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Journal of Sound and Vibration 279 (2005) 723–737
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

On triply coupled vibrations of thin-walled beams with arbitrary cross-section

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Received 1 September 2003; accepted 13 November 2003

Abstract

The purpose of this paper is to analyze triply coupled vibrations of thin-walled beams with arbitrary open cross-section. Starting from the Vlasov's theory, the governing differential equations for coupled bending and torsional vibrations were performed using the principle of virtual displacements. In the case of a simply supported thin-walled beam, a closed-form solution for the natural frequencies of free harmonic vibrations was derived. The significance of neglecting cross-sectional warping and rotary inertia on the accuracy of results was analyzed. A recent paper on the same subject is discussed, with a critical review of it.

1. Introduction

Thin-walled beams of open cross-section are widely used as structural components within the fields of mechanical, civil, aeronautical engineering, etc., offering a high performance in terms of minimum weight for a given strength. The vibration characteristics of those elements are of fundamental importance in the design of thin-walled structures.

In the general case of arbitrary cross-section of thin-walled beams, lateral vibrations in two perpendicular directions are coupled with torsional vibrations, and the frequency equations of such elements should be considered simultaneously. The resulting coupling is referred to as triple coupling.

Many authors have investigated the free vibration characteristics of thin-walled beams [1–5], but only a few studies deal with triply coupled vibration. Friberg [6] proposed a numerical method to evaluate exactly a frequency-dependent stiffness matrix. Natural frequencies were

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.11.053

found using the algorithm of Wittrick and Williams. The effect of warping was not taken into account. Yaman [7] investigated the triply coupled vibrations of channels by a wave propagation approach. Various frequency response curves of coupled vibrations were presented for a variety of different end boundary conditions. In the analysis of coupled vibrations, the proposed method can serve as a convenient alternative to techniques such as Vlasov's theory and the finite element method. Tanaka and Bercin [8] have solved the governing differential equations for coupled bending and torsional vibrations in an exact sense. The natural frequency values were derived using the computer software Mathematica. But, the mentioned authors [8] have not included the product of inertia term EI_{xy} in non-principal co-ordinate system. This gap has been corrected in the paper by Arpaci and Bazdag [9].

Recently, Arpaci et al. [10] presented exact analytical method for predicting the undamped natural frequencies of a thin walled beam having no axis of symmetry. The influence of rotary inertia (the fourth mixed derivative term in the equation of torsional vibration has not been considered) and warping was included in triply coupled vibration analysis. One of the author's conclusion is that the rotary inertia effect may considerably alter the natural frequencies, the relative error associated with the neglecting of it, for some conditions, especially for free ends boundary conditions, reaching 170%.

The purpose of the present paper is to point out that the conclusion above is incorrect. At the simplest example of a beam with simply supported ends, it is shown that the rotary inertia effect, except in the cases of high frequencies of vibration, may be neglected.

2. Equations of motion

A straight thin-walled beam of an arbitrary open cross-section, whose length is L, is considered. As it is well known from Vlasov's theory the displacements u_*, v_* and w_* of an arbitrary point S_* of cross-section can be described by only four components, three translations u_P, v_P and w of

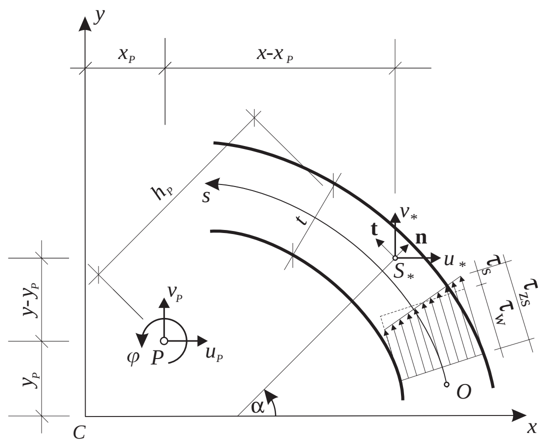


Fig. 1. Section geometry.

arbitrarily taken pole P and the cross-section rotation φ about the same pole, Fig. 1:

$$
\begin{aligned}
u_* &= u_P - \varphi(y - y_P),\\
v_* &= v_P + \varphi(x - x_P),\\
w_* &= w - u'_P x - v'_P y - \varphi' \omega_P,
\end{aligned}
\tag{1}
$$

where ω_P is the warping function with respect to pole P.

Component deformations different from zero are given by

$$
\begin{aligned}
\varepsilon_z &= w' - u''_P x - v''_P y - \varphi'' \omega_P,\\
\gamma_s &= 2\varphi' e,
\end{aligned}
\tag{2}
$$

where e is the distance of the observed point from the middle surface measured along the normal $\mathbf{n}$.

Reducing the normal stresses on the center of gravity and shear stresses on the pole P, for stress resultants the following expressions are obtained:

$$
\begin{aligned}
N &= \int\int_{\mathbf{F}} \sigma_z \, \mathrm{d}\mathbf{F},\\
M_x &= \int\int_{\mathbf{F}} \sigma_z y \, \mathrm{d}\mathbf{F},\\
M_y &= -\int\int_{\mathbf{F}} \sigma_z x \, \mathrm{d}\mathbf{F},\\
V_x &= -\int\int_{\mathbf{F}} \tau_{zs} \sin\alpha \; \mathrm{d}\mathbf{F},\\
V_y &= \int\int_{\mathbf{F}} \tau_{zs} \cos\alpha \; \mathrm{d}\mathbf{F},\\
T_P &= \int\int_{\mathbf{F}} \tau_{zs} h_p \, \mathrm{d}\mathbf{F},\\
T_s &= 2\int\int_{\mathbf{F}} \tau_s \, \mathrm{e} \, \mathrm{d}\mathbf{F},\\
M_{\omega_P} &= \int\int_{\mathbf{F}} \sigma_z \omega_P \, \mathrm{d}\mathbf{F}.
\end{aligned}
\tag{3}
$$

In Eqs. (3), N represents the axial force, M_x and M_y the bending moments with respect to the x and y-axis, V_x and V_y the shear forces in the x and y directions T_P the torsion moment, T_s the Saint Venant torque, M_{ω_P} the bimoment and $\mathbf{F}$ the area of the cross-section.

