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# Low-rank damping modifications and defective systems

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**Abstract**

The structural modification of dynamical systems is an important issue in a wide range of applications, for example in vibration suppression or in active and passive control. It is well known that for a proportionally (or classically) damped system there always exists a real matrix of eigenvectors which simultaneously diagonalizes the three system matrices of inertia, damping and stiffness, even if the system possesses repeated eigenvalues. For general viscously damped systems the eigenvalue analysis must be performed in state space, and for systems with distinct eigenvalues the corresponding eigenvectors diagonalize the state space matrices. However, with general viscous damping, systems with repeated complex eigenvalues may have insufficient linearly independent complex eigenvectors. These systems are termed *defective*. In contrast to non-defective (or *simple*) systems, for defective systems only a Jordan decomposition exists. In this paper conditions on rank 1, rank 2 and higher rank modifications are derived which ensure that the modified system is simple. If none of the eigenvalues of the unmodified system is an eigenvalue of the modified system then every rank 1 modification that produces a pair of repeated complex eigenvalues leads to a defective system. Under the same assumptions there exist higher rank modifications which lead to simple systems. Either these modifications produce a proportionally damped system, or the restrictions on these modifications are rather strict which suggests that in practical cases every rank 2 or higher modification that produces repeated pairs of complex eigenvalues will lead to a defective system. The findings are demonstrated by simulated examples.


## 1. Introduction

The investigation of the structural modification of dynamical systems has applications in areas as such as model updating, system design and system control. Of particular interest are low-rank

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modifications [1,2]. The issue of defective systems has been neglected and is rarely mentioned in the literature. Mottershead [3] has found that defectiveness can occur in cross receptances using a rank 1 modification for zero-placements of undamped systems. Veselić [4] has shown that if a system with a rank 1 damping matrix has a pair of repeated complex eigenvalues, that are not eigenvalues of the original system, it is necessarily defective. This result has recently been extended to proportionally damped systems [5], where it is demonstrated that a unit rank modification to a classical damping matrix can produce a defective system. Based on this method the scope of this paper is to extend these findings to higher rank modifications of the damping matrix.

Defective systems are relatively rare in practice, since exactly repeated eigenvalues are difficult to obtain. However, systems with closely spaced eigenvalues are very common, and the accuracy with which the eigenvalues may be calculated is significantly diminished if the corresponding system with repeated roots is defective. Friswell and Champneys [6] demonstrated this using the pseudospectra for simple examples, and motivated the investigation of defective systems.

Before discussing the system modification some notation and preliminary results will be highlighted. Consider the general characteristic polynomial $q(s) := \det(\mathbf{A} + s\mathbf{B})$ of a matrix pencil $(\mathbf{A}, \mathbf{B})$. A second order system defined by the mass, damping and stiffness matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, may be written in state space form. One possibility for $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}. \tag{1}$$

Alternatively the characteristic polynomial may be obtained directly as $q(s) := \det(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})$. If $\lambda$ is a root of $q$ then the smallest number $n_A = n_A(\lambda) \in \mathbb{N}$ for which

$$\left.\frac{\partial^{n_A} q(s)}{\partial s^{n_A}}\right|_{s=\lambda} \neq 0 \tag{2}$$

is called the *algebraic multiplicity* of $\lambda$, and the natural number

$$n_G(\lambda) = n_G := n - rank(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}) \tag{3}$$

is called the *geometric* multiplicity of $\lambda$. Note, that in general $n_G \leqslant n_A$. A root $\lambda$ is called *defective* if $n_G < n_A$. The second order system $(\mathbf{M}, \mathbf{C}, \mathbf{K})$ is called *defective* if $q$ has a defective root; otherwise the system is termed *simple* (or not defective) (see Ref. [7, p. 17]). Since the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real, the complex eigenvalues will occur in conjugate pairs. Thus if a complex eigenvalue is repeated, then so will the corresponding complex conjugate eigenvalue, and these will be referred to as a pair of repeated complex eigenvalues.

The following analysis relies on the formula for the adjugate of a matrix. This expression is given here for the general case.

**Theorem 1.** *Suppose* $\mathbf{D}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ *represents the* $n \times n$ *matrix pencil of the original system. Providing* $\lambda$ *is not an eigenvalue of the original system, so that* $\det(\mathbf{D}) \neq 0$, *the adjugate of* $\mathbf{S}(\lambda) = [\mathbf{D}(\lambda) + \lambda\mathbf{Y}\mathbf{Y}^\top]$, *where* $\mathbf{Y}$ *is an* $n \times m$ *matrix, is given by*

$$\mathbf{S}^{ad} = \det(\mathbf{D})[\mathbf{D}^{-1}\det(\mathbf{I}_m + \lambda\mathbf{Y}^\top\mathbf{D}^{-1}\mathbf{Y}) - \lambda\mathbf{D}^{-1}\mathbf{Y}[\mathbf{I}_m + \lambda\mathbf{Y}^\top\mathbf{D}^{-1}\mathbf{Y}]^{ad}\mathbf{Y}^\top\mathbf{D}^{-1}], \tag{4}$$

*where the explicit dependence of* $\mathbf{S}$ *and* $\mathbf{D}$ *on* $\lambda$ *has been dropped.*

**Proof.** For non-singular **S**, the Sherman–Morrison–Woodbury [8] formula gives

$$\mathbf{S}^{-1} = [\mathbf{D} + \lambda \mathbf{Y}\mathbf{Y}^\top]^{-1} = \mathbf{D}^{-1} - \lambda \mathbf{D}^{-1}\mathbf{Y}[\mathbf{I}_m + \lambda \mathbf{Y}^\top \mathbf{D}^{-1}\mathbf{Y}]^{-1}\mathbf{Y}^\top \mathbf{D}^{-1}. \tag{5}$$

From Ref. [9, pp. 45–46] it is also known that

$$\det(\mathbf{S}) = \det(\mathbf{D})\det(\mathbf{I}_n + \lambda \mathbf{D}^{-1}\mathbf{Y}\mathbf{Y}^\top) = \det(\mathbf{D})\det(\mathbf{I}_m + \lambda \mathbf{Y}^\top \mathbf{D}^{-1}\mathbf{Y}). \tag{6}$$

Combining these two formulae by using,

$$\mathbf{S}^{ad} = \det(\mathbf{S})\mathbf{S}^{-1} \tag{7}$$

for non-singular **S**, gives Eq. (4). Some care needs to be exercised when $\lambda$ is an eigenvalue of the modified system, so that $\det(\mathbf{S}) = 0$ and thus **S** is singular. However, it is readily apparent that the expression for the adjugate on the right side of Eq. (4) is continuous except when $\lambda$ is an eigenvalue of the original system and thus **D** is singular. Since it is assumed that the repeated eigenvalue of the modified system is not an eigenvalue of the original system, Eq. (4) is continuous and thus Eq. (4) holds at these values of $\lambda$. □

## 2. Unit-rank modifications

Prells and Friswell [5] showed that a unit rank modification of a proportionally damped system that produces pairs of repeated complex eigenvalues always leads to a defective system. In this case $m = 1$ and **Y** becomes the vector **y**. If $\lambda$ is an eigenvalue of the modified system then $\det(\mathbf{S}) = 0$. However, if $\lambda$ is not an eigenvalue of original system, then **D** is non-singular and Eq. (6) implies that

$$\det(1 + \lambda \mathbf{y}^\top \mathbf{D}^{-1}\mathbf{y}) = 0 \tag{8}$$

and thus Eq. (4) implies that

$$\mathbf{S}^{ad} = -\lambda \det(\mathbf{D})\mathbf{D}^{-1}\mathbf{y}\mathbf{y}^\top \mathbf{D}^{-1}. \tag{9}$$

Since $\lambda$ is not an eigenvalue of the original system and since $\mathbf{y} \neq \mathbf{0}$ the adjugate of **S** is always finite and never zero. Thus it is now possible to prove the following theorem.

