Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 279 (2005) 1121–1129
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Extension of the resonant beam technique to highly anisotropic materials

S. Puchegger[a,*], D. Loidl[a], K. Kromp[a], H. Peterlik[a], R. Weiß[b]

[a] Ceramics Group, Institute of Material Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria
[b] Schunk Kohlenstofftechnik GmbH, Rodheimer Straße 59, D-35452 Heuchelheim, Germany

Received 5 August 2003; accepted 28 November 2003

Abstract

The resonant beam technique is a simple and effective method to measure the elastic moduli of rectangular or cylindrical isotropic beams. This paper proposes an extension of this method to highly anisotropic materials with orthotropic or higher symmetry via measurement of one or more samples, which are cut out of the material in different directions. The method is validated with computer simulated samples and measured unidirectionally carbon fibre reinforced carbon samples.

1. Introduction

The resonant beam technique (RBT) is widely used to determine simultaneously both Young's modulus and the shear modulus of isotropic materials from the resonance frequencies of flexural vibrations [1–3]. Timoshenko [4,5] established the basic differential equation of the theory. The solutions of this equation for different end configurations (i.e., supported, clamped or free at the respective end) can be found in a work by Huang [6].

The measurement and evaluation of the elastic moduli of anisotropic materials with the resonant beam technique is possible, but is subject to the following drawbacks. The evaluation procedure, which fits the calculated eigenfrequencies of a beam to the measured eigenfrequencies via Young's modulus E and the parameter E/kG, gives accurate results for Young's modulus, but more or less diverging or even wrong results for the shear modulus. The reason for this behaviour is that the shear correction factor k for isotropic materials is not applicable to anisotropic samples. The difference between the experimentally measured and the real value is insignificant if the shear modulus is small compared to Young's modulus, but can be arbitrarily large if the shear

*Corresponding author.
E-mail address: stephan.puchegger@ap.univie.ac.at (S. Puchegger).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.11.049

modulus is of the same order as Young's modulus. The second drawback is that not all elastic moduli can be calculated with this simple method, even if several sample beams, oriented along different directions in space, are measured.

This paper presents an extension of the RBT to the measurement of the elastic moduli of anisotropic materials. This extension is based on the recently derived anisotropic shear correction factor [7], which in turn is based on the work of Hutchinson [8] for isotropic beams. The paper will focus on the sample requisites, evaluation problems and possible approaches to solve these problems. The new method is tested experimentally with a simulated mono-crystalline copper like material and a unidirectional carbon fibre reinforced carbon material, which has been manufactured by Schunk Kohlenstofftechnik.

2. The transverse eigenfrequencies of an anisotropic rectangular beam

The basis for the RBT is the frequency equation (1) [6], which is the solution of the fourth order differential equation of the Timoshenko beam theory with the correct boundary conditions (i.e., in this case the free–free solution). The roots of the frequency equation are the resonant frequencies of the flexural vibrations, which can be determined numerically via a suitable method like bracketing. Since the derivation of the differential equation of the Timoshenko beam theory given in Ref. [9] does not depend on any specific symmetry of elasticity, it does hold true for anisotropic materials if the following conditions are met: (The references to equations in this paragraph refer to Cowper's publication [9]. The co-ordinate system is the same as in Fig. 1.) the nature of the elastic symmetry must be so, that it can be described with elastic moduli (isotropic, cubic, hexagonal, etc. but not trigonal, triclinic, etc.). Then the stress–strain relation $\varepsilon_{xz} = \sigma_{xz}/G$ given in Eq. (17) holds true with $G = G_{xy}$ and the isotropic Hooke's law in Eq. (20) can be replaced with its anisotropic counterpart. It must still be assumed, that the transverse stresses σ_{yy} and σ_{zz} are small in comparison with σ_{xx}, so that the integral in Eq. (22c) can be neglected. The