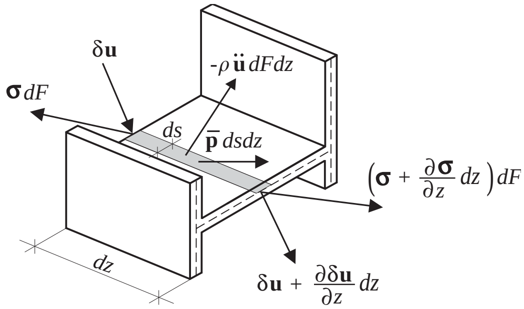


Fig. 2. Differential element of beam.

The equations of motion of thin-walled beam can be obtained using the principle of virtual displacements [11]. All vector and matrix quantities are defined with respect to the right-handed rectangular co-ordinate system (x, y, z). The z-axis is parallel with the longitudinal centroidal axis of the beam, while x and y are arbitrarily taken.

A small element between cross-sections $z_1 = z$ and $z_2 = z + \mathrm{d}z$ (Fig. 2) subjected to external loads $\bar{\mathbf{p}}(\bar{p}_x,\ \bar{p}_y,\ \bar{p}_z)$ per unit area of midplane is considered.

At any point on the cross-section z_1 acts as a stress vector

$$\boldsymbol{\sigma} = \tau_{zs}\mathbf{t} + \sigma_z\mathbf{i}_z = -\tau_{zs}\sin\alpha\mathbf{i}_x + \ \tau_{zs}\cos\alpha\mathbf{i}_y + \sigma_z\mathbf{i}_z. \tag{4}$$

The vector of virtual displacements $\delta\mathbf{u}$, which satisfies the necessary continuity conditions and displacement boundary conditions, may be adopted in the same form as a vector of real displacements

$$\begin{aligned}\delta\mathbf{u} &= \delta u_*\mathbf{i}_x + \delta v_*\mathbf{i}_y + \delta w_*\mathbf{i}_z \\ &= \ [\delta u_P - \delta\varphi(y - y_P)]\mathbf{i}_x + [\delta v_P + \delta\varphi(x - x_P)]\mathbf{i}_y \\ &\quad + \ (\delta w - \delta u'_P x - \delta v'_P y - \delta\varphi'\omega_P)\,\mathbf{i}_z.\end{aligned} \tag{5}$$

Virtual displacement parameters, for distinction from real displacements are marked with prefix δ, are arbitrary functions of co-ordinates and do not depend upon external loads.

The virtual work expression is

$$\delta W + \delta U = 0, \tag{6}$$

where δW is the virtual work of external load and inertia forces through virtual displacements $\delta\mathbf{u}$: and δU the virtual work of actual stresses $\boldsymbol{\sigma}$ realized through virtual strains $\delta\boldsymbol{\varepsilon} = [\delta\varepsilon_z\ \ \delta\gamma_T]$.

The virtual work of the external load and inertia forces per unit length of the element is

$$\delta W = \int\int_{\mathbf{F}} (\boldsymbol{\sigma}_{,z}\delta\mathbf{u} + \boldsymbol{\sigma}\delta\mathbf{u}_{,z})\,\mathrm{d}\mathbf{F} + \int_s \bar{\mathbf{p}}\delta\mathbf{u}\,\mathrm{d}s - \rho\int\int_{\mathbf{F}} \ddot{\mathbf{u}}\delta\mathbf{u}\,\mathrm{d}\mathbf{F}, \tag{7}$$

where ρ is the density (mass per unit volume), and $\ddot{\mathbf{u}}$ is the acceleration vector given by

$$\begin{aligned}\ddot{\mathbf{u}} &= \ddot{u}_*\mathbf{i}_x + \ddot{v}_*\mathbf{i}_y + \ddot{w}_*\mathbf{i}_z \\ &= [\ddot{u}_P - \ddot{\varphi}(y - y_P)]\mathbf{i}_x + [\ddot{v}_P + \ddot{\varphi}(x - x_P)]\mathbf{i}_y + (\ddot{w} - \ddot{u}'_P x - \ddot{v}'_P y - \ddot{\varphi}'\omega_P)\mathbf{i}_z.\end{aligned} \tag{8}$$

A dot denotes differentiation with respect to time τ.