**Theorem 2.** *If the damping matrix is modified by the addition of a rank* 1 *component such that the resulting system has a repeated eigenvalue not present in the original system, then the modified system is defective.*

**Proof.** Since **y** is a non-zero vector, Eq. (9) shows that the rank of the adjugate of **S** is 1. If $\lambda$ is a repeated eigenvalue of the modified system then it is known that

$$\mathbf{S}\mathbf{S}^{ad} = \det(\mathbf{S})\mathbf{I}_n = \mathbf{0} \tag{10}$$

and hence $rank(\mathbf{S}^{ad}) = 1$ implies that $rank(\mathbf{S}) = n - 1$. Thus there is at most one linearly independent eigenvector associated with the eigenvalue $\lambda$, and the unit-rank modification leads to a defective system if it produces a repeated eigenvalue. □

## 3. Rank 2 modifications

The procedure adopted in this section is as follows. For a full-rank matrix $\mathbf{Y} \in \mathbb{R}^{n\times 2}$, the determinant of $\mathbf{S}$ and the adjugate $\mathbf{S}^{ad}$ of $\mathbf{S}$ may be evaluated. Under the assumption $\det(\mathbf{S}(\lambda)) = 0$ but $\det(\mathbf{D}(\lambda)) \neq 0$ it may then be shown $\mathbf{S}^{ad}(\lambda) \neq 0$ if $\mathbf{Y}$ satisfies certain conditions. These conditions define a class of modifications $\mathbf{Y}$ which lead to defective systems if repeated eigenvalues are produced.

In a similar way to the rank 1 modification case, from Eq. (6) since $\det(\mathbf{S}) = 0$ and $\det(\mathbf{D}) \neq 0$,

$$\det(\mathbf{I}_2 + \lambda \mathbf{Y}^\top \mathbf{D}^{-1} \mathbf{Y}) = 0. \tag{11}$$

Thus Eq. (4) implies that

$$\mathbf{S}^{ad} = -\lambda \det(\mathbf{D}) \mathbf{D}^{-1} \mathbf{Y} \mathbf{Q}^{ad} \mathbf{Y}^\top \mathbf{D}^{-1}, \tag{12}$$

where

$$\mathbf{Q} = \mathbf{I}_2 + \lambda \mathbf{Y}^\top \mathbf{D}^{-1} \mathbf{Y}. \tag{13}$$

Using Eq. (10), a system with a repeated eigenvalue is simple if it has a full set of eigenvalues, which is given by the condition $rank(\mathbf{S}) = n - 2$, or $rank(\mathbf{S}^{ad}) \geqslant 2$. Since $rank(\mathbf{Y}) = 2$ and $\mathbf{Q}$ is a $2 \times 2$ matrix, the system is simple if $rank(\mathbf{Q}^{ad}) = 2$. However from Eq. (11), $\det(\mathbf{Q}) = 0$ and this implies that the adjugate can only be zero if $\mathbf{Q} = \mathbf{0}$. Thus the following theorem has been proved.

**Theorem 3.** *If the damping matrix is modified by the addition of a rank* 2 *component such that the resulting system has a repeated eigenvalue not present in the original system, then the modified system is defective unless the modification satisfies the condition*

$$\lambda \mathbf{Y}^\top \mathbf{D}^{-1} \mathbf{Y} = -\mathbf{I}_2. \tag{14}$$

Of course since the damping matrix is real, $\mathbf{Y} \in \mathbb{R}^{n\times 2}$ and Eq. (14) is equivalent to

$$\mathbf{Y}^\top \operatorname{Re}\{\lambda \mathbf{D}^{-1}\} \mathbf{Y} = -\mathbf{I}_2, \tag{15}$$

$$\mathbf{Y}^\top \operatorname{Im}\{\lambda \mathbf{D}^{-1}\} \mathbf{Y} = \mathbf{0}. \tag{16}$$

Two questions arise:

(1) Does there exist a $\mathbf{Y} \in \mathbb{R}^{n\times 2}$ satisfying Eqs. (15) and (16)?
(2) If there exists such a modification $\mathbf{Y}$, does it correspond to a meaningful and/or realisable damper configuration?

It is clear that modifications do exist that can satisfy Eqs. (15) and (16), however this is a low-dimensional subspace of all modifications that produce repeated eigenvalues. Thus unless the modification is carefully designed, any modified system that has pairs of repeated complex eigenvalues is very likely to be defective. As a consequence the questions given above are somewhat academic, although computing modifications that lead to simple systems do highlight the fact that such modifications are relatively rare. The particular case of modifications to a proportionally damped system is considered in a later section.

## 4. Higher rank modifications

For modifications higher than rank 2 a similar procedure may be adopted to obtain the conditions required for a system to be simple. However a number of different cases arise depending on algebraic and geometric multiplicities of the repeated eigenvalue. Consider the case where $m = 3$, that is when the modification is of rank 3. Define,

$$\mathbf{Q} = \mathbf{I}_3 + \lambda \mathbf{Y}^\top \mathbf{D}^{-1} \mathbf{Y}. \tag{17}$$

In a similar manner to the rank 2 modification, from Eq. (6), $\det(\mathbf{Q}) = 0$. Assume that the repeated eigenvalue has an algebraic multiplicity of 2. The system is simple if the geometric multiplicity is also 2, which occurs if $rank(\mathbf{S}) = n - 2$, or $rank(\mathbf{S}^{ad}) \geqslant 2$. From Eq. (12), this implies that $rank(\mathbf{Q}^{ad}) \geqslant 2$ and hence the condition for a simple system is that $rank(\mathbf{Q}) \leqslant 1$. This condition may be written as

$$\lambda \mathbf{Y}^\top \mathbf{D}^{-1} \mathbf{Y} = \mathbf{a}\mathbf{a}^\mathrm{T} - \mathbf{I}_3, \tag{18}$$

for some vector $\mathbf{a} \in \mathbb{R}^3$. If the algebraic multiplicity is 3, then by a similar argument the condition for a simple system is that $\mathbf{Q} = \mathbf{0}$. Of course in both cases, the modified system must also yield pairs of repeated complex eigenvalues, that were not eigenvalues of the unmodified system. The conditions required to produce a simple system with repeated eigenvalues for these higher rank modifications is rather complicated, and will not be considered further in this paper.

## 5. Low-rank modifications of a proportionally damped system

The conditions derived in the previous sections have made no assumptions about the damping in the original system. This section will restrict attention to cases where the original system is proportionally damped. Finding simple systems with repeated pairs of complex eigenvalues is difficult, and this section will give a method to generate simple systems with repeated eigenvalues. It should be emphasized that the proposed procedure will not generate all possible solutions.

Caughey and O'Kelly [10] called the system $\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K}$, where $\mathbf{M}$ is non-singular, *proportionally* (or *classically*) damped if scalars $c_i,\ i = 0, \ldots, n-1$, exist such that

$$\mathbf{C} = \sum_{i=0}^{n-1} c_i \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^i. \tag{19}$$

If $\mathbf{M}\in\mathbb{R}^{n\times n}$ is positive definite and $\mathbf{C},\mathbf{K}$ positive semi-definite then for proportionally damped systems the matrix $\mathbf{X}_0\in\mathbb{R}^{n\times n}$ of eigenvectors of the undamped system will also diagonalize the damping matrix $\mathbf{C}$. Such damping matrices are called *proportional* or *classical*.

