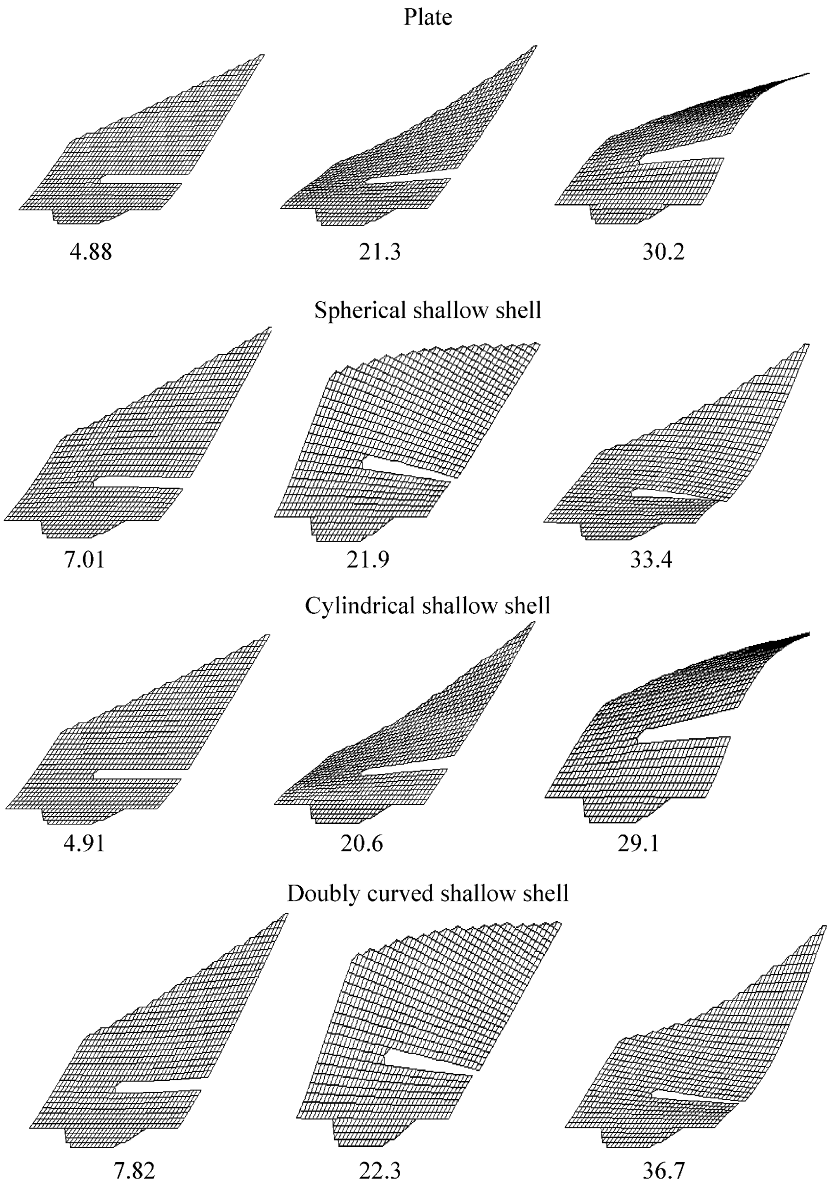

Fig. 5. Forms of vibration and three first modes for console shallow shells $a/R_y = 0.5$, $PF/a = 0.3$, $d/a = 0.25$, $b/a = 1$, $r/a = 0.125$, $a_1/a = 0.5$, $v = 0.3$, $\alpha = 0$.

The normalized equation of crack/cut line is presented as

$$\omega_2 = \left(\sqrt{l^2 \vee_0 \bar{\varphi}} = 0\right),$$

where $l = y - d$; $\bar{\varphi}(x, y) = ((x-a)^2 + (y-d)^2 - r_l^2)/(2r_l)$, where $r_l = |PF|$.

Frequency parameters $\Lambda_i = \lambda_i a^2 \sqrt{\rho h/D}$ and according to them forms of vibrations are shown in Fig. 5. Parameters of the shell are the next: $a/R_y = 0.5$, $PF/a = 0.3$, $d/a = 0.25$, $b/a = 1$, $r/a = 0.125$, $a_1/a = 0.5$, $\nu = 0.3$, $\alpha = 0$.

5. Conclusion

In the paper, an effective RFM method was presented. This method is worked out for solving the vibration problems of shallow shells with an arbitrary plan forms and different boundary conditions, including mixed ones. It uses the well-known Ritz method in combination with the R-functions theory. The sequences of basic functions satisfying different boundary conditions and taking into account crack/cuts are constructed by the R-functions method. Computations are carried out in specialized problem-oriented system POLE-SHELL. Several shallow shells problems are solved to demonstrate the effectiveness of the method for shells having various geometrical planforms. The results obtained by the proposed algorithm are compared with those known in literature. There was good agreement in all cases.