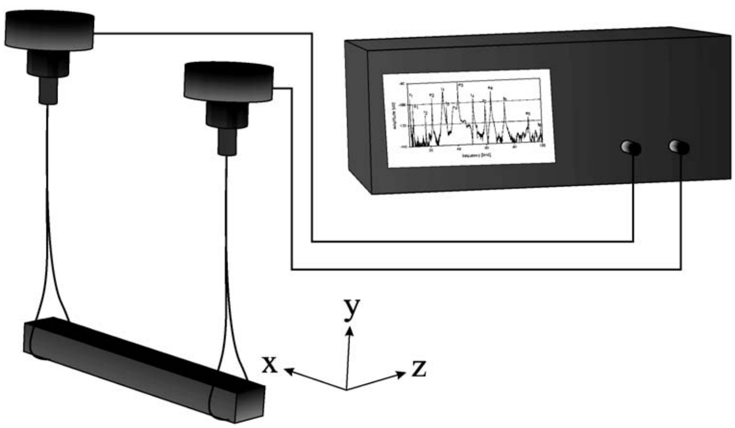

Fig. 1. The experimental set-up and the definition of the co-ordinate system. The network analyser stimulates the sample, which is suspended in two carbon fibre loops, via a piezo electric transducer in a given frequency range and records the answer signal via a piezo electric receiver.

shear stresses, the displacement and the harmonic function in the second assumption (Eq. (24)) must be replaced with their anisotropic counterparts (see Ref. [7] for a similar procedure).

The frequency equation $f(\omega)$ for an anisotropic beam with free–free boundary conditions can, if all these conditions hold true, be written as

$$f(\omega) = 2 - 2\cosh(b\alpha)\cos(b\beta) + \frac{b}{\sqrt{1 - b^2 r^2 s^2}} \times (b^2 r^2 (r^2 - s^2)^2 + (3r^2 - s^2))\sinh(b\alpha)\sin(b\beta) \tag{1}$$

with

$$\alpha = \frac{\sqrt{-(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + 4/b^2}}}{\sqrt{2}},$$

$$\beta = \frac{\sqrt{(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + 4/b^2}}}{\sqrt{2}},$$

$$b^2 = \frac{\rho A l^4 \omega^2}{E_x I_z}, \quad r^2 = \frac{I_z}{A l^2}, \quad s^2 = \frac{E_x I_z}{k A G_{xy} l^2},$$

$$A = wh, \quad I_z = \frac{wh^3}{12}, \quad I_y = \frac{hw^3}{12}$$

and l, w, h being the length, the width and the height of the specimen, I_y and I_z the second moment of area with respect to the y- and z-axis, ρ the density, k the shear correction factor [10] (see Appendix A) and A the cross-sectional area. E_x and G_{xy} are Young's modulus in the direction of the beam and the shear modulus in the plane of vibration. The compliance matrix has to be rotated in such a way, that the co-ordinate system of a sample coincides with the co-ordinate system of the moduli used in Eq. (1). Please consult Ref. [11] for further information on this topic.

3. The samples and the experimental set-up

3.1. The samples

If the sample material has a high elastic symmetry, then it is advantageous to produce samples with a non-square cross-section, since that will separate the eigenfrequencies of the transverse vibrations in the xy and in the xz plane. This maximizes the number of measured eigenfrequencies which are available for the fitting procedure. A lower elastic symmetry (e.g., orthotropic) can separate the eigenfrequencies even in the case of a square cross-section.

Due to the large sample size it is of utmost importance, that the sample material is homogeneous.

The first sample type is a simulated cubic crystal with a shear modulus, that is of the same order as Young's modulus of the material. The second sample type is a unidirectional carbon-fibre reinforced carbon, which features a very large Young's modulus in the direction of the reinforcement, a very low Young's modulus perpendicular to the direction of reinforcement and low shear moduli.

Cubic crystal: The first sample is a cubic crystal, whose eigenfrequencies are simulated via calculations based on resonant ultrasound spectroscopy (RUS) [12]. The crystal is similar to a cubic copper single crystal, but has a lower shear modulus. The elastic properties of the crystal are the following: a Young's modulus of 66.6 GPa, a shear modulus of 55 GPa and a Poisson ratio of 0.41. The sample has a length of 50 mm, a width of 7 mm, a height of 6 mm and a density of 8920 kg/m^3. The simulated specimen is oriented along the [1 0 0] direction.

Unidirectional carbon–carbon (C/C) composite: The second sample type are plates of unidirectional carbon–fibre reinforced carbon produced by Schunk Kohlenstofftechnik. The plates are approximately 300 mm long, 100 mm wide and 5 mm high. The direction of reinforcement is along the longest axis of the sample plate. The sample bars have been cut parallel and perpendicular to the direction of reinforcement and are approximately 100 mm long, 8 mm wide, 5 mm high and have a density of 1600 kg/m^3. The samples cut perpendicular to the reinforcement direction often contain cracks, since the material is very brittle. The samples chosen for the measurement are those with a minimum number of visible cracks.