Denote the modally transformed unmodified system by

$$\mathbf{D}(\lambda_0) := \lambda_0^2\mathbf{I}_n + 2\lambda_0\mathbf{\Gamma}_0\mathbf{\Omega}_0 + \mathbf{\Omega}_0^2, \tag{20}$$

where $\mathbf{\Gamma}_0 = \mathrm{diag}[\zeta_{0i}]$ and $\mathbf{\Omega}_0 = \mathrm{diag}[\omega_{0i}]$ are diagonal matrices of the damping ratios and natural frequencies respectively. Hence, $\mathbf{X}_0^\top\mathbf{M}\mathbf{X}_0 = \mathbf{I}_n$, $\mathbf{X}_0^\top\mathbf{C}\mathbf{X}_0 = \mathbf{\Gamma}_0\mathbf{\Omega}_0$ and $\mathbf{X}_0^\top\mathbf{K}\mathbf{X}_0 = \mathbf{\Omega}_0^2$, where $\mathbf{X}_0\in\mathbb{R}^{n\times n}$ is the matrix of mass normalised eigenvectors of the proportionally damped system. The modified system is denoted by $\mathbf{S}(\lambda) := \mathbf{D}(\lambda) + \lambda\mathbf{Y}\mathbf{Y}^\top$ where $\mathbf{Y}\in\mathbb{R}^{n\times m}, m\in\{1,2\}$.

Without loss of generality, the following discussion will assume that the unmodified system has been transformed to modal co-ordinates. This is possible because Eq. (14) is invariant with respect to the modal transformation. Thus,

$$\mathbf{Y}^\top\mathbf{D}^{-1}\mathbf{Y} = \mathbf{Y}^\top\mathbf{X}_0^{-1}(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})^{-1}\mathbf{X}_0^{-\top}\mathbf{Y} \tag{21}$$

and $\mathbf{X}_0^{-\top}\mathbf{Y}$ is the corresponding rank 2 modification of $\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$.

This section will concentrate on the existence of modifications to produce a simple system, rather than the physical realization of these modifications. Since $\mathbf{D}$ is diagonal, $\mathbf{D}(\lambda) = \mathrm{diag}[p_i(\lambda)]$, where $p_i(\lambda) = \lambda^2 + 2\zeta_{0i}\omega_{0i}\lambda + \omega_{0i}^2$. By definition $\lambda\mathbf{D}^{-1} = \mathrm{diag}[\lambda/p_i(\lambda)]$ and

$$\mathrm{Re}\left\{\frac{\lambda}{p_i(\lambda)}\right\} = -\frac{\omega}{|p_i|^2}(\zeta[\omega^2 + \omega_{0i}^2] - 2\omega_{0i}\zeta_{0i}\omega), \tag{22}$$

$$\mathrm{Im}\left\{\frac{\lambda}{p_i(\lambda)}\right\} = -\frac{\mathrm{Im}\{\lambda\}}{|p_i|^2}(\omega^2 - \omega_{0i}^2), \tag{23}$$

where $\lambda = -\zeta\omega + \mathrm{j}\omega\sqrt{1-\zeta^2}$.

In the light of Eq. (16) the diagonal matrix given by Eq. (23) has to have at least 2 entries equal to zero, in which case $\omega = \omega_{0i} = \omega_{0k}$ for at least 2 indices $i \neq k$, or alternatively $\omega_{0i} < \omega < \omega_{0i+1}$ for some $i \geqslant 2$, if the eigenvalues are ordered as $\omega_{01} \leqslant \cdots \leqslant \omega_{0n}$. Before exploring this in detail a special case will be studied.

### 5.1. Results for 2 degrees of freedom systems

Consider the case $n = 2$ then, by assumption, $\mathbf{Y}\in\mathbb{R}^{2\times 2}$ is non-singular and hence Eq. (14) is equivalent to

$$\mathbf{Y}\mathbf{Y}^\top = -\frac{1}{\lambda}\mathbf{D}(\lambda). \tag{24}$$

This equation holds true only if the imaginary part of the right side is zero and if the real part is positive definite. By definition

$$-\frac{p_i(\lambda)}{\lambda} = -\frac{\bar{\lambda}}{|\lambda|^2}(\lambda - \lambda_{0i})(\lambda - \bar{\lambda}_{0i}) \tag{25}$$

$$= -\frac{1}{|\lambda|^2}(|\lambda|^2\,\mathrm{Re}\{\lambda\} - 2|\lambda|^2\,\mathrm{Re}\{\lambda_{0i}\} + |\lambda_{0i}|^2\,\mathrm{Re}\{\lambda\}) - \mathrm{j}\frac{\mathrm{Im}\{\lambda\}}{|\lambda|^2}(|\lambda|^2 - |\lambda_{0i}|^2), \tag{26}$$

where the bar denotes the complex conjugate. The imaginary part becomes zero if $|\lambda| = \omega = \omega_{0i} = |\lambda_{0i}|$. Hence a necessary condition is that the unmodified system has equal natural frequencies $\omega_{01} = \omega_{02}$. Inserting this result into the expression for the real part the condition to be positive definite is equivalent to

$$\mathrm{Re}\{\lambda_{0i}\} > \mathrm{Re}\{\lambda\} \tag{27}$$

which is, by definition, equivalent to

$$\zeta > \max(\zeta_{01}, \zeta_{02}). \tag{28}$$

However, in this case the modification $\mathbf{Y}\mathbf{Y}^\top = -\mathbf{D}(\lambda)/\lambda$ is diagonal, and hence the modified system is classically damped. This is summarized in the following theorem.

**Theorem 4.** *If the damping matrix of any 2 degree-of-freedom proportionally damped system is modified by the addition of a rank 2 component such that the resulting system is non-proportionally damped and has a pair of repeated complex eigenvalues, then the modified system is defective.*

### 5.2. The general case

The findings of the preceding example can be extended to the general case of a modification $\mathbf{Y} \in \mathbb{R}^{n\times 2}$. The purpose for this extension is to outline a procedure to generate a simple non-proportionally damped system with repeated eigenvalues. This procedure will be demonstrated by a numerical example later in this paper.

First suppose, without loss of generality, that $\omega = \omega_{01} = \omega_{02}$. Then the first two diagonal elements of the diagonal matrix given by Eq. (23) are zero and the remaining $n-2$ diagonal entries are positive. Hence Eq. (16) holds true only if the last $n-2$ rows of $\mathbf{Y} \in \mathbb{R}^{n\times 2}$ are zero. For arbitrary $\zeta > \max(\zeta_{01}, \zeta_{02})$ the first two diagonal elements of the diagonal matrix given by Eq. (22) are positive and for an arbitrary orthonormal matrix $\mathbf{Z} \in \mathbb{R}^{2\times 2}$ the matrix

$$\mathbf{Y} = \begin{bmatrix} \mathbf{GZ} \\ \mathbf{0} \end{bmatrix}, \tag{29}$$

where

$$\mathbf{G} = \sqrt{2\omega} \begin{bmatrix} \sqrt{\zeta - \zeta_{01}}/|p_1(\lambda)| & 0 \\ 0 & \sqrt{\zeta - \zeta_{02}}/|p_2(\lambda)| \end{bmatrix} \tag{30}$$

leads to a diagonal modification of the damping matrix. Hence also in this general case the modified damping matrix is diagonal and therefore the modified system is classically damped.

