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Enforcing nodes at required locations in a harmonically excited structure using simple oscillators

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Abstract

With a suitably chosen set of spring–mass parameters, a single or multiple points of zero vibration (or otherwise referred to as nodes) can be induced anywhere along a general elastic structure during forced harmonic excitations. In application, however, the actual selection of the oscillator parameters also depends on the tolerable vibration amplitudes of the absorber masses, because if the vibration amplitudes of these masses are large, then theoretically feasible solutions could not be implemented in practice. In this paper, spring–mass systems are used as a means to impose single or multiple nodes anywhere along a harmonically forced structure, subjected to the constraints of tolerable vibration amplitudes for the masses. When the node locations are chosen so that they are closely spaced, a region of nearly zero amplitudes can be induced, effectively quenching vibration in that segment of the structure. Numerical experiments show that the required mass and its vibration amplitude for each oscillator are inversely related. This observation serves as a guide for the proper selection of the oscillator parameters in order to induce multiple nodes and to meet the tolerable vibration amplitudes of the oscillator masses. An efficient procedure for choosing the required oscillator parameters is outlined in detail, and numerical experiments are performed to verify the proposed methodology of imposing nodes at multiple locations along any arbitrary structure during harmonic excitations.

1. Introduction

Spring–mass systems are frequently used as vibration absorbers to control and to minimize excess vibration in structural systems [1–9]. In a recent paper [10], the present author used sprung masses to impose the points of zero vibration for general elastic structures during forced harmonic excitations. For convenience, such points are referred to as nodes. It was found that when the

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oscillator attachment locations and the node locations coincide (or collocated), it is always possible to tune the spring–mass parameters such that their attachment locations can be made to coincide exactly with the nodes of the structure, thereby allowing nodes to be imposed at multiple locations anywhere along the combined assembly, for any excitation frequency. When the oscillators and the node locations are not collocated, however, it is only possible to induce nodes at certain locations along the elastic structure for a given driving frequency. Finally, to induce S nodes along a structure for the non-collocated case, a set of S non-linear algebraic equations needs to be solved.

In general, the selection of the sprung masses in order to induce nodes is not unique, and the eventual choice is often dictated by the tolerable vibration amplitudes of the oscillator masses. If the vibration amplitudes of these elastically mounted masses are excessively large, then an arbitrarily chosen set of oscillator parameters may not be feasible to be implemented in application. In Ref. [10], the maximum allowable absorber amplitudes were not considered in the selection of spring–mass parameters, while in this paper, they will be specified as additional design objectives, making the problem more challenging and practical. With the additional constraints, the problem becomes substantially more complicated analytically, because the node locations and the amplitude constraints lead to a total of $2S$ non-linear algebraic equations that must be satisfied simultaneously. Numerically, the solution to these $2S$ equations is very computationally intensive because the convergence is often very slow. To make the selection of the oscillator parameters more efficient, a procedure is proposed that requires only the solution of S equations, thereby reducing the number of equations that need to be solved by half and offering substantial computational savings over the direct approach of solving a set of $2S$ non-linear algebraic equations. Numerical experiments are performed to verify the utility of the proposed scheme of imposing nodes at multiple locations during harmonic excitations, subjected to the constraints of tolerable vibration amplitudes of the oscillator masses. The ability to enforce node locations has practical benefits because it would allow sensitive instruments to be placed near or at points where there are little or no vibration. In addition, the proposed scheme allows certain locations along the structure to remain stationary without using any rigid supports.

2. Theory

2.1. Governing equations

Consider the system of Fig. 1, which consists of an arbitrarily supported elastic structure to which S-sprung masses are attached. A localized harmonic force of forcing amplitude F and excitation frequency ω, $f(t) = F\mathrm{e}^{\mathrm{j}\omega t}$, is applied to the structure at x_f, where $\mathrm{j} = \sqrt{-1}$. Utilizing the assumed-modes method, the physical deflection of the structure at any point x is given by

$$w(x,t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \tag{1}$$

where the $\phi_i(x)$ are the eigenfunctions of the unloaded structure (the elastic structure without any sprung masses) that form the trial functions for this approximate solution, the $\eta_i(t)$ are the corresponding generalized co-ordinates, and N is the number of modes used in the assumed-modes

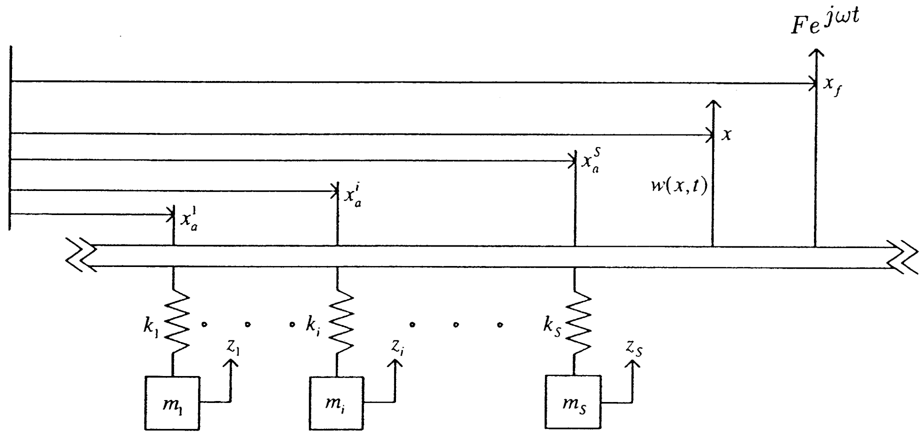

Fig. 1. An arbitrarily supported elastic structure that is subjected to a localized harmonic excitation and carrying any number of sprung masses.