Substituting Eqs. (4), (5) and (8) into Eq. (7), the following expression for δW is obtained:

$$\begin{aligned}\delta W = \int\int_{\mathbf{F}} \{ & -\tau'_{zs}\sin\alpha[\delta u_P - \delta\varphi(y - y_P)] \\ & + \tau'_{zs}\cos\alpha[\delta v_P - \delta\varphi(x - x_P)] \\ & + \sigma'_z(\delta w - \delta v'_P y - \delta u'_P x - \delta\varphi'\omega_P) \\ & - \tau_{zs}\sin\alpha[\delta u'_P - \delta\varphi'(y - y_P)] \\ & + \tau_{zs}\cos\alpha[\delta v'_P - \delta\varphi'(x - x_P)] \\ & + \sigma_z(\delta w' - \delta v''_P y - \delta u''_P x - \delta\varphi''\omega_P)\}\,\mathrm{d}\mathbf{F} \\ + \int_s \{ & \bar{p}_x[\delta u_P - \delta\varphi(y - y_P)] \\ & + \bar{p}_y[\delta v_P + \delta\varphi(x - x_P)] \\ & + \bar{p}_z(\delta w - \delta u'_P x - \delta v'_P y - \delta\varphi'\omega_P)\}\,\mathrm{d}s \\ - \rho\int\int_{\mathbf{F}} & (\delta u_*\ddot{u}_* + \delta v_*\ddot{v}_* + \delta w_*\ddot{w}_*)\,\mathrm{d}\mathbf{F}.\end{aligned} \tag{9}$$

The virtual work of the internal load due to the corresponding variation of deformation, per unit length of the element, is

$$\delta U = -\int\int_{\mathbf{F}} (\sigma_z\delta\varepsilon_z + \tau_s\delta\gamma_s)\,\mathrm{d}\mathbf{F}. \tag{10}$$

Using expressions (2) for virtual strains, where real displacement should be replaced by virtual displacement, one gets for δU

$$\delta U = -\int\int_{\mathbf{F}} [\sigma_z(\delta w' - \delta u''_P x - \delta v''_P y - \delta\varphi''\omega_P) + \tau_s 2\delta\varphi' e]\,\mathrm{d}\mathbf{F}. \tag{11}$$

By suitable rearrangement of Eqs. (9) and (11) in accordance with virtual displacement parameters, the principle of virtual work may be expressed as

$$\begin{aligned}
&\delta w\left\{\int\int_{\mathbf{F}} \sigma_z' \,\mathrm{d}\mathbf{F} - \rho\int\int_{\mathbf{F}} \ddot{w}_* \,\mathrm{d}\mathbf{F} + \int_s \bar{p}_z \,\mathrm{d}s\right\}\\
&+ \delta u_P\left\{-\int\int_{\mathbf{F}} \tau_{zs}' \sin\alpha \,\mathrm{d}\mathbf{F} - \rho\int\int_{\mathbf{F}} \ddot{u}_* \,\mathrm{d}\mathbf{F} + \int_s \bar{p}_x \,\mathrm{d}s\right\}\\
&+ \delta v_P\left\{\int\int_{\mathbf{F}} \tau_{zs}' \cos\alpha \,\mathrm{d}\mathbf{F} - \rho\int\int_{\mathbf{F}} \ddot{v}_* \,\mathrm{d}\mathbf{F} + \int_s \bar{p}_y \,\mathrm{d}s\right\}\\
&+ \delta\varphi\left\{\int\int_{\mathbf{F}} \tau_{zs}' h_P \,\mathrm{d}\mathbf{F} + \rho\int\int_{\mathbf{F}} [(y-y_P)\ddot{u}_* - (x-x_P)\ddot{v}_*]\,\mathrm{d}\mathbf{F}\right.\\
&\quad\left. + \int_s [\bar{p}_y(x-x_P) - \bar{p}_x(y-y_P)]\,\mathrm{d}s\right\}\\
&- \delta u_P'\left\{\int\int_{\mathbf{F}} (\sigma_z' x + \tau_{zs}\sin\alpha)\,\mathrm{d}\mathbf{F} - \rho\int\int_{\mathbf{F}} x\ddot{w}_* \,\mathrm{d}\mathbf{F} + \int_s \bar{p}_z x \,\mathrm{d}s\right\}\\
&- \delta v_P'\left\{\int\int_{\mathbf{F}} (\sigma_z' y - \tau_{zs}\cos\alpha)\,\mathrm{d}\mathbf{F} - \rho\int\int_{\mathbf{F}} y\ddot{w}_* \,\mathrm{d}\mathbf{F} + \int_s \bar{p}_z y \,\mathrm{d}s\right\}\\
&- \delta\varphi'\left\{\int\int_{\mathbf{F}} (\sigma_z'\omega_P - \tau_{zs}h_P + 2\tau_s e)\,\mathrm{d}\mathbf{F}\right.\\
&\quad\left. - \rho\int\int_{\mathbf{F}} \omega_P\ddot{w}_* \,\mathrm{d}\mathbf{F} + \int_s \bar{p}_z\omega_P \,\mathrm{d}s\right\} = 0.
\end{aligned} \tag{12}$$

To satisfy these equations identically for any virtual displacement parameter $\delta w_o, \delta u_P, \delta v_P, \ldots$, it is necessary that the expressions in the great brackets vanish. Now, using the expressions for stress resultants (3), one obtains

$$\begin{aligned}
&N' - \rho\int\int_{\mathbf{F}} \ddot{w}_* \,\mathrm{d}\mathbf{F} + p_z = 0,\\
&V_x' - \rho\int\int_{\mathbf{F}} \ddot{u}_* \,\mathrm{d}\mathbf{F} + p_x = 0,\\
&V_y' - \rho\int\int_{\mathbf{F}} \ddot{v}_* \,\mathrm{d}\mathbf{F} + p_y = 0,\\
&T_P' + \rho\int\int_{\mathbf{F}} [(y-y_P)\ddot{u}_* - (x-x_p)\ddot{v}_*]\,\mathrm{d}\mathbf{F} + m_P = 0,\\
&M_y' + V_x + \rho\int\int_{\mathbf{F}} x\ddot{w}_* \,\mathrm{d}\mathbf{F} + m_y = 0,\\
&M_x' - V_y - \rho\int\int_{\mathbf{F}} y\ddot{w}_* \,\mathrm{d}\mathbf{F} + m_x = 0,\\
&M_{\omega_P}' - T_P + T_s - \rho\int\int_{\mathbf{F}} \omega_P\ddot{w}_* \,\mathrm{d}\mathbf{F} + m_{\omega_P} = 0.
\end{aligned} \tag{13}$$