Now consider the case when the diagonal matrix given by Eq. (23) is non-singular. Suppose $\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_p \\ \mathbf{Y}_s \end{bmatrix}$, where $\mathbf{Y}_p \in \mathbb{R}^{2\times 2}$ is non-singular. Also assume that $\omega_{02} < \omega < \omega_{03}$. Then the diagonal matrix defined by Eq. (23) may be written as

$$\mathrm{Im}\{\lambda\mathbf{D}^{-1}\} = \mathbf{\Delta} = \begin{bmatrix} -\mathbf{\Delta}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{\Delta}_s \end{bmatrix}, \tag{31}$$

where $\mathbf{\Delta}_p$ is $2 \times 2$ and $\mathbf{\Delta}_s$ is $(n-2) \times (n-2)$, and both matrices are positive definite. Now the condition given by Eq. (16) becomes

$$\mathbf{Y}^\top \mathbf{\Delta} \mathbf{Y} = -\mathbf{Y}_p^\top \mathbf{\Delta}_p \mathbf{Y}_p + \mathbf{Y}_s^\top \mathbf{\Delta}_s \mathbf{Y}_s = \mathbf{0} \tag{32}$$

and since $\mathbf{Y}_p$ is non-singular this equation is equivalent to

$$\mathbf{\Delta}_p = \mathbf{Z}^\top \mathbf{\Delta}_s \mathbf{Z}, \tag{33}$$

where $\mathbf{Z} = \mathbf{Y}_s \mathbf{Y}_p^{-1}$. For $n \geqslant 4$ let $\mathbf{\Theta} \in \mathbb{R}^{(n-2)\times 2}$ be an arbitrary orthonormal matrix, that is $\mathbf{\Theta}^\top \mathbf{\Theta} = \mathbf{I}_2$. Then the matrix

$$\mathbf{Z} = \mathbf{\Delta}_s^{-1/2} \mathbf{\Theta} \mathbf{\Delta}_p^{1/2} \tag{34}$$

is a solution of Eq. (33) and hence for arbitrary non-singular $\mathbf{H} \in \mathbb{R}^{2\times 2}$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{\Delta}_s^{-1/2} \mathbf{\Theta}\, \mathbf{\Delta}_p^{1/2} \end{bmatrix} \mathbf{H} \tag{35}$$

satisfies Eq. (16). In order to determine $\mathbf{H}$, first note that the diagonal matrix given by Eq. (22) is positive definite if

$$\zeta > \frac{2\omega_{0i}\omega}{(\omega^2 + \omega_{0i})} \zeta_{0i} \tag{36}$$

and since

$$\frac{2\omega_{0i}\omega}{(\omega^2 + \omega_{0i})} \leqslant 1 \tag{37}$$

a sufficient condition is

$$\zeta > \max_i(\zeta_i). \tag{38}$$

Let the diagonal matrix $\mathbf{\Upsilon} = -\mathrm{Re}\{\lambda \mathbf{D}^{-1}\}$. Using the same partition as for $\mathbf{\Delta}$ then the condition given by Eq. (15) becomes

$$\mathbf{Y}^\top \mathbf{\Upsilon} \mathbf{Y} = \mathbf{H}^\top [\mathbf{\Upsilon}_p + \mathbf{\Delta}_p^{1/2} \mathbf{\Theta}^\top \mathbf{\Delta}_s^{-1} \mathbf{\Upsilon}_s \mathbf{\Theta}\, \mathbf{\Delta}_p^{1/2}] \mathbf{H} = \mathbf{I}_2. \tag{39}$$

Since the matrix in the square brackets is positive definite it can be written as $\mathbf{P}^\top \mathbf{P}$ for some non-singular matrix $\mathbf{P} \in \mathbb{R}^{2\times 2}$ and hence $\mathbf{H} = \mathbf{P}^{-1}$. Note that this solution requires

(1) $\omega_{02} < \omega < \omega_{03}$,
(2) $\zeta > \max_i(\zeta_i)$,
(3) $n \geqslant 4$.

Although $\mathbf{Y} = \mathbf{Y}(\mathbf{\Theta})$ represents a set of solutions, it may well be that there are other solutions which do not satisfy the above conditions. Some of these alternative solutions will be considered in Section 5.4.

### 5.3. A numerical example

Consider an undamped, 4 degree-of-freedom, mass–spring chain system, clamped at one end. The masses are assumed to be equal, $m_i = 1$ kg, $i = 1, 2, 3, 4$, and the spring stiffnesses are

$k_1 = k_4 = 3$ N/m and $k_2 = k_3 = 2$ N/m. Thus the stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} 3+2 & -2 & 0 & 0 \\ -2 & 2+2 & -2 & 0 \\ 0 & -2 & 2+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}. \tag{40}$$

The natural frequencies of this system are, to 4 decimal places,

$$\mathbf{\Omega}_0 = \text{diag}(0.5354, 1.6031, 2.4495, 2.8536). \tag{41}$$

The natural frequency and damping ratio of the repeated eigenvalue $\lambda = -\omega\zeta + \text{j}\omega\sqrt{1-\zeta^2}$ is chosen to be

$$\omega = 2, \quad \zeta = 0.4. \tag{42}$$

Of course, the eigenvalues occur in conjugate pairs, and choosing $\lambda$ to be repeated also means that $\bar{\lambda}$ is repeated. With $\mathbf{\Theta} = \mathbf{I}_2$,

$$\mathbf{Y} = \begin{bmatrix} -0.8913 & 0 \\ 0 & 1.0009 \\ -1.4022 & 0 \\ 0 & 1.2345 \end{bmatrix} \tag{43}$$

and the modified system has eigenvalues

$$\mathbf{\Lambda} = \text{diag}(-0.5802 + 0.3056\text{j}, -0.8000 + 1.8330\text{j}, -0.8000 + 1.8330\text{j}, -0.4629 + 2.2401\text{j}) \tag{44}$$

which corresponds to

$$\mathbf{\Omega} = \text{diag}(0.6558, 2, 2, 2.2874), \tag{45}$$

$$\mathbf{\Gamma} = \text{diag}(0.8848, 0.4, 0.4, 0.2024). \tag{46}$$

Note that the modification has increased the two lowest frequencies from 0.5354 and 1.6031 to 0.6558 and 2.0 rad/s, respectively; a phenomenon that is characteristic for non-proportional, general viscous damping.

The matrix of eigenvectors of the modified system, $\mathbf{X} \in \mathbb{C}^{4\times 4}$, is indeed non-singular and given to 4 decimal places by

$$\mathbf{X} = \begin{bmatrix} -1.1266 + 1.1180\text{j} & -0.2563 - 0.2918\text{j} & 0 & 0 \\ 0 & 0 & 0.6405 - 0.3477\text{j} & -0.0896 - 0.5453\text{j} \\ -0.0645 + 0.2463\text{j} & -0.4102 + 0.3343\text{j} & 0 & 0 \\ 0 & 0 & -0.0226 - 0.4276\text{j} & -0.4804 + 0.2202\text{j} \end{bmatrix}. \tag{47}$$

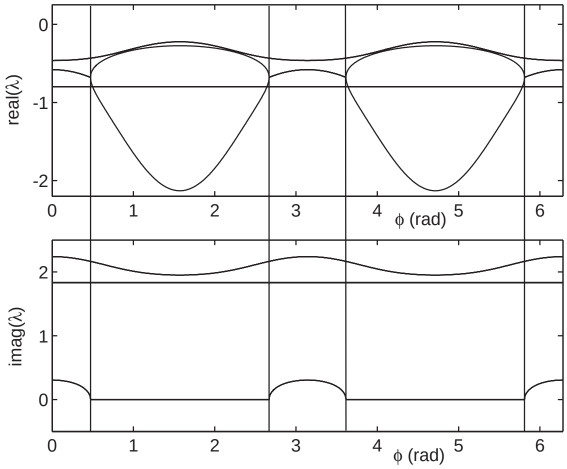


Fig. 1. Dependency of the eigenvalues on the choice of the rotation to define the modification.