3.2. The experimental set-up

Only a short summary of the equipment and experimental set-up will be given here, because it has been described in more detail in a previous publication [3]. The specimen is suspended on two carbon fibre loops, which are attached to two piezoelectric transducers, and excited to flexural vibrations via the loops (see Fig. 1). The resonant frequencies are identified by sweeping the frequency range with a network analyser. Even very weak resonances can be detected, since the noise level is at about -140 dBm[1].

4. The evaluation

The evaluation procedure is very similar to the evaluation procedures of the Resonant Beam Technique, Resonant Ultrasound Spectroscopy or the Electro magnetic Acoustic Resonance (EMAR) [13]. It is an inverse process that tries to acquire the elastic constants by fitting a set of calculated eigenfrequencies to the measured eigenfrequencies. The calculated eigenfrequencies of a sample are given by the roots of Eq. (1). The two problems with this process are that not all eigenfrequencies might be present in the measured spectrum or that surplus eigenfrequencies (e.g., from the transducers) could appear and that good starting values are required for the fitting procedure.

The problem of the surplus eigenfrequencies can be tackled by choosing different or specifically constructed transducers but the problem of missing eigenfrequencies, which have not been excited, is present in all three measurement procedures. This can make it harder to assign the calculated eigenfrequencies to their corresponding measured counterparts, if no previous knowledge about

[1] dBm: The symbol is an abbreviation for 'decibels relative to one milliwatt'.

the elastic constants is present. This method tries to simplify this problem via a two stage evaluation, but cannot solve it altogether.

4.1. The isotropic evaluation

The evaluation of the samples is a two stage process. The first stage is the evaluation of the anisotropic samples under the assumption, that the elastic symmetry is isotropic. This will give a shear modulus which is slightly too small if the shear modulus of the material is low and far too high a result if the modulus is of the same order or higher than the Young's modulus. The result is thus possibly but not necessarily a practical starting value for the second evaluation stage. An example of a material with a low shear modulus and high Young's modulus is carbon-fibre reinforced carbon.

Using the isotropic evaluation as a first stage has two advantages:

(1) Young's moduli calculated in the isotropic evaluation are close to the real value. This offers two possibilities for the second evaluation stage. The first is to keep Young's moduli constant to simplify the fitting procedure. The alternative is to use them as good starting values if Young's moduli are included in the fit.

    Care should therefore be taken, that the main sample axes are aligned with the directions of elastic symmetry. Examples are [1 0 0] for a cubic system, [1 0 0] and [0 0 1] for a transversely isotropic system oriented along the z-axis and [1 0 0], [0 1 0] and [0 0 1] for an orthotropic system. It is not always possible to produce beams oriented along all of the main symmetry directions of the sample material due to limitations of its size and shape.

(2) This isotropic two parameter fitting makes it far easier to identify the eigenfrequencies of the transverse vibrations. The mapping from the calculated to the measured eigenfrequencies gained in this first evaluation stage is used in the second evaluation stage, which in turn can concentrate on the multi-parameter fitting to the anisotropic model.

4.2. The anisotropic evaluation

The second stage of the evaluation is to fit the calculated eigenfrequencies to the measured eigenfrequencies for all samples simultaneously via the anisotropic elastic moduli. It is of crucial importance at this point of the evaluation that the sample is homogeneous, since the algorithm is based on the assumption that all the samples originate from the same homogeneous material and are likely to produce meaningless elastic moduli if this is not the case.

If Young's moduli in the directions of elastic symmetry have been measured and evaluated in the first stage, it is reasonable to keep them fixed in the second stage of the evaluation, since the higher-dimensional the parameter space is, the more problematic the fitting procedure becomes. A second fitting run can be done with no fixed parameters after approximate values for the not so well known constants have been found in a first anisotropic fitting run.

The condition that the compliance matrix must always be positive definite bounds the region of the valid set of elastic constants. The fitting algorithm must not be allowed to leave this region or one has to make certain that the final point of the procedure is within the allowed region. Depending on the fitting algorithm used, either the penalty function method or the projection method is more suitable [14]. The penalty function method is simpler to implement, but a penalty function suitable for one task is very likely not suitable for a different task.