The forces V_x, V_y and T_P can be eliminated from Eq. (13) in order to obtain four equations

$$
\begin{aligned}
&N' - \rho \int\int_{\mathbf{F}} \ddot{w}_* \, \mathrm{d}\mathbf{F} + p_z = 0,\\
&M_y'' + \rho \int\int_{\mathbf{F}} x\ddot{w}_*' \, \mathrm{d}\mathbf{F} + \rho \int\int_{\mathbf{F}} \ddot{u}_* \, \mathrm{d}\mathbf{F} - p_x + m_y' = 0,\\
&M_x'' - \rho \int\int_{\mathbf{F}} y\ddot{w}_*' \, \mathrm{d}\mathbf{F} - \rho \int\int_{\mathbf{F}} \ddot{v}_* \, \mathrm{d}\mathbf{F} + p_y + m_x' = 0,\\
&M_{\omega_P}'' + T_s' - \rho \int\int_{\mathbf{F}} \omega_P \ddot{w}_*' \, \mathrm{d}\mathbf{F} + \rho \int\int_{\mathbf{F}} [(y - y_P)\ddot{u}_* - (x - x_p)\,\ddot{v}_*] \, \mathrm{d}\mathbf{F} + m_P + m_{\omega_P}' = 0. \qquad (14)
\end{aligned}
$$

The stress resultants can be expressed directly in terms of the displacements [12]. The equations are written in matrix form

$$
\begin{bmatrix} N \\ M_y \\ -M_x \\ -M_{\omega_P} \\ T_s \end{bmatrix} = E \begin{bmatrix} \mathbf{F} & -S_x & -S_y & -S_{\omega_P} & 0 \\ -S_x & I_{xx} & I_{xy} & I_{x\omega_P} & 0 \\ -S_y & I_{xy} & I_{yy} & I_{y\omega_P} & 0 \\ -S_{\omega_P} & I_{x\omega_P} & I_{y\omega_P} & I_{\omega_P\omega_P} & 0 \\ 0 & 0 & 0 & 0 & \frac{GK}{E} \end{bmatrix} \begin{bmatrix} w' \\ u_P'' \\ v_P'' \\ \varphi'' \\ \varphi' \end{bmatrix}. \qquad (15)
$$

The equations of motion can be obtained by substituting for the stress resultants from Eq. (15) into Eq. (14)

$$
\begin{aligned}
&E \left[\begin{array}{c|c|c|c} \mathbf{F} & -S_x & -S_y & -S_{\omega_P} \\ \hline -S_x & I_{xx} & I_{xy} & I_{x\omega_P} \\ \hline -S_y & I_{xy} & I_{yy} & I_{y\omega_P} \\ \hline -S_{\omega_P} & I_{x\omega_P} & I_{y\omega_P} & I_{\omega_P\omega_P} \end{array}\right] \begin{bmatrix} w''' \\ u_P'''' \\ v_P'''' \\ \varphi'''' \end{bmatrix} - GK \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w' \\ u_P'' \\ v_P'' \\ \varphi'' \end{bmatrix} \\
&- \rho \left[\begin{array}{c|c|c|c} \mathbf{F} & -S_x & -S_y & -S_{\omega_P} \\ \hline -S_x & I_{xx} & I_{xy} & I_{x\omega_P} \\ \hline -S_y & I_{xy} & I_{yy} & I_{y\omega_P} \\ \hline -S_{\omega_P} & I_{x\omega_P} & I_{y\omega_P} & I_{\omega_P\omega_P} \end{array}\right] \begin{bmatrix} \ddot{w}' \\ \ddot{u}_P'' \\ \ddot{v}_P'' \\ \ddot{\varphi}'' \end{bmatrix} \\
&+ \rho \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \mathbf{F} & 0 & y_P\mathbf{F} - S_y \\ \hline 0 & 0 & \mathbf{F} & -x_P\mathbf{F} + S_x \\ \hline 0 & y_P\mathbf{F} - S_y & -x_P\mathbf{F} + S_x & I_P \end{array}\right] \begin{bmatrix} \ddot{w} \\ \ddot{u}_P \\ \ddot{v}_P \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} -p_z \\ p_x - m_y' \\ p_y + m_x' \\ m_P + m_\omega' \end{bmatrix}. \qquad (16)
\end{aligned}
$$

To achieve the compact form the order of Eq. (14)–(1) is artificially raised by one.