The distribution of zeros in $\mathbf{Y}$ and $\mathbf{X}$ is due to the choice of $\mathbf{\Theta} = \mathbf{I}_2$. To investigate the effect of this choice on the eigenvalues of the modified system attention is restricted to a rotation

$$\mathbf{\Theta}(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}. \tag{48}$$

Since the modification $\mathbf{Y} = \mathbf{Y}(\phi)$ depends on $\phi$ the eigenvalues of the modified system $\lambda = \lambda(\phi)$ also depends on $\phi$. Fig. 1 shows this dependency for the real part (top) and the imaginary part (bottom) of $\lambda$. Note that for certain ranges of $\phi$, indicated by the vertical lines, the resulting system has one real eigenvalue. Note also that the chosen repeated eigenvalue (horizontal lines) is not affected by the rotation angle.

Although the above modification leads to a non-defective system with a repeated eigenvalue the realization of such a modification is an open problem. The back-transformation of $\mathbf{YY}^\top$, i.e. $\mathbf{X}_0^{-\top}\mathbf{YY}^\top\mathbf{X}_0^{-1}$, leads to a fully occupied damping matrix which would be difficult to implement physically. Whether a non-defective system with a repeated eigenvalue can be generated by varying a given damper arrangement is an interesting problem but beyond the scope of this investigation.

### 5.4. A theorem on rank 2 modifications

The discussion of alternative solutions of Eqs. (15) and (16) is now continued. The detailed study of the general solution requires consideration of conic sections, and is beyond the scope of this paper. Instead solutions that do not require the first two conditions given at the end of Section 5.2 will be derived.

**Theorem 5.** *Let* $2 \leqslant r \leqslant n-2$, *and suppose* $\omega_{0r} < \omega < \omega_{0r+1}$, *then the diagonal matrix defined by Eq.* (23) *can be partitioned as*

$$-\lambda\mathbf{D}^{-1} = \mathbf{\Delta} = \begin{bmatrix} \mathbf{\Delta}_1 & \mathbf{0} \\ \mathbf{0} & -\mathbf{\Delta}_2 \end{bmatrix}, \tag{49}$$

*where* $\mathbf{\Delta}_1 \in \mathbb{R}^{r\times r}$ *and* $\mathbf{\Delta}_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ *are positive definite. Moreover, let* $\mathbf{U} \in \mathbb{R}^{r\times 2}$ *and* $\mathbf{V} \in \mathbb{R}^{(n-r)\times 2}$ *be orthonormal matrices. Then*

$$\mathbf{Y} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{\Delta}_1^{-1/2}\mathbf{U} \\ \mathbf{\Delta}_2^{-1/2}\mathbf{V} \end{bmatrix} \mathbf{H} \tag{50}$$

*satisfies Eq.* (16) *for arbitrary* $\mathbf{H} \in \mathbb{R}^{2\times 2}$. *Let* $\mathbf{\Upsilon}_1$ *and* $\mathbf{\Upsilon}_2$ *denote the corresponding partitions of the diagonal matrix defined by Eq.* (22). *If*

$$\mathbf{U}^\top \mathbf{\Delta}_1^{-1} \mathbf{\Upsilon}_1 \mathbf{U} + \mathbf{V}^\top \mathbf{\Delta}_2^{-1} \mathbf{\Upsilon}_2 \mathbf{V} \tag{51}$$

*is positive definite, then* $\mathbf{H}$ *can be chosen in such a way that* $\mathbf{Y}$ *satisfies Eq.* (15).

**Proof.** That $\mathbf{Y}$ given in Eq. (50) satisfies Eq. (16) follows directly from calculation. If the matrix in Eq. (51) is positive definite, its singular value decomposition is of the form $\mathbf{L\Xi}^2\mathbf{L}^\top$ and hence $\mathbf{H} = \mathbf{L\Xi}^{-1}$. Note that $\mathbf{Y} = \mathbf{Y}(\mathbf{U}, \mathbf{V})$ represents a solution space generated by varying the two orthonormal matrices $\mathbf{U}$ and $\mathbf{V}$ which can be used to ensure condition given by Eq. (51) is satisfied if the number of positive diagonal elements of $\mathbf{\Upsilon}_1$ and $\mathbf{\Upsilon}_2$ is at least 2. This is necessary for the non-singularity of $\mathbf{U}$ and $\mathbf{V}$. The rows in $\mathbf{U}$ and $\mathbf{V}$ which are associated with the negative diagonal elements of $\mathbf{\Upsilon}_1$ and $\mathbf{\Upsilon}_2$, respectively, may be set to zero. □

This approach will be demonstrated by a non-trivial example.

## 6. Two simulation examples

In this section two examples of undamped finite element models will be studied. In the first example, a repeated eigenvalue of a 33 degree-of-freedom model will be placed by using a rank 2 non-proportional damping matrix which has been calculated from Eqs. (15) and (16). Indeed, the resulting system is simple. The second example is a 46 degree-of-freedom model with a rank 2 non-proportional damping matrix which produces some clustered eigenvalues. The conditions given in Eqs. (15) and (16) will be checked for the clustered eigenvalues. The singular values of the associated eigenvectors reveals that one pair of the clustered eigenvalues is indeed defective.

### *6.1. Non-defective bridge model with a repeated eigenvalue*

A small bridge of length 6 m is modelled by 12 Euler–Bernoulli beam elements as shown in Fig. 2. The unmodified system is undamped. Each beam element has the same physical properties: length 0.5 m, density $10^5$ kg/m$^3$, cross-sectional area $10^{-4}$ m$^2$, area moment of inertia $10^{-4}$ m$^4$, moment of inertia 40 kg m$^2$ and modulus of elasticity 70 GN/m$^2$. Each of the 11 nodes has 3 degree-of-freedom, two translational $u_x, u_y$ and one rotational $t_{xy}$. Hence the entire model has 33 degree-of-freedom. The first 12 modes are shown in Figs. 3 and 4, together with the corresponding eigenfrequencies.

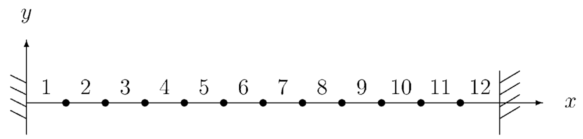


Fig. 2. Simple beam model of a small bridge consisting of 12 Euler–Bernoulli beam elements.

Using the real-valued modal matrix of the undamped model the mass and stiffness matrix are transformed to diagonal form. Since the eighth and ninth eigenfrequencies are adjacent the repeated eigenvalue, $\lambda = -\omega\zeta + \mathrm{j}\omega\sqrt{1-\zeta^2}$ is chosen with $\omega = 650\pi$ rad/s and $\zeta = 0.4$. This choice of the repeated eigenvalue implies $\omega_{07} < \omega < \omega_{08}$ and hence the two orthonormal matrices **U** and **V** are $7 \times 2$ and $26 \times 2$, respectively. The following simple case is chosen.