5. Results and discussion

Cubic crystal: Fig. 2 shows the simulation and evaluation procedure of the cubic crystal. The results from the isotropic evaluation show clearly that Young's modulus is reproduced very accurately and that the shear modulus is too large. The shear modulus is nevertheless successfully used as a starting value for the second part of the fitting procedure. The results ($E = 66.6$ GPa, $G = 55.1$ GPa and $\nu = 0.38$) of the anisotropic fit reproduce the initial moduli ($E = 66.6$ GPa, $G = 55$ GPa and $\nu = 0.41$), that were used to calculate the eigenfrequency–spectrum via RUS, closely.

Unidirectional carbon–carbon composite: The measurement and evaluation procedure of the two transversely isotropic unidirectional carbon–carbon composite samples is shown in Fig. 3.

The isotropic evaluation of the two samples, which are oriented perpendicular to each other, gives Young's moduli that are very close to the final value. The beam perpendicular to the reinforcement has two different shear moduli. The shear modulus in the plane parallel to the reinforcement should be the same as the two shear moduli given by the beam parallel to the reinforcement, but has a larger numerical value ($\approx$12.4 GPa versus $\approx$7.5 GPa). The lower of the two values has been used in the second evaluation stage, since it is closer to the final value, which is due to the higher Young's modulus in the direction of the corresponding beam.

The starting value for ν_{xz} for the second part of the evaluation, is based on the fact, that a larger ν_{xz} would produce a compliance matrix that is not positive definite. The results of the second evaluation stage are thus $E_x = 5.2$ GPa, $E_z = 210$ GPa, $G_{xy} = 1.87$ GPa, $G_{xz} = 8.9$ GPa and

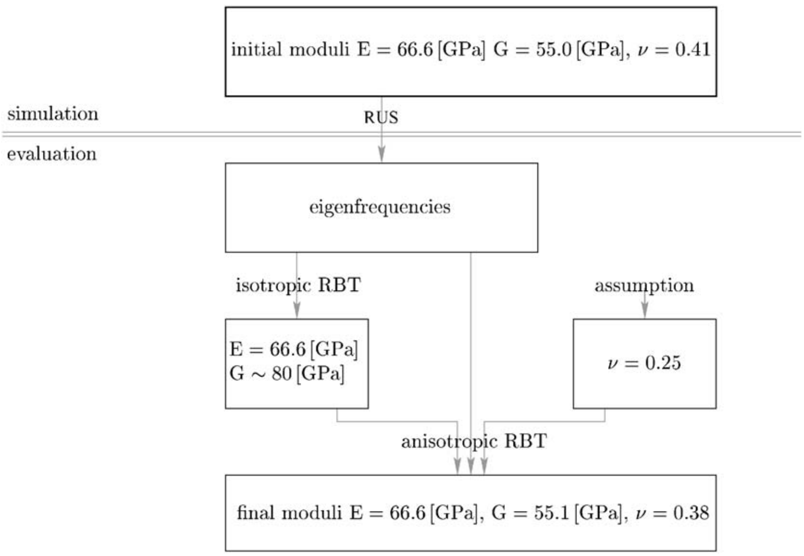

Fig. 2. The simulation and evaluation procedure of the cubic crystal.