In principal co-ordinates and by selecting the shear centre D as the pole, instead of some arbitrary point P as before, matrix equation (16) becomes

$$
\begin{aligned}
&E\begin{bmatrix} \mathbf{F} & 0 & 0 & 0 \\ 0 & I_{xx} & 0 & 0 \\ 0 & 0 & I_{yy} & 0 \\ 0 & 0 & 0 & I_{\omega_D\omega_D} \end{bmatrix}\begin{bmatrix} w''' \\ u_D'''' \\ v_D'''' \\ \varphi'''' \end{bmatrix} - GK\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} w' \\ u_D'' \\ v_D'' \\ \varphi'' \end{bmatrix} \\
&- \rho\begin{bmatrix} \mathbf{F} & 0 & 0 & 0 \\ 0 & I_{xx} & 0 & 0 \\ 0 & 0 & I_{yy} & 0 \\ 0 & 0 & 0 & I_{\omega_D\omega_D} \end{bmatrix}\begin{bmatrix} \ddot{w}' \\ \ddot{u}_D'' \\ \ddot{v}_D'' \\ \ddot{\varphi}'' \end{bmatrix} \\
&+ \rho\mathbf{F}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_D \\ 0 & 0 & 1 & -x_D \\ 0 & y_D & -x_D & \frac{I_D}{\mathbf{F}} \end{bmatrix}\begin{bmatrix} \ddot{w} \\ \ddot{u}_D \\ \ddot{v}_D \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} -p_z \\ p_x - m_y' \\ p_y + m_x' \\ m_D + m_\omega' \end{bmatrix}.
\end{aligned} \tag{17}
$$

The first equation in Eq. (17), describing axial vibration, is uncoupled from the rest of the system and may be analyzed independently.

The free harmonic transverse and torsional vibrations are defined by the coupled homogeneous Eqs. $(17)_{2,3,4}$. The solution may be expressed in the form

$$
\begin{bmatrix} u_D(z,t) \\ v_D(z,t) \\ \varphi(z,t) \end{bmatrix} = \begin{bmatrix} U(z) \\ V(z) \\ \Phi(z) \end{bmatrix} \sin pt, \tag{18}
$$

where p is the radian frequency and U, V and Φ are amplitudes of the transverse displacements and torsional rotation. Substituting Eq. (18) into homogeneous equations (17) yields

$$
\begin{aligned}
&E\begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{\omega_D\omega_D} \end{bmatrix}\begin{bmatrix} U'''' \\ V'''' \\ \Phi'''' \end{bmatrix} - GK\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} U'' \\ V'' \\ \Phi'' \end{bmatrix} \\
&+ \rho p^2\begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{\omega_D\omega_D} \end{bmatrix}\begin{bmatrix} U'' \\ V'' \\ \Phi'' \end{bmatrix} \\
&- \rho\mathbf{F}p^2\begin{bmatrix} 1 & 0 & y_D \\ 0 & 1 & -x_D \\ y_D & -x_D & \frac{I_D}{\mathbf{F}} \end{bmatrix}\begin{bmatrix} U \\ V \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned} \tag{19}
$$

In the case of a beam with simply supported ends (fork supports at each end which prevent rotation and can warp freely) the end conditions are

$$\begin{bmatrix} U \\ V \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} U'' \\ V'' \\ \Phi'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{20}$$

These requirements are satisfied by taking

$$\begin{bmatrix} U(z) \\ V(z) \\ \Phi(z) \end{bmatrix} = \begin{bmatrix} C_U \\ C_V \\ C_\Phi \end{bmatrix} \sin \lambda_n z, \tag{21}$$

where C_u, C_v and C_Φ are constants and $\lambda_n = n\pi/L$, $n = 1, 2, \ldots$.

Substituting Eq. (21) into Eq. (19) results in

$$\left(\lambda_n^4 E \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{\omega_D \omega_D} \end{bmatrix} + \lambda_n^2 GK \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda_n^2 \rho p^2 \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{\omega_D \omega_D} \end{bmatrix} - \rho \mathbf{F} p^2 \begin{bmatrix} 1 & 0 & y_D \\ 0 & 1 & -x_D \\ y_D & -x_D & \dfrac{I_D}{\mathbf{F}} \end{bmatrix} \right) \begin{bmatrix} C_U \\ C_V \\ C_\Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{22}$$

Setting the determinant of the above system equal to zero

$$\begin{vmatrix} \lambda_n^4 I_{xx} - (\lambda_n^2 I_{xx} + \mathbf{F}) p_* & 0 & -y_D \mathbf{F} p_* \\ 0 & \lambda_n^4 I_{yy} - (\lambda_n^2 I_{yy} + \mathbf{F}) p_* & x_D \mathbf{F} p_* \\ -y_D \mathbf{F} p_* & x_D \mathbf{F} p_* & \lambda_n^4 I_{\omega_D \omega_D} + \lambda_n^2 \dfrac{GK}{E} - (\lambda_n^2 I_{\omega_D \omega_D} + I_D) p_* \end{vmatrix} = 0, \tag{23}$$

where

$$p_* = \frac{\rho}{E} p^2 \tag{24}$$

yields the following algebraic frequency equation:

$$a p_*^3 + b p_*^2 + c p_* + d = 0 \tag{25}$$

with the coefficients

$$
\begin{aligned}
a &= -(\lambda_n^2 I_{xx} + \mathbf{F})(\lambda_n^2 I_{yy} + \mathbf{F})(\lambda_n^2 I_{\omega_D\omega_D} + I_D) + (\lambda_n^2 I_{xx} + \mathbf{F}) x_D^2 \mathbf{F}^2 + (\lambda_n^2 I_{yy} + \mathbf{F}) y_D^2 \mathbf{F}^2,\\
b &= (\lambda_n^2 I_{xx} + \mathbf{F})(\lambda_n^2 I_{yy} + \mathbf{F}) \left(\lambda_n^4 I_{\omega_D\omega_D} + \lambda_n^2 \frac{GK}{E} \right)\\
&\quad + \lambda_n^4 [(\lambda_n^2 I_{xx} + \mathbf{F}) I_{yy} + (\lambda_n^2 I_{yy} + \mathbf{F}) I_{xx}](\lambda_n^2 I_{\omega_D\omega_D} + I_D)\\
&\quad - \lambda_n^4 \mathbf{F}^2 (I_{yy} y_D^2 + I_{xx} x_D^2),\\
c &= -\lambda_n^8 I_{xx} I_{yy} (\lambda_n^2 I_{\omega_D\omega_D} + I_D)\\
&\quad - \lambda_n^4 [(\lambda_n^2 I_{xx} + \mathbf{F}) I_{yy} + (\lambda_n^2 I_{yy} + \mathbf{F}) I_{xx}] \left(\lambda_n^4 I_{\omega_D\omega_D} + \lambda_n^2 \frac{GK}{E} \right),\\
d &= \lambda_n^{12} I_{xx} I_{yy} I_{\omega_D\omega_D} + \lambda_n^{10} \frac{GK}{E} I_{xx} I_{yy}.
\end{aligned}
\tag{26}
$$