expansion. Formulating the total kinetic and potential energies of the combined system, applying Lagrange's equations, and assuming simple harmonic motion with the same response frequency as the driving frequency, the following matrix equations are obtained [10]:

$$([\mathscr{K}] - \omega^2[\mathscr{M}])\underline{\bar{\eta}} + [R]\underline{\bar{z}} = F\underline{\phi}(x_f) \tag{2}$$

and

$$[R]^{\mathrm{T}}\underline{\bar{\eta}} + ([k] - \omega^2[m])\underline{\bar{z}} = \underline{0}, \tag{3}$$

where $\underline{\bar{\eta}} = [\bar{\eta}_1\ \bar{\eta}_2 \ldots \bar{\eta}_N]^{\mathrm{T}}$ and $\underline{\bar{z}} = [\bar{z}_1\ \bar{z}_2 \ldots \bar{z}_S]^{\mathrm{T}}$. The $N \times N$ $[\mathscr{M}]$ and $[\mathscr{K}]$ matrices of Eq. (2) are

$$[\mathscr{M}] = [M^d], \quad [\mathscr{K}] = [K^d] + \sum_{i=1}^{S} k_i \underline{\phi}(x_a^i)\underline{\phi}^{\mathrm{T}}(x_a^i), \tag{4}$$

where $[M^d]$ and $[K^d]$ are diagonal matrices whose ith elements are M_i and K_i (the generalized masses and stiffnesses of the elastic structure), vectors $\underline{\phi}(x_a^i)$ and $\underline{\phi}(x_f)$ consist of the eigenfunctions of the elastic structure evaluated at x_a^i and x_f, respectively,

$$\underline{\phi}(x_a^i) = [\phi_1(x_a^i)\ \phi_2(x_a^i) \ldots \phi_N(x_a^i)]^{\mathrm{T}}, \quad \underline{\phi}(x_f) = [\phi_1(x_f)\ \phi_2(x_f) \ldots \phi_N(x_f)]^{\mathrm{T}}, \tag{5}$$

and the $N \times S$ matrix $[R]$ is given by

$$[R] = [-k_1\underline{\phi}(x_a^1) \quad \ldots \quad -k_i\underline{\phi}(x_a^i) \quad \ldots \quad -k_S\underline{\phi}(x_a^S)]. \tag{6}$$

Note that $[\mathscr{M}]$ is a diagonal matrix and $[\mathscr{K}]$ is a diagonal matrix modified by S rank one matrices. The $S \times S$ matrices $[m]$ and $[k]$ are both diagonal, whose ith elements are given by m_i and k_i (the mass and spring stiffness of the ith oscillator). Eqs. (2) and (3) can be written in the compact form

$$\begin{bmatrix} [\mathscr{K}] - \omega^2[\mathscr{M}] & [R] \\ [R]^{\mathrm{T}} & [k] - \omega^2[m] \end{bmatrix} \begin{bmatrix} \underline{\bar{\eta}} \\ \underline{\bar{z}} \end{bmatrix} = \begin{bmatrix} F\underline{\phi}(x_f) \\ \underline{0} \end{bmatrix} \tag{7}$$

which consists of $N+S$ equations with $N+S$ unknowns in the generalized co-ordinates $\underline{\bar{\eta}}$ and the oscillator displacements $\underline{\bar{z}}$.

Eq. (2) can be reduced in terms of $\underline{\bar{\eta}}$ only by simple algebraic manipulation. Using Eq. (3), the $\bar{z}_i$ are found to be

$$\bar{z}_i = \frac{k_i \underline{\phi}^{\mathrm{T}}(x_a^i)}{k_i - \omega^2 m_i} \underline{\bar{\eta}}, \quad i = 1, \ldots, S. \tag{8}$$

Substituting the expressions of Eq. (8) into Eq. (2), the following matrix equation, of size $N \times N$, is obtained:

$$\left\{ [K^d] + \sum_{i=1}^{S} \sigma_i \underline{\phi}(x_a^i) \underline{\phi}^{\mathrm{T}}(x_a^i) - \omega^2 [M^d] \right\} \underline{\bar{\eta}} = F \underline{\phi}(x_f), \tag{9}$$

where

$$\sigma_i = \frac{k_i m_i \omega^2}{m_i \omega^2 - k_i}. \tag{10}$$

Finally, the natural frequencies of the modified structure correspond to the zeros of the characteristic determinant of the coefficient matrix of $\underline{\bar{\eta}}$ of Eq. (9).