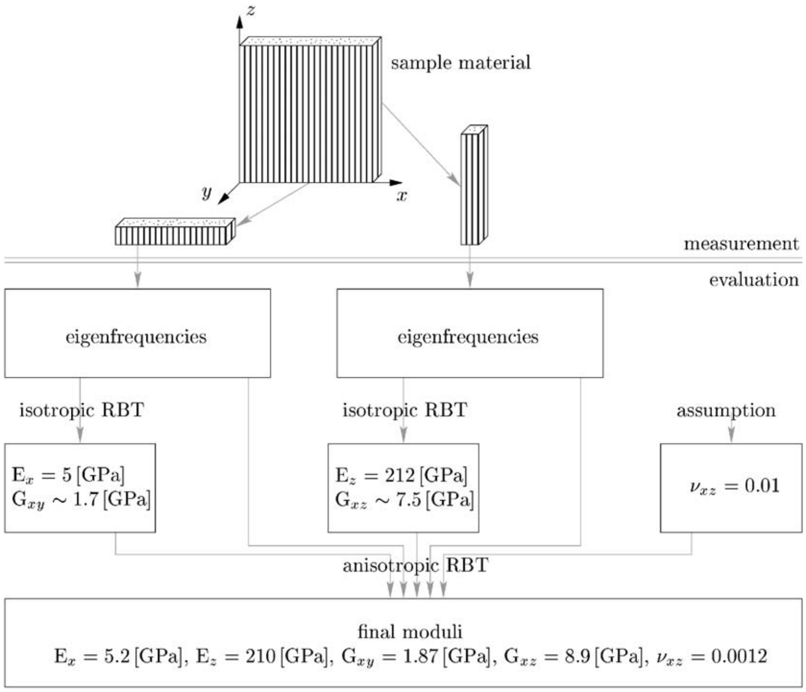

Fig. 3. The measurement and evaluation procedure of the transversly isotropic sample material.

$\nu_{xz} = 0.0012$ for this material. Young's modulus in the direction of the reinforcement is dominated by Young's modulus of the carbon fibres of around 300 GPa, which is due to the production process of the composite at about 2100°C. The measured Young's modulus follows from this value, since the volume fraction of the reinforcement is usually 60–70% and Young's modulus of the matrix material ($\approx E_x$) can be neglected. The shear modulus in a plane parallel to the reinforcement (G_{xz}) is expectedly larger than in a plane normal to the reinforcement (G_{xy}).

6. Final remarks

It can thus be concluded, that the method is theoretically and experimentally feasible, but very much depends on the material being homogeneous, because the samples need to be long beams. The evaluations of both the cubic sample simulated with RUS and the measured transversely isotropic samples reproduce the initial moduli respectively give meaningful and consistent moduli for the sample material.

Another possible application of the results of the isotropic evaluation is as starting values for the fitting procedure of a RUS sample. RUS is based on the stiffness matrix and not the compliance matrix, but it is possible to separate the fitting and the calculation part of RUS, so that the fitting is based on elastic moduli or the compliance matrix and the calculation of the eigenfrequencies is based on the corresponding stiffness matrix. This would again result in a reduction of the number of unknown values in the fitting procedure.

Acknowledgements

The support of the Austrian Research Funds (FWF), project No. P 15670, is gratefully acknowledged.

Appendix A. The shear correction factor

The shear correction factor k for an orthotropic rectangular beam [7] is

$$k = -\frac{E_x}{G_{xy}\left[\frac{A}{I_z^2} C_4 + \nu_{xy} - \frac{I_y}{I_z}\nu_{xz}\right]} \tag{A.1}$$