Thin-walled beams are elements that satisfy condition

$$
\frac{f}{L} < 0.1, \quad f = \text{characteristic dimension of cross-section} \tag{27}
$$

so that

$$
\left.
\begin{aligned}
&\lambda_n^2 I_{xx} \ll \mathbf{F}, \quad \text{i.e.,}\ n^2\pi^2 \int\int_{\mathbf{F}} \left(\frac{x}{L}\right)^2 \mathrm{d}\mathbf{F} \ll \int\int_{\mathbf{F}} \mathrm{d}\mathbf{F}\\
&\lambda_n^2 I_{yy} \ll \mathbf{F}, \quad \text{i.e.,}\ n^2\pi^2 \int\int_{\mathbf{F}} \left(\frac{y}{L}\right)^2 \mathrm{d}\mathbf{F} \ll \int\int_{\mathbf{F}} \mathrm{d}\mathbf{F}\\
&\lambda_n^2 I_{\omega_D\omega_D} \ll I_D, \quad \text{i.e.,}\ n^2\pi^2 \int\int_{\mathbf{F}} \left(\frac{\omega_D}{L}\right)^2 \mathrm{d}\mathbf{F} \ll \int\int_{\mathbf{F}} r_D^2\, \mathrm{d}\mathbf{F}
\end{aligned}
\right\}\ \text{for low modes of vibration.} \tag{28}
$$

From the relationships above, one can see that the fourth mixed derivative terms (effect of rotary inertia) in Eqs. (22), except in the cases of very high frequencies of vibration, may be neglected. Therefore, coefficients (26) may be written in simplified form

$$
\begin{aligned}
a &= -\mathbf{F}^2 I_D + \mathbf{F}^3 (x_D^2 + y_D^2),\\
b &= \mathbf{F}^2 \left(\lambda_n^4 I_{\omega_D\omega_D} + \lambda_n^2 \frac{GK}{E} \right) + \lambda_n^4 (I_{xx} + I_{yy})\, \mathbf{F} I_D - \lambda_n^4 \mathbf{F}^2 (I_{yy} y_D^2 + I_{xx} x_D^2),\\
c &= -\lambda_n^8 I_{xx} I_{yy} I_D - \lambda_n^4 \mathbf{F} (I_{xx} + I_{yy}) \left(\lambda_n^4 I_{\omega_D\omega_D} + \lambda_n^2 \frac{GK}{E} \right),\\
d &= \lambda_n^{12} I_{xx} I_{yy} I_{\omega_D\omega_D} + \lambda_n^{10} \frac{GK}{E} I_{xx} I_{yy}.
\end{aligned}
\tag{29}
$$

3. Numerical examples

The first three modes ($n = 1, 2, 3$) of natural frequencies of a simply supported thin-walled beam, for three kinds of cross-section, were determined. Three numerical values which characterize three different types of natural frequencies: predominantly torsional, predominantly flexural in x direction and predominantly flexural in y direction correspond to every mode. The natural frequencies are obtained by including/excluding the effect of rotary inertia and warping. The relative error is defined as $\varepsilon = |(p - \check{p})/p|100$, where p is the exact frequency and $\check{p}$ is the frequency obtained by excluding the rotary inertia/warping effect.

The geometrical and material properties of the beam are

$$E = 2.10 \times 10^8 \text{ kN/m}^2,$$
$$G = 8.07 \times 10^7 \text{ kN/m}^2,$$
$$\rho = \frac{78.50}{9.81} = 8.002 \text{ kN s}^2/\text{m}^4,$$
$$L = 10.0 \text{ m}.$$

The geometrical properties of the cross-sections of the beam, in the examples below, were calculated using the computer program given in Ref. [13].

3.1. Example 1

A thin-walled beam with unsymmetrical channel section, Fig. 3, is considered as the first example. The results are listed in Table 1.

3.2. Example 2

The cross-section characteristics and data used in the analysis are given in Fig. 4. The results are shown in Table 2.

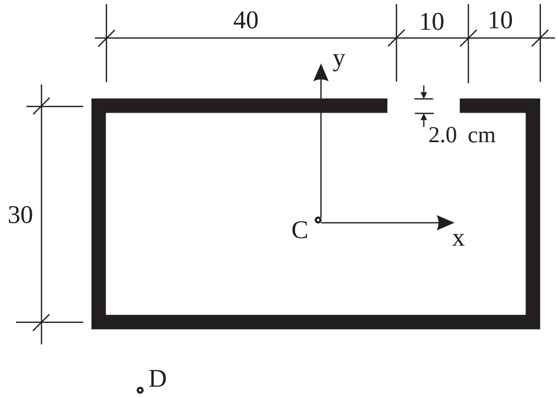
3.3. Example 3

The cross-section characteristics and other details are shown in Fig. 5. The results are given in Table 3.

