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Short Communication

Transcendental dynamic stability functions for beams carrying rigid bodies

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Abstract

This paper shows how the transcendental dynamic stability functions can be used to determine the natural frequencies of a beam system carrying a rigid body. The dynamic stability functions used here satisfy the partial differential equation governing the flexural motion of Euler–Bernoulli beams exactly. The boundary and continuity conditions are expressed in a convenient matrix form by assembling the dynamic stiffness coefficients. This yields a determinantal frequency equation.

1. Introduction

For several decades structural and civil engineers have been using transcendental static and dynamic stiffness matrices, also called static and dynamic stability functions, for the determination of buckling loads and natural frequencies (see for example Refs. [1–12]). In vibration analysis, this method is convenient for determining the exact natural frequencies of continuous systems. Its potential use in the analysis of a common mechanical system, namely a beam carrying a rigid body, seems to have been largely overlooked by researchers. A recent journal publication gives exact results for some of the natural frequencies of a two-part beam system carrying a rigid body by solving the equation of motion subject to given boundary and continuity conditions from first principles [13]. The purpose of this paper is to demonstrate how

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such problems may be solved more conveniently using the transcendental dynamic stability functions, which give the same, exact results.

Static stability functions [1,2] have been used extensively by civil engineers for calculating the critical loads of frameworks. These functions give the forces and moments at the ends of skeletal members due to unit displacements at the same or the other end. Similar functions for calculating natural frequencies, based on the method outlined by Veletsos and Newmark [3], were published in a table form by Armstrong [4]. Dynamic stability functions that included the effect of axial force (geometric stiffness) were presented by Mohsin and Sadek [5]. An experimental study on the vibration behaviour of axially loaded skeletal frames by the author while he was an undergraduate student at the University of Manchester [6], gave results that were in agreement with the theoretical results using the functions in [5]. This method was extended to solve a variety of continuous systems which include tapered beams, Timoshenko beam columns and folded plates [7–9] to name a few, by Williams and a number of other researchers. The method involves setting up a dynamic stiffness matrix that takes into account the stiffness and mass distribution in a structure. The coefficients of this matrix are transcendental functions of the frequency, and the natural frequencies may be calculated by searching the roots of the determinantal equation. This may be done by several trial and error search procedures, but to ensure that no modes are missed in a given frequency range, the Wittrick–Williams algorithm [10,11] should be used. In a comprehensive book entitled Dynamic stiffness and substructures [12], Leung has outlined several very useful computational procedures for efficient and exact dynamic analysis of different types of structures, using transcendental dynamic stiffness coefficients. The dynamic stiffness method is generally efficient and reliable, and gives a better understanding of the structural behaviour. Therefore, it is useful to derive the dynamic stiffness coefficients for Euler–Bernoulli beams carrying a rigid body.

2. Derivations

Dynamic stability functions are stiffness coefficients that give the dynamic actions (forces or moments) at the ends of a structural member (beam, bar, shaft, etc.) due to a prescribed unit displacement (translation or rotation at one of the ends of the structural member) taking into account the inertial effect of the member.

To illustrate the use of these functions in the vibration analysis of a beam system carrying a rigid body, first a particular system that has been reported in a recent publication [13] will be considered. Fig. 1 shows a system that has been used by the author as an example in his vibration

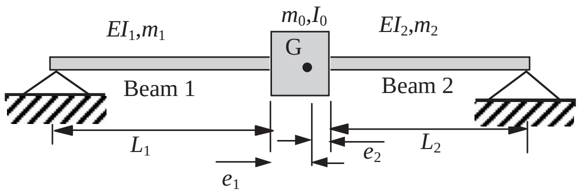

Fig. 1.

lectures. Results for the special case of $e_1 = e_2$ have been published by Kopmaz and Telli [13]. This consists of two Euler–Bernoulli beams that are pinned at one end and rigidly connected to a rigid body at the other end. The centroid of the rigid body is located at distances e_1 and e_2 from the tips of the left and right beams. The body has a mass m_0 and moment of inertia I_0 about its centroid. The beams on the left and right will be referred to as beam 1 and beam 2, respectively, and have the following properties: Intensities of mass (mass per unit length) m_1, m_2, flexural rigidities EI_1, EI_2, and lengths L_1, L_2. Non-dimensional frequency parameters for the beams λ_1 and λ_2 are given by

$$\lambda_i^4 = \frac{m_i \omega^2 L_i^4}{EI_i} \quad \text{for } i = 1, 2. \tag{1}$$