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{7\times 2}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{26\times 2}. \tag{52}$$

Since the unmodified model is undamped the matrix given by Eq. (51) is positive definite for every choice of **U** and **V**. The rank 2 matrix **Y** has the corresponding block-diagonal structure

$$\mathbf{Y} = [4.1252\mathbf{e}_1 + 28.582\mathbf{e}_8, 5.8525\mathbf{e}_2 + 28.575\mathbf{e}_9] \in \mathbb{R}^{33\times 2}, \tag{53}$$

where $\mathbf{e}_i$ denotes a vector of zeros except a 1 in the $i$th entry.

The resulting non-proportionally damped modified system is, indeed, non-defective. To compare the effect on the mode shapes the rank 2 damping modification has been back-transformed using the real-valued modal matrix of the undamped model. The solution of the eigenvalue problem yields a complex modal matrix $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_{33}] \in \mathbb{C}^{33\times 33}$. The first 12 modes are shown in Figs. 5 and 6 where $\mathrm{Re}\{\mathbf{x}_i\}$ and $-\mathrm{Im}\{\mathbf{x}_i\}$ are plotted in each frame, except the second and third frame of Fig. 6 where the real and imaginary parts of the mode shapes are shown. Obviously, there is no significant difference between the real and the negative imaginary part for all modes except for modes 8 and 9 which are associated with the repeated eigenvalue. Indeed, these two mode shapes are strongly correlated and similar to the second mode shape. Although the modeshapes of $\mathbf{x}_8$ and $\mathbf{x}_9$ look alike, the eigenvectors $\mathbf{x}_8$ and $\mathbf{x}_9$ are linearly independent which is confirmed by the condition number of the matrix $[\mathbf{x}_8\ \mathbf{x}_9]$, which is $\approx 354$. The differences are mainly due to the vertical displacements $u_y$ and the slopes $t_{xy}$.

Note that only eigenvalues $\lambda_1$ and $\lambda_2$ and the repeated eigenvalue $\lambda = \lambda_8 = \lambda_9$ have non-zero real parts and that the rank 2 damping has no significant effect on the eigenfrequencies. It is amazing that the mode shapes associated with the eigenfrequencies $\omega_{08}$ and $\omega_{09}$ of the unmodified system have disappeared (Fig. 4, frames 2 and 3). All other modeshapes are preserved.

Clearly, the correlation between modes 2, 8 and 9 is due to the structure of **Y** which depends on the choice of the orthonormal matrices **U** and **V**. It seems that the choice of **U** and **V** enables some control over the correlated modes. But this needs further investigation. Again, it should be emphasized that the realization of such a damping configuration is difficult. Although **Y** has a simple sparse structure the back-transformation yields an almost fully populated damping matrix

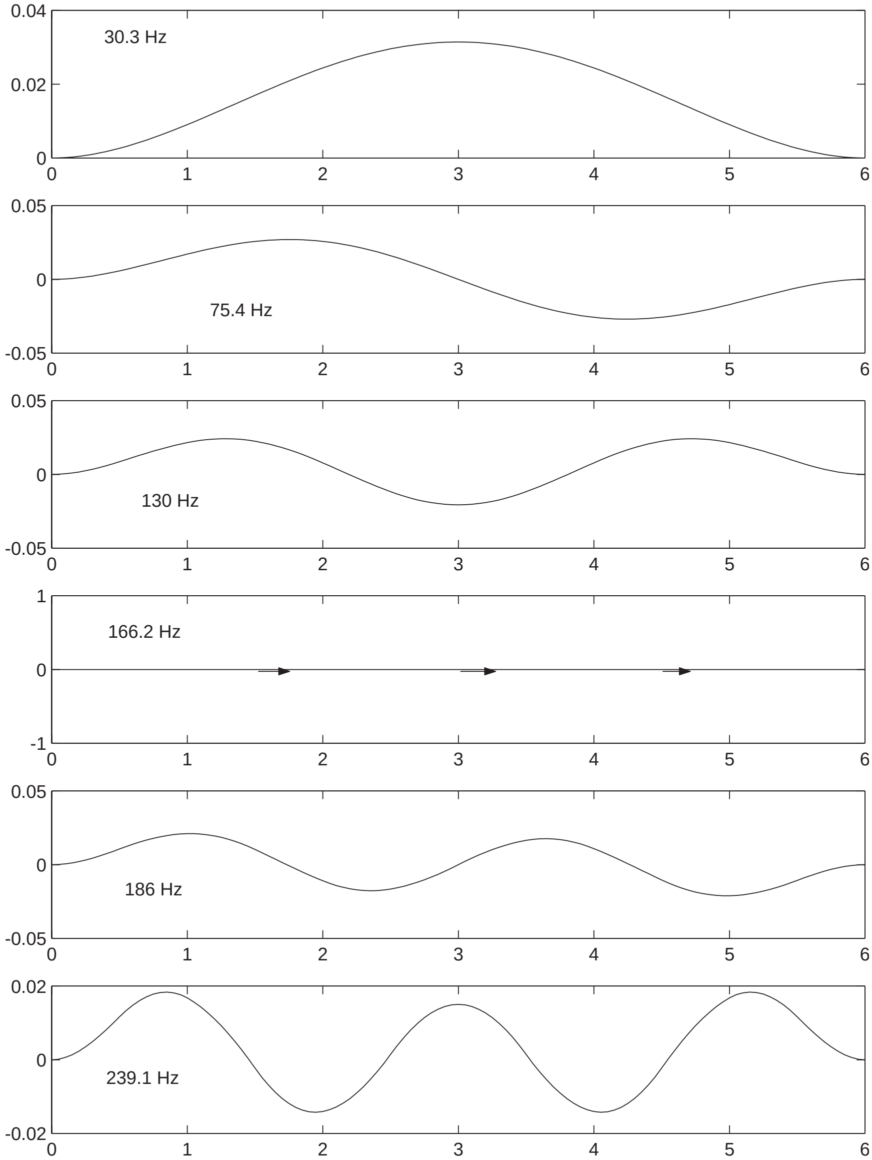


Fig. 3. The first 6 eigenmodes of the bridge model.

which corresponds to a rather unrealistic damper configuration. Whether a different choice of **U** and **V** can guarantee a realistic (simple) damper arrangement needs to be investigated.