It is evident from the results in Tables 1–3 that warping has a profound influence on the natural frequencies of the simply supported beam. Indeed, the errors are unacceptably large, which has been confirmed by many authors. But, the rotary inertia has relatively marginal effect on the natural frequencies, and can be disregarded, except in the rare cases involving very high frequencies of vibration (modes for $n = 3, 4, \ldots$). It may be supposed that it is reasonable enough to extend this conclusion to the beam with other boundary conditions, but this demands further examination.

4. Conclusions

Using the principal of virtual displacements the system of equations for triply coupled vibrations of thin-walled beams with open cross-sections is derived. In the example of simply supported beam it is shown that the fourth mixed derivative term in governing differential equations of motion may be neglected, except in the cases involving very high frequencies of vibration, which has been theoretically proved.



$$F = 0.034\ m^2$$
$$I_{yy} = 5.81146 x 10^{-4}\ m^4$$
$$I_{xx} = 1.75303 x 10^{-3}\ m^4$$
$$I_{\omega\omega} = 1.28016 x 10^{-4}\ m^6$$
$$I_D = 7.51444 x 10^{-3}\ m^4$$
$$K = 4.53333 x 10^{-6}\ m^4$$
$$x_D = -0.23526\ m$$
$$y_D = -0.31147\ m$$

Fig. 3. Cross section layout for Example 1.

Table 1
Natural frequencies (Hz) of beam studied as Example 1

Mode	Exact	Rotary inertia ignored	Error (%)	Warping ignored	Error (%)
$n = 1$	54.54	54.57	0.06	23.83	56.31
	81.83	81.93	0.12	74.18	9.35
	212.57	214.38	0.85	187.31	11.88
$n = 2$	212.73	213.19	0.22	48.67	77.12
	322.45	324.07	0.50	293.59	8.95
	819.01	846.58	3.37	725.02	11.48
$n = 3$	475.05	477.35	0.48	73.29	84.57
	719.57	727.55	1.11	656.24	8.80
	1768.75	1900.35	7.44	1578.54	10.75

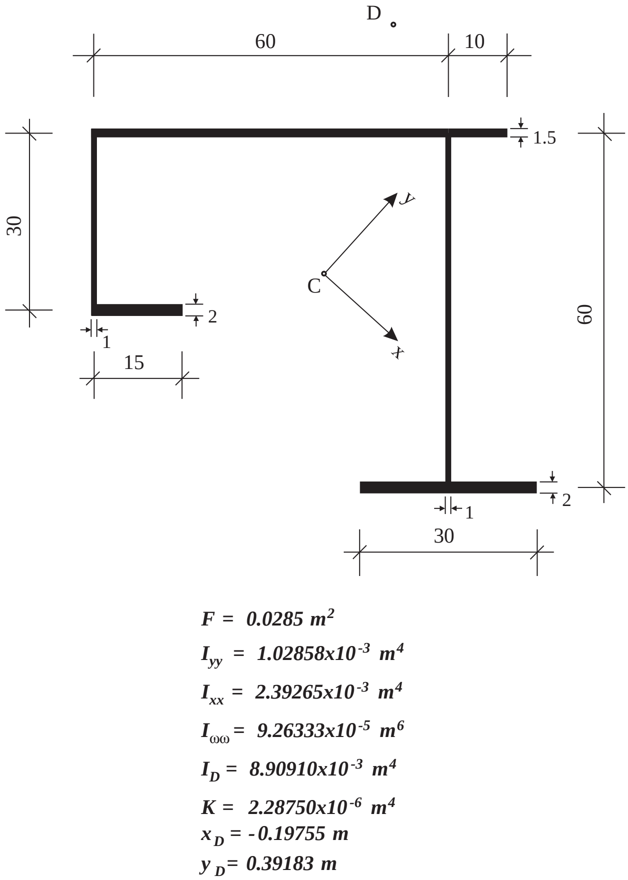
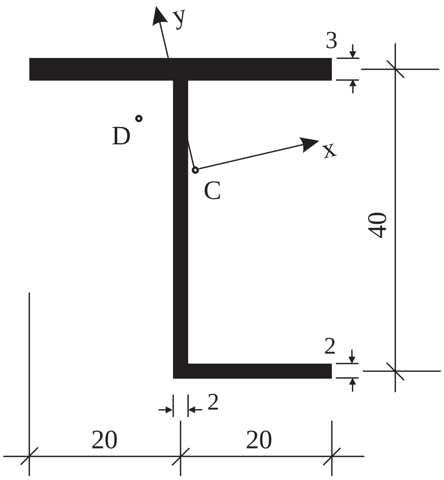


Fig. 4. Cross section layout for Example 2.

Table 2
Natural frequencies (Hz) of beam studied as Example 2

Mode	Exact	Rotary inertia ignored	Error (%)	Warping ignored	Error (%)
$n = 1$	51.05	51.07	0.04	15.91	68.83
	101.41	101.61	0.20	100.40	0.99
	233.71	236.20	1.06	224.57	3.91
$n = 2$	197.83	198.19	0.18	31.94	83.86
	402.91	406.10	0.79	399.05	0.96
	904.15	941.96	4.18	870.88	3.68
$n = 3$	441.48	443.32	0.42	47.93	89.14
	897.74	913.53	1.76	889.22	0.95
	1939.99	2118.33	9.19	1874.56	3.32



$F = 0.024\ m^2$

$I_{yy} = 6.65827x10^{-4}\ m^4$

$I_{xx} = 1.82139x10^{-4}\ m^4$

$I_{\omega\omega} = 3.94255x10^{-6}\ m^6$

$I_D = 1.09990x10^{-3}\ m^4$

$K = 5.20000x10^{-6}\ m^4$

$x_D = -0.054809\ m$

$y_D = 0.086555\ m$

Fig. 5. Cross section layout for Example 3.