Let the centre of mass be given a translation Δ and rotation ϕ. This would result in translations of $\Delta + e_1\phi$ and $\Delta - e_2\phi$ of the left and right side beams at the point of attachment. They would both rotate by an angle ϕ. By Newton's third law of motion, the forces and moments acting on the rigid body are equal and opposite to those on the beams (see Figs. 2a–c). These actions may be related to the displacement and rotation using the dynamic stability functions. To do this, it is first instructive to consider the effect of end displacements in a single beam unit.

Consider an Euler–Bernoulli beam of length L, flexural rigidity EI and mass per unit length m, which is pinned at the left end and is given a rotation θ at the other end (see Fig. 3a). The resulting actions at its ends may be expressed in terms of a nominal elastic stiffness factor $k = EI/L$ and dynamic stability functions (given in [4,5] and defined later) S'', Q'' and q'' as shown in Fig. 3a. Similarly, the actions due to a translation (δ) of the right end are expressed in terms of dynamic stability functions, Q'', q'', T'' and t'' as shown in Fig. 3b [4,5]. Using the principle of superposition, the net actions due to the translation and rotation of the rigid body are given as follows.

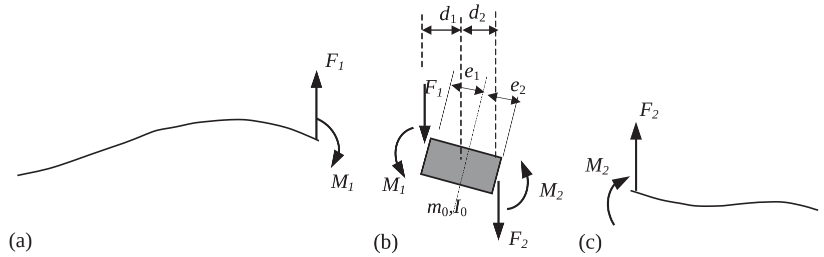

Fig. 2.

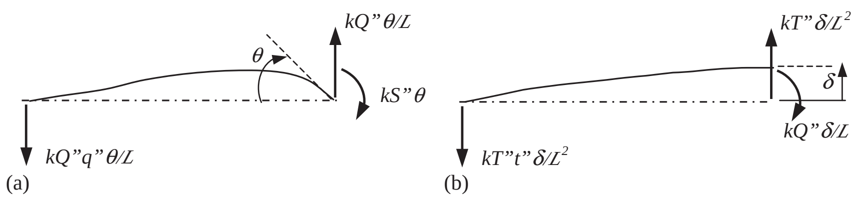

Fig. 3.

The net force acting at the right end of Beam 1 is

$$F_1 = \left(\frac{k_1 Q_1''}{L_1}\right)\phi + \left(\frac{k_1 T_1''}{L_1^2}\right)(\Delta + e_1\phi). \tag{2}$$

Similarly the force at the left end of Beam 2 is

$$F_2 = \left(-\frac{k_2 Q_2''}{L_2}\right)\phi + \left(\frac{k_2 T_2''}{L_2^2}\right)(\Delta - e_2\phi). \tag{3}$$

The moments in Beams 1 and 2 are

$$M_1 = \left(k_1 S_1''\right)\phi + \left(\frac{k_1 Q_1''}{L_1}\right)(\Delta + e_1\phi) \tag{4}$$

and

$$M_2 = (k_2 S_2'')\phi - \left(\frac{k_2 Q_2''}{L_2}\right)(\Delta - e_2\phi). \tag{5}$$