### *6.2. Two cantilevers connected by two dashpots*

In practical applications defective eigenvalues are rarely repeated exactly. It is more common that eigenvalues are clustered, i.e., they are close but numerically not equal. The question of

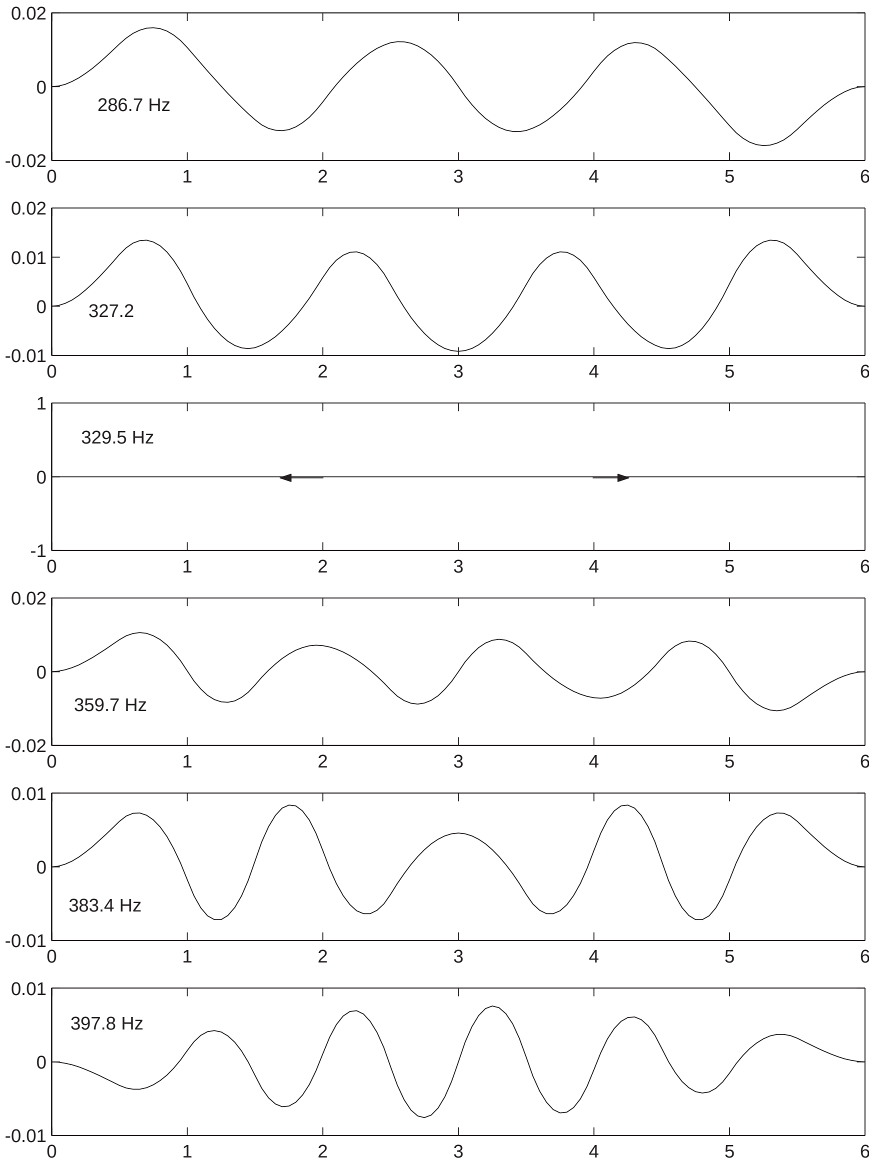


Fig. 4. Eigenmodes 7 to 12 of the bridge model.

whether a set of clustered eigenvalues can lead to a defective system may be answered by checking the conditions given by Eqs. (15) and (16). This will be demonstrated by an example.

Fig. 7 shows the model of a beam element structure. Both beams are made of steel, and are of length 1 m, width 50 mm and depth 25 mm. Only bending in the most flexible plane is considered. The first beam is clamped at both ends, whereas the second beam is only clamped at the left end. Both beams are divided into 12 Euler–Bernoulli beam elements. At the fourth and eighth nodes from the left end the beams are joined by two dashpots, whose damping values are calculated to

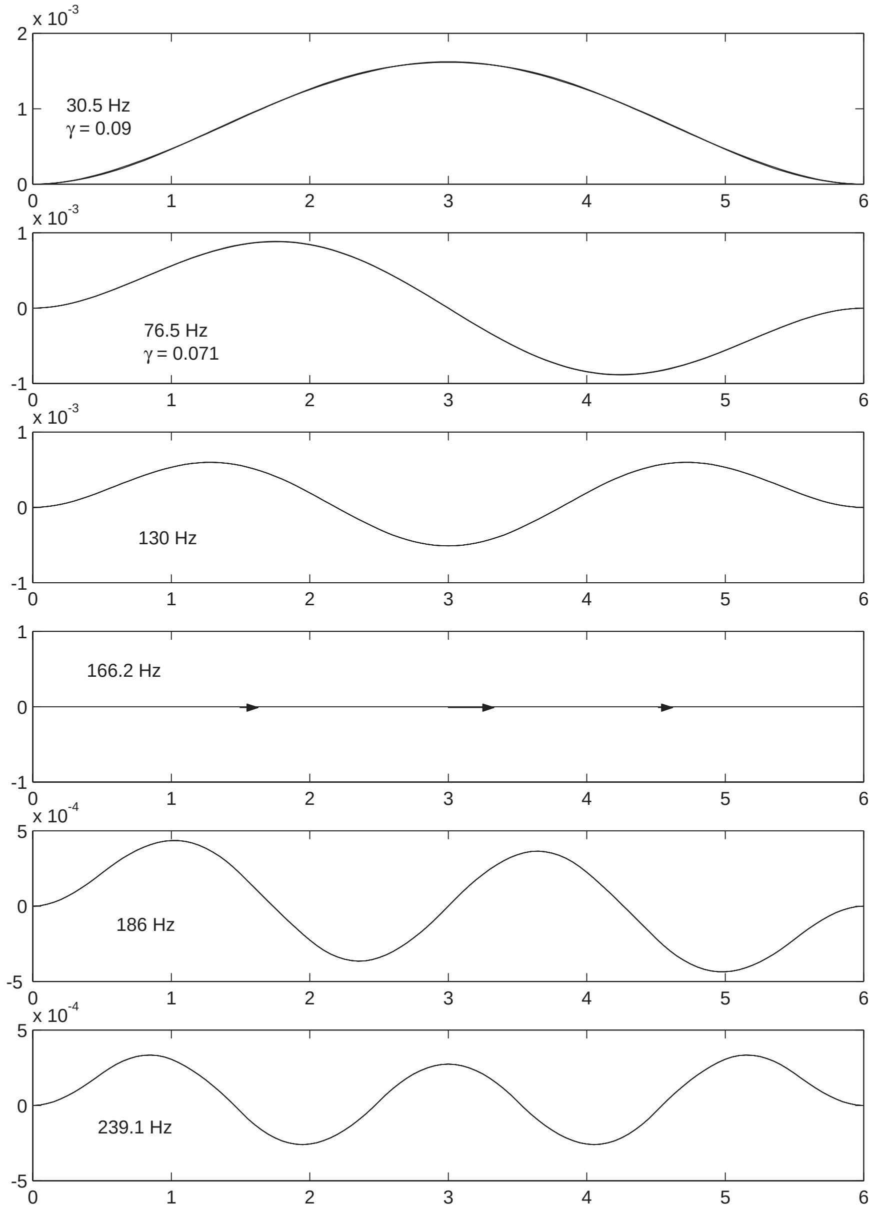


Fig. 5. The real and negative imaginary parts of the first 6 modeshapes of the modified bridge model.

produce repeated eigenvalues. The values are approximately $c_1 = 4.0238$ N s/m and $c_2 = 0.24994$ N s/m. The corresponding rank 2 modification is

$$\mathbf{Y} = \mathbf{X}_0^\top \left[\sqrt{c_1}(\mathbf{e}_7 - \mathbf{e}_{29}), \sqrt{c_2}(\mathbf{e}_{15} - \mathbf{e}_{37})\right], \tag{54}$$

where $\mathbf{X}_0 \in \mathbb{R}^{46\times 46}$ is the modal matrix of the undamped model. Table 1 shows the first 15 pairs of eigenvalues of the modified system (to five significant figures).