Table 3
Natural frequencies (Hz) 0f beam studied as Example 3

Mode	Exact	Rotary inertia ignored	Error (%)	Warping ignored	Error (%)
$n = 1$	42.41	42.43	0.05	41.99	0.99
	74.91	74.96	0.07	71.78	4.18
	99.48	99.59	0.11	95.95	3.55
$n = 2$	150.87	150.99	0.08	126.13	16.40
	227.12	227.53	0.18	206.30	9.17
	357.85	360.00	0.60	353.63	1.18
$n = 3$	304.81	305.27	0.15	199.82	34.44
	474.64	476.70	0.43	441.39	7.01
	792.13	802.83	1.35	784.81	0.92

Appendix A

The values that determine geometrical properties of cross-section are given by

$$S_x = \int\int_{\mathbf{F}} x\,\mathrm{d}\mathbf{F}, \quad S_y = \int\int_{\mathbf{F}} x\,\mathrm{d}\mathbf{F}, \quad S_{\omega_P} = \int\int_{\mathbf{F}} \omega_P\,\mathrm{d}\mathbf{F},$$

$$I_{xx} = \int\int_{\mathbf{F}} x^2\,\mathrm{d}\mathbf{F}, \quad I_{yy} = \int\int_{\mathbf{F}} y^2\,\mathrm{d}\mathbf{F}, \quad I_{xy} = \int\int_{\mathbf{F}} xy\,\mathrm{d}\mathbf{F},$$

$$I_{x\omega_P} = \int\int_{\mathbf{F}} x\omega_P\,\mathrm{d}\mathbf{F}, \quad I_{y\omega_P} = \int\int_{\mathbf{F}} y\omega_P\,\mathrm{d}\mathbf{F}, \quad I_{\omega_P\omega_P} = \int\int_{\mathbf{F}} \omega_P^2\,\mathrm{d}\mathbf{F},$$

$$I_P = \int\int_{\mathbf{F}} [(x - x_P)^2 + (y - y_P)^2]\,\mathrm{d}\mathbf{F},$$

$$K = \frac{1}{3}\int_s t^3\,\mathrm{d}s.$$

Externally applied loads and moments per unit length of a beam are as follows:

$$p_x = \int_s \bar{p}_x\,\mathrm{d}s, \quad p_y = \int_s \bar{p}_y\,\mathrm{d}s, \quad p_z = \int_s \bar{p}_z,$$

$$m_x = \int_s \bar{p}_z y\,\mathrm{d}s, \quad m_y = -\int_s \bar{p}_z x\,\mathrm{d}s, \quad m_P = \int_s [\bar{p}_y(x - x_P) - \bar{p}_x(y - y_P)]\,\mathrm{d}s,$$

$$m_\omega = \int_s \bar{p}_z\omega_P\,\mathrm{d}s.$$

References

[1] S.P. Timoshenko, D.H. Young, W. Weaver, *Vibration Problems in Engineering*, Wiley, New York, 1974.

[2] C.F. Kollbrunner, N. Hajdin, S. Brčić, *Undamped Vibrations of Elastic Thin-Walled Beams of Deformable Cross Sections*, Institute for Engineering Research, Verlag Leemann, Zurich, 1978.

[3] E. Dokumaci, An exact solution for coupled bending and torsion vibrations of uniform beams having single cross-sectional symmetry, *Journal of Sound and Vibration* 119 (3) (1987) 443–449.

[4] R.E.D. Bishop, S.M. Cannon, S. Miao, On coupled bending and torsional vibration of uniform beams, *Journal of Sound and Vibration* 131 (3) (1989) 457–464.

[5] J.R. Banerjee, Coupled bending-torsional dynamic stiffness matrix for beam elements, *International Journal for Numerical Methods in Engineering* 28 (1989) 1283–1298.

[6] P.O. Friberg, Coupled vibrations of beams—an exact dynamic element stiffness matrix, *International Journal for Numerical Methods in Engineering* 19 (1983) 479–493.

[7] Y. Yaman, Vibrations of open-section channels: a coupled flexural and torsional wave analysis, *Journal of Sound and Vibration* 204 (1) (1997) 131–158.

[8] M. Tanaka, A.N. Bercin, Free vibration for uniform beams of nonsymmetrical cross section using Mathematica, *Computer and Structures* 71 (1999) 1–8.

[9] A. Arpaci, E. Bozdag, On free analysis of thin-walled beams with nonsymmetrical open cross-sections, *Computer and Structures* 80 (2002) 691–695.

[10] A. Arpaci, S.E. Bozdag, E. Sunbuloglu, Triply coupled of thin-walled open cross-section beams including rotary inertia effects, *Journal of Sound and Vibration* 260 (2003) 889–900.

[11] A. Prokić, *Matrix Method of Thin-Walled Structures*, Izgradnja, Beograd, 1999 (in Serbian).

[12] A. Gjesvik, *The Theory of Thin-Walled Bars*, Willey, New York, 1981.

[13] A. Prokić, Computer program for determination of geometrical properties of thin-walled beams with open-closed section, *Computer and Structures* 74 (2000) 705–715.