Consider the motion of the rigid body in Fig. 2b. Applying Newton's second law of motion in the translational and rotational sense gives the following equations:

$$F_1 + F_2 = -m_0\ddot{\Delta} = \omega^2 m_0 \Delta, \tag{6}$$

$$M_1 + M_2 + F_1 d_1 - F_2 d_2 = -I_0\ddot{\phi} = \omega^2 I_0 \phi.$$

Noting that for small rotations, $d_1 = e_1$ and $d_2 = e_2$, the above equation may be rewritten as

$$M_1 + M_2 + F_1 e_1 - F_2 e_2 - \omega^2 I_0 \phi = 0. \tag{7}$$

Substituting Eqs. (2)–(5) into Eqs.(6) and (7), and carrying out some simplifications including non-dimensionalisation of the parameters, results in the following matrix equation:

$$[\mathbf{D}]\{\boldsymbol{\delta}\} = \{\mathbf{0}\}, \tag{8}$$

where the elements of the dynamic stiffness matrix $[\mathbf{D}]$ are given by

$$\begin{aligned} D_{1,1} &= \gamma^3 T_1'' + \psi T_2'' - \alpha\gamma^3\lambda_1^4; \\ D_{1,2} &= D_{2,1} = \gamma^3 Q_1'' - \psi\gamma Q_2'' + \gamma^3\eta_1 T_1'' - \psi\gamma\eta_2 T_2''; \\ D_{2,2} &= \gamma^3(S_1'' + 2\eta_1 Q_1'') + \psi(\gamma^2 S_2'' + 2\gamma^2\eta_2 Q_2'') \\ &\quad + (\gamma^3 T_1''\eta_1^2 + \psi\gamma^2 T_2''\eta_2^2) - \gamma^3\beta\lambda_1^4. \end{aligned} \tag{9a–d}$$

Here,

$$\gamma = \frac{L_2}{L_1}; \quad \psi = \frac{EI_2}{EI_1}; \quad \alpha = \frac{m_0}{m_1 L_1}; \quad \beta = \frac{I_0}{m_1 L_1^3};$$
$$\eta_i = \frac{e_i}{L_1} \quad \text{for } i = 1, 2. \tag{10a–e}$$

The non-dimensional displacement vector $\{\boldsymbol{\delta}\}$ is defined by

$$\{\boldsymbol{\delta}\} = \begin{Bmatrix} (\Delta/L_1) \\ \phi \end{Bmatrix}. \tag{11}$$

The dynamic stiffness coefficients are [4]:

$$S_i'' = \frac{2\lambda_i \sinh(\lambda_i)\sin(\lambda_i)}{(\cosh(\lambda_i)\sin(\lambda_i) - \cos(\lambda_i)\sinh(\lambda_i))} \quad \text{for } i = 1, 2, \tag{12a}$$

$$Q_i'' = \frac{\lambda_i^2(\cosh(\lambda_i)\sin(\lambda_i) + \cos(\lambda_i)\sinh(\lambda_i))}{(\cosh(\lambda_i)\sin(\lambda_i) - \cos(\lambda_i)\sinh(\lambda_i))} \quad \text{for } i = 1, 2, \tag{12b}$$

$$T_i'' = \frac{2\lambda_i^3 \cosh(\lambda_i)\cos(\lambda_i)}{(\cosh(\lambda_i)\sin(\lambda_i) - \cos(\lambda_i)\sinh(\lambda_i))} \quad \text{for } i = 1, 2 \tag{12c}$$

and

$$\lambda_2 = \lambda_1 \gamma \sqrt[4]{\frac{\zeta}{\psi}}, \quad \text{where } \zeta = \frac{m_2}{m_1}. \tag{13a, b}$$

The natural frequency parameters are found by solving $|D| = 0$. As expected, the solution of this equation agrees with the results in [13] which can be generated using a computer program available on the worldwide web [14].

The effect of partial lateral and rotational restraints may also be included in the above formulation. For example, if the rigid body is partially restrained against lateral motion and rotation by springs of stiffness k, and k', respectively (see Fig. 4), then the body would be subject to an additional restraining force F_r and a restraining moment M_r as shown in Fig. 5. Once again, assuming that the flexural rotations are small, the distance between the spring force and the centre of mass is given by

$$d_r = e_r. \tag{14}$$

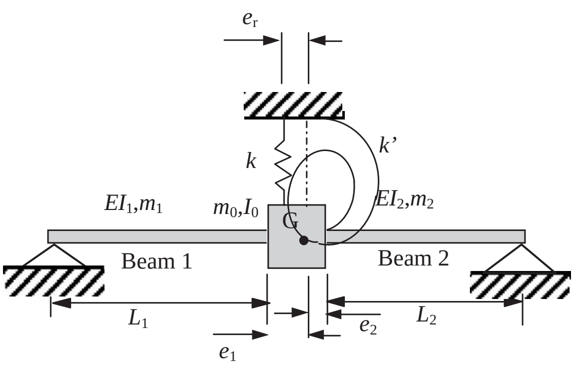

Fig. 4.