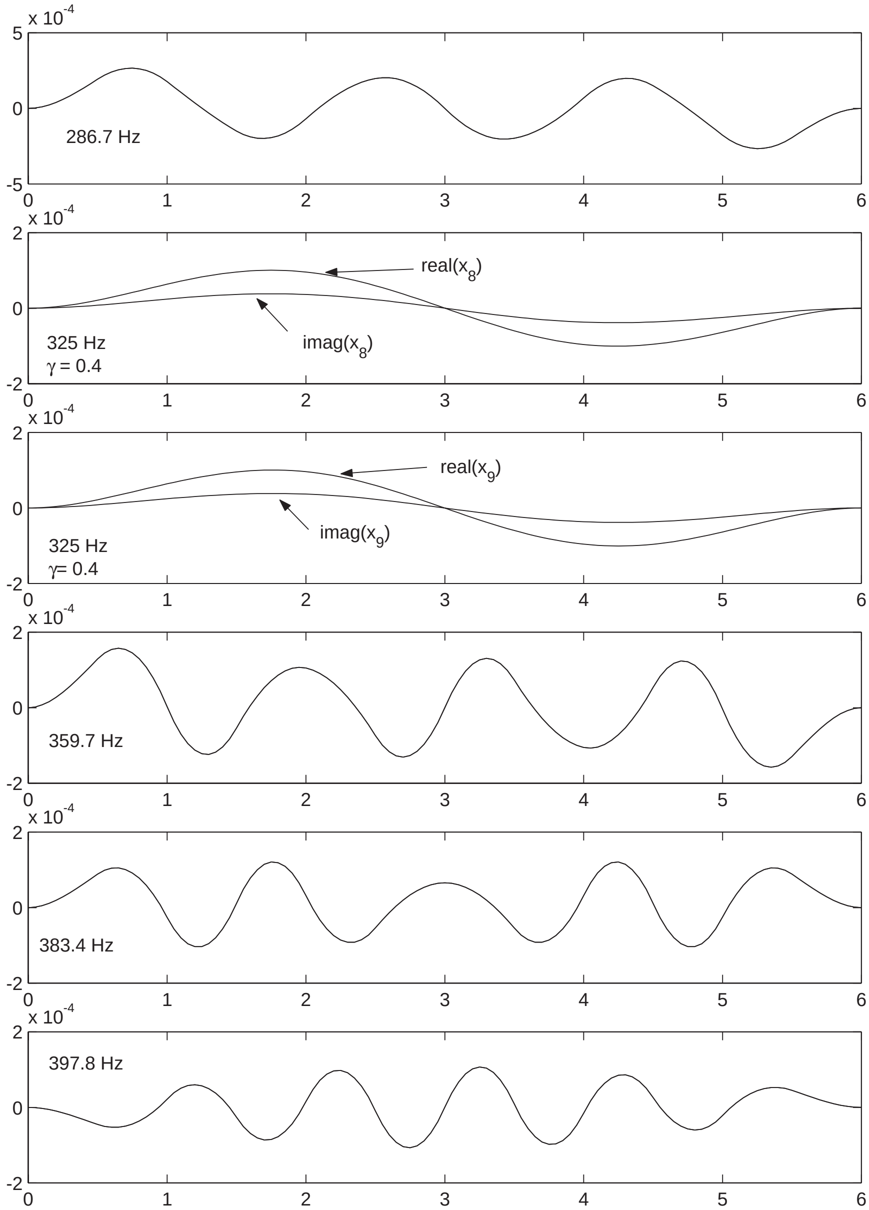


Fig. 6. The real and negative imaginary parts of modeshapes 7 and 10 to 12, and the real and imaginary parts of modeshapes 8 and 9, of the modified bridge model.

Clearly, the pairs (4,5), (6,7), (10,11), (12,13) and (14,15) of eigenvalues are clustered. As expected, none of the corresponding ten eigenvalues satisfy the condition given by Eq. (14). In particular, for the first pair the maximum absolute value of the error in Eq. (14) is about 1. Taking the singular value decomposition of the five matrices each consisting of the two corresponding

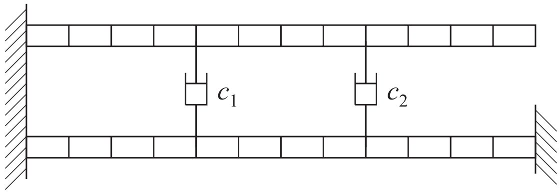


Fig. 7. Two cantilever beams connected by two dashpots.

Table 1
The first 15 eigenvalues for the beam example

| Mode | | Mode | |
|---|---|---|---|
| 1 | $-3.7955 \times 10^{-2} \pm j1.3072 \times 10^{2}$ | 8 | $-3.7557 \times 10^{-1} \pm j7.4399 \times 10^{3}$ |
| 2 | $-2.9613 \times 10^{-1} \pm j8.1922 \times 10^{2}$ | 9 | $-5.7551 \times 10^{-2} \pm j7.4399 \times 10^{3}$ |
| 3 | $-3.3462 \times 10^{-1} \pm j8.3180 \times 10^{2}$ | 10 | $-4.1203 \times 10^{-1} \pm j1.1130 \times 10^{4}$ |
| 4 | $-4.5131 \times 10^{-1} \pm j2.2936 \times 10^{3}$ | 11 | $-4.1157 \times 10^{-1} \pm j1.1131 \times 10^{4}$ |
| 5 | $-4.5130 \times 10^{-1} \pm j2.2936 \times 10^{3}$ | 12 | $-2.9985 \times 10^{-2} \pm j1.5583 \times 10^{4}$ |
| 6 | $-3.3016 \times 10^{-2} \pm j4.4970 \times 10^{3}$ | 13 | $-2.9832 \times 10^{-2} \pm j1.5587 \times 10^{4}$ |
| 7 | $-3.3876 \times 10^{-2} \pm j4.4971 \times 10^{3}$ | 14 | $-2.2667 \times 10^{-1} \pm j2.0821 \times 10^{4}$ |
| | | 15 | $-2.2775 \times 10^{-1} \pm j2.0831 \times 10^{4}$ |

eigenvectors leads to the following condition numbers:

$$\begin{aligned}
&(4,5) \to cond \approx 2.2 \times 10^{5},\\
&(6,7) \to cond \approx 2.1,\\
&(10,11) \to cond \approx 2.3,\\
&(12,13) \to cond \approx 1.02,\\
&(14,15) \to cond \approx 1.05.
\end{aligned}$$

This means that the eigenvectors associated with pair (4, 5) are close to being linearly dependent, and thus the system is, in practice, defective.

## 7. Conclusions

This paper has investigated low-rank modifications of the classical damping matrix and determined whether, if the modified system possesses pairs of repeated complex eigenvalues, the resulting system is defective. For unit rank modifications, if the system has repeated eigenvalues (that were not eigenvalues of the unmodified system), and the damping is non-proportional, then the system will be defective. Extending this result for the unit rank case, the condition on a rank 2 modification has been presented that ensures the modified system is simple, even if it possesses pairs of repeated complex eigenvalues and non-proportional damping. A procedure to generate modifications that produce simple systems with repeated eigenvalues has been given and demonstrated by simulated examples. This condition is readily extended to higher rank modifications. The important conclusion from this exercise is that defective systems may be much more common in practice than originally thought. Indeed, if a modified system has repeated eigenvalues and non-proportional damping, then it is very likely to be defective. The ramifications for the dynamic analysis of such systems is under investigation. However, at the very least, the results highlight that care is required in the calculation of the eigenvalues of damped structures with pairs of repeated or clustered complex eigenvalues.

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