It can thus be concluded that the proposed R-function method and software created on its base is a useful addition to the methods for the analysis of plates and shallow shells.

References

[1] A.W. Leissa, *Vibration of Shells*, Nasa SP-288, US Government Printing Office, Washington, DC, 1973 (reprinted by the Acoustical Society of America, 1993).

[2] M.S. Qatu, Vibration of doubly cantilevered laminated-composite thin shallow shells, *Thin-Walled Structures* 15 (3) (1993) 235–248.

[3] K.M. Liew, C.W. Lim, S. Kitipornchai, Vibration of shallow shells: a review with bibliography, *Transactions of the American Society of Mechanical Engineers, Applied Mechanics Review* 50 (1997) 431–444.

[4] O.C. Zienkiewicz, *The Finite Element Method*, 3rd Edition, McGraw-Hill, New York, 1979.

[5] Y.K. Cheung, *Finite Strip Method in Structure Analysis*, Pergamon Press, Oxford, 1976.

[6] P.K. Banerjee, R. Butterfield, *Boundary Element Methods in Engineering Science*, McGraw-Hill, Maidenhead, 1981.

[7] C.W. Lim, K.M. Liew, A *pb*-2 Ritz formulation for flexural vibration of shallow cylindrical shells of rectangular planform, *Journal of Sound and Vibration* 173 (3) (1994) 343–375.

[8] K.M. Liew, S. Kitipornchai, C.M. Wang, Research developments in analyses of plates and shells, *Journal of Constructional Steel Research* 26 (1993) 231–248.

[9] K.M. Liew, M.K. Lim, C.W. Lim, D.B. Li, Y.R. Zhang, Effects of initial twist and thickness variations on the vibration behaviour of shallow conical shells, *Journal of Sound and Vibration* 180 (2) (1995) 271–296.

[10] K.M. Liew, C.W. Lim, A Ritz vibration analysis of doubly-curved rectangular shallow shells using a refined first-order theory, *Computer Methods in Applied Mechanics amd Engineering* 127 (1995) 145–162.

[11] A.W. Leissa, J.K. Lee, A.J. Wang, Vibration of cantilevered shallow cylindrical shells of rectangular form, *Journal of Sound and Vibration* 78 (1981) 311–328.

[12] A.W. Leissa, J.K. Lee, A.J. Wang, Rotating blade vibration analysis using shells, *Transactions of the American Society of Mechanical Engineers, Journal of Engineering for Power* 104 (1982) 296–302.

[13] J.K. Lee, A.W. Leissa, A.J. Wang, Vibrations of cantilevered circular cylindrical shells: shallow versus deep shell theory, *International Journal of Mechanical Sciences* 25 (1983) 361–383.

[14] V.L. Rvachev, T.I. Sheiko, R-functions in boundary value problems in mechanics. *Applied Mechanics Review* 48(4) (1995).

[15] V.L. Rvachev, *The R-Functions Theory and its Some Applying*, Naukova Dumka, Kiev, 1982 (in Russian).

[16] V.L. Rvachev, L.V. Kurpa, *The R-Functions in Tasks of the Theory of Plates*, Naukova Dumka, Kiev, 1987 (in Russian).

[17] V.L. Rvachev, L.V. Kurpa, A.V. Shmatko, Free vibration of shallow shells of arbitrary form in plan, *Mashinoznavstvo* 2 (2000) 3–6 (in Ukrainian).

[18] V.Z. Vlasov, *The General Theory of Shells and its Applying in Engineering*, Gostechizdat, Moscow, 1949 (in Russian).

[19] L.V. Kurpa, A.V. Shmatko, About one approach to problem of computer modeling of tasks of the plates and shallow shells theory of the complex plan-form, *Collection of Scientific Works of the VII International Conference MICROCAD-99*, Kharkov, 1999, pp. 299–302 (in Russian).

[20] V.L. Rvachev, A.N. Shevchenko, *Problem-Oriented Languages and Systems for Engineering Calculations*, Teknika, Kiev, 1988 (in Russian).

[21] D. Bucco, J. Mazumdar, Estimation of the fundamental frequencies of shallow shells by a finite element—isodeflection contour method, *Computers & Structures* 17 (1983) 441–447.

[22] Mohamad S. Qatu, Vibration analysis of cantilevered shallow shells with triangular and trapezoidal planforms, *Journal of Sound and Vibration* 191 (2) (1996) 219–231.