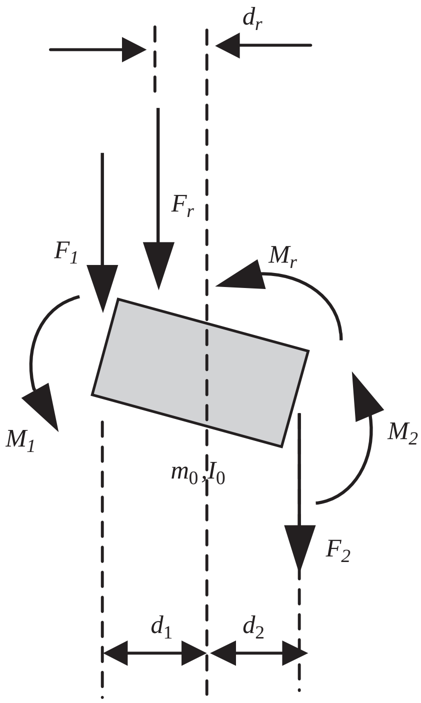

Fig. 5.

Applying Newton's second law of motion gives Eqs. (15) and (16) which replace Eqs. (6) and (7) for the structure with the unrestrained rigid body:

$$F_1 + F_2 + F_r = -m_0\ddot{\Delta} = \omega^2 m_0 \Delta, \tag{15}$$

$$M_1 + M_2 + M_r + F_1 e_1 - F_2 e_2 + F_r e_r = -I_0\ddot{\phi} = \omega^2 I_0 \phi. \tag{16}$$

Note that it is assumed here that the lateral spring is located at distance e_r to the left of the centre of the rigid body. The location of the rotational restraint has no effect on the moment.

The constitutive equations for the restraints are

$$F_r = k(\Delta + e_r\phi) \tag{17a}$$

and

$$M_r = k'\phi_r. \tag{17b}$$

The stiffness parameters and the location of the lateral restraint may be expressed in terms of non-dimensional parameters defined as follows:

$$\rho = k'\, EI_1/L_1; \quad \tau = k\, EI_1/L_1^3; \quad \varepsilon = e_r/L_1. \tag{18a–c}$$

Substituting Eqs. (17) and (18) into Eqs. (15) and (16) yields, after some algebraic manipulations, the following expressions for the elements of the dynamic stiffness matrix:

$$D_{1,1} = \gamma^3 T_1'' + \psi T_2'' + \tau\gamma^3 - \alpha\gamma^3\lambda_1^4, \tag{19a}$$

$$D_{1,2} = D_{2,1} = \gamma^3 Q_1'' - \psi\gamma Q_2'' + \gamma^3\eta_1 T_1'' - \psi\gamma\eta_2 T_2'' + \tau\varepsilon\gamma^3, \tag{19b, c}$$

$$\begin{aligned} D_{2,2} = {} & \gamma^3\left(S_1'' + 2\eta_1 Q_1''\right) + \psi\gamma^2\left(S_2'' + 2\eta_2 Q_2''\right) \\ & + \left(\gamma^3 T_1''\eta_1^2 + \psi\gamma^2 T_2''\eta_2^2\right) + \gamma^2(\rho + \tau\varepsilon^2) - \gamma^3\beta\lambda_1^4. \end{aligned} \tag{19d}$$

If one of the beams were clamped at the extreme end, instead of being simply supported, the corresponding equations for the stiffness coefficients may be obtained by replacing S'', Q'', T'' with S, Q and T, respectively. These functions are available in the literature [4–5].

Natural frequencies of other mechanical systems consisting of skeletal elements, rigid bodies and partial support restraints may also be determined conveniently in this manner. With the use of the W–W algorithm, the use of dynamic stability functions remains a convenient method for natural frequencies of many common continuous systems.

3. Conclusions

A simple procedure using transcendental dynamic stability functions to determine the natural frequencies of a beam system connected to a rigid body subject to elastic restraints has been presented. This method alleviates the need to derive the solution to the equation of motion for structural members for any given set of boundary or continuity conditions from first principles.

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