with C_4 defined as

$$\begin{aligned}
C_4 = hw^5 \Bigg(& -\frac{2}{5} - \frac{E_x G_{xy}}{5G_{xz}^2} + \frac{E_x}{5G_{xz}} + \frac{2G_{xy}}{5G_{xz}} - \frac{\nu_{xy}}{5} + \frac{2G_{xy}^2\nu_{xy}}{5G_{xz}^2} \\
& - \frac{G_{xy}\nu_{xy}}{5G_{xz}} - \frac{2G_{xy}^2\nu_{xy}}{5E_x G_{xz}} + \frac{2G_{xz}\nu_{xy}}{5E_x} + \frac{G_{xy}\nu_{xy}^2}{5E_x} - \frac{G_{xy}^3\nu_{xy}^2}{5E_x G_{xz}^2}\Bigg) \\
& + h^5 w \Bigg(-\frac{8E_x}{15G_{xy}} - \frac{2\nu_{xy}}{15} + \frac{8G_{xz}\nu_{xy}}{15G_{xy}} + \frac{G_{xy}\nu_{xy}^2}{5E_x} \\
& + \frac{2G_{xz}\nu_{xy}^2}{15E_x} - \frac{2\nu_{xz}}{3} - \frac{G_{xy}\nu_{xy}\nu_{xz}}{3E_x}\Bigg) \\
& + h^3 w^3 \Bigg(\frac{8}{9} + \frac{2E_x}{9G_{xy}} - \frac{2E_x}{9G_{xz}} - \frac{8G_{xz}}{9G_{xy}} - \frac{7\nu_{xy}}{9} - \frac{2G_{xy}\nu_{xy}}{3E_x} + \frac{2G_{xy}\nu_{xy}}{3G_{xz}} \\
& + \frac{4G_{xz}\nu_{xy}}{9E_x} - \frac{2G_{xz}\nu_{xy}}{9G_{xy}} + \frac{8G_{xz}^2\nu_{xy}}{9E_x G_{xy}} + \frac{2G_{xy}\nu_{xy}^2}{9E_x} - \frac{4G_{xy}^2\nu_{xy}^2}{9E_x G_{xz}} \\
& + \frac{5G_{xz}\nu_{xy}^2}{9E_x} + \frac{4\nu_{xz}}{3} - \frac{2G_{xy}\nu_{xz}}{3E_x} + \frac{G_{xy}\nu_{xz}}{3G_{xz}} - \frac{8G_{xz}\nu_{xz}}{9G_{xy}} \\
& - \frac{8G_{xy}\nu_{xy}\nu_{xz}}{9E_x} - \frac{G_{xy}^2\nu_{xy}\nu_{xz}}{3E_x G_{xz}} + \frac{8G_{xz}^2\nu_{xy}\nu_{xz}}{9E_x G_{xy}}\Bigg) \\
& + w^5 \sum_{n=1}^{\infty} \frac{(n\pi h - w\tanh(n\pi h/w))}{(n\pi)^5}\Bigg(-\frac{32G_{xy}\nu_{xz}}{E_x} + \frac{16G_{xy}\nu_{xz}}{G_{xz}} \\
& - 16\nu_{xz} + \frac{32G_{xz}\nu_{xz}}{E_x} + \frac{16G_{xy}\nu_{xy}\nu_{xz}}{E_x} - \frac{16G_{xy}^2\nu_{xy}\nu_{xz}}{E_x G_{xz}} + \frac{32G_{xz}\nu_{xz}^2}{E_x}\Bigg),
\end{aligned} \tag{A.2}$$

E_x, G_{xy}, G_{xz}, ν_{xy} and ν_{xz} are Young's modulus, the shear moduli and the Poisson ratios of the beam used in the theory. w and h are the width and the height of the sample. The co-ordinate system is defined in Fig. 1.

References

[1] F. Förster, Ein neues Messverfahren zur Bestimmung des Elastizitätsmoduls und der Dämpfung, *Zeitschrift für Metallkunde* 29 (1937) 109–115.

[2] A. Wanner, K. Kromp, Young's and shear moduli of laminated carbon/carbon composites by a resonant beam method, in: *Brittle Matrix Composites*, Vol. 2, Elsevier, Amsterdam, 1988, pp. 280–289.

[3] W. Lins, G. Kaindl, H. Peterlik, K. Kromp, A novel resonant beam technique to determine the elastic moduli in dependence on orientation and temperature up to 2000°C, *Review of Scientific Instruments* 90 (7) (1999) 3052–3058.

[4] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philosophical Magazine* 41 (1921) 744–746.

[5] S.P. Timoshenko, On the transverse vibrations of bars of uniform cross-section, *Philosophical Magazine* 43 (1922) 125–131.

[6] T.C. Huang, The effect of rotary inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end condition, *American Society of Mechanical Engineers, Journal of Applied Mechanics* 18 (1961) 579–584.

[7] S. Puchegger, D. Loidl, K. Kromp, H. Peterlik, Hutchinson's shear coefficient for anisotropic beams, *Journal of Sound and Vibration* 266 (2003) 207–216.

[8] J.R. Hutchinson, Shear coefficients for Timoshenko beam theory, *American Society of Mechanical Engineers, Journal of Applied Mechanics* 68 (2001) 87–92.

[9] G.R. Cowper, The shear coefficient in Timoshenko's beam theory, *American Society of Mechanical Engineers, Journal of Applied Mechanics* 33 (1966) 335–340.

[10] T. Kaneko, On Timoshenko's correction for shear in vibrating beams, *Journal of Physics D* 8 (1975) 1927–1936.

[11] S.G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Body*, MIR Publishers, Moscow, 1981.

[12] R.G. Leisure, F.A. Willis, Resonant ultrasound spectroscopy, *Journal of Physics: Condensed Matter* 9 (1997) 6001–6029.

[13] M. Hirao, H. Ogi, Electromagnetic acoustic resonance and materials characterization, *Ultrasonics* 35 (6) (1997) 413–421.

[14] J. Bard, *Nonlinear Parameter Estimation*, Academic Press, New York, 1974.