In application, the number of imposed nodes can be less than the number of sprung masses. However, it is more efficient to use as few sprung masses as possible to impose the desired number of nodes. Thus, in the subsequent analysis, only S sprung masses will be used to impose S nodes. Now, to impose S nodes at any desired locations, x_n^r, along the elastic structure requires that

$$w(x_n^r, t) = \sum_{i=1}^{N} \phi_i(x_n^r) \eta_i(t) = \underline{\phi}^{\mathrm{T}}(x_n^r) \underline{\eta} = \underline{\phi}^{\mathrm{T}}(x_n^r) \underline{\bar{\eta}}\, \mathrm{e}^{\mathrm{j}\omega t} = 0, \quad r = 1, \ldots, S. \tag{11}$$

Assuming that the excitation frequency does not coincide with any natural frequencies of the modified system, the coefficient matrix of Eq. (9) can be inverted to give

$$\underline{\bar{\eta}} = \left\{ [K^d] + \sum_{i=1}^{S} \sigma_i \underline{\phi}(x_a^i) \underline{\phi}^{\mathrm{T}}(x_a^i) - \omega^2 [M^d] \right\}^{-1} F \underline{\phi}(x_f) \tag{12}$$

which allows Eq. (11), the constraint equations that dictate the location of nodes, to be rewritten as

$$\underline{\phi}^{\mathrm{T}}(x_n^r) \left\{ [K^d] + \sum_{i=1}^{S} \sigma_i \underline{\phi}(x_a^i) \underline{\phi}^{\mathrm{T}}(x_a^i) - \omega^2 [M^d] \right\}^{-1} F \underline{\phi}(x_f) = 0, \quad r = 1, \ldots, S. \tag{13}$$

In practice, the selection of m_i and k_i is often governed by the maximum allowable vibration amplitudes of the oscillator masses. From Eq. (3), the vector of mass amplitudes, of length S, is given by

$$\underline{\bar{z}} = -([k] - \omega^2 [m])^{-1} [R]^{\mathrm{T}} \underline{\bar{\eta}} \tag{14}$$

or alternatively,

$$\underline{\bar{z}} = -([k] - \omega^2[m])^{-1}[R]^{\mathrm{T}}\left\{[K^d] + \sum_{i=1}^{S} \sigma_i \underline{\phi}(x_a^i)\underline{\phi}^{\mathrm{T}}(x_a^i) - \omega^2[M^d]\right\}^{-1} F\underline{\phi}(x_f). \tag{15}$$

Incidentally, Eq. (15) yields a closed form expression for $\underline{\bar{z}}$, which is convenient for analytical manipulations. Numerically, $\underline{\bar{z}}$ is evaluated instead by solving the matrix equation given by Eq. (7), which can be used to determine $\underline{\bar{z}}$ even when $\omega^2 = k_r/m_r$. Let the vector of tolerable vibration amplitudes of the sprung masses be given $\underline{\bar{z}}_{max}$, where $\underline{\bar{z}}_{max}$ is defined as $\underline{\bar{z}}_{max} = |\underline{\bar{z}}|$ (vertical bars denote the absolute value). Once the elastic structure and its boundary conditions are specified, the attachment locations x_a^i are given, and the excitation frequency ω and the excitation location x_f are known, Eq. (13) and the amplitude constraints lead to a set of $2S$ equations that must be solved simultaneously for the required $2S$ spring–mass parameters in order to induce a single or multiple nodes at x_n^r.

The MATLAB routine *fsolve* can be used to find the roots, the m_i and the k_i, that satisfy t he aforementioned system of $2S$ non-linear algebraic equations. Numerically, *fsolve* requires a set of initial guess of the unknown variables to be provided. For a set of initial guesses, if *fsolve* does not converge to a solution, then *fsolve* is run again with a different set of starting values until a solution is obtained. The proposed technique of solving for the mass and stiffness parameters in order to impose nodes at x_n^r, subjected to the mass amplitude constraints, is very robust but highly computationally intensive. Later in the paper, an efficient procedure will be proposed that allows the oscillator parameters to be easily determined for any given set of amplitude constraints.

Two cases deserve special attention: one when the attachment location and the node location coincide, and one in which there is only one oscillator and one node. The former case results in having to satisfy only S equations, and the latter case leads to closed-form expressions for the required mass parameter and its vibration amplitude, m and $\bar{z}$, respectively.

2.2. Collocated

Consider the special case where the attachment and the node locations coincide. For this case, the oscillators and the nodes are said to be *collocated*. From Eq. (3), note that if

$$k_r = m_r\omega^2, \quad r = 1, \ldots, S, \tag{16}$$

then

$$[R]^{\mathrm{T}}\underline{\bar{\eta}} = \underline{0}. \tag{17}$$

Because the oscillators and the node locations are collocated, $x_a^r = x_n^r$, in which case the rth row of Eq. (17) yields

$$-k_r\underline{\phi}^{\mathrm{T}}(x_a^r)\underline{\bar{\eta}} = -k_r\underline{\phi}^{\mathrm{T}}(x_n^r)\underline{\bar{\eta}} = 0, \quad r = 1, \ldots, S \tag{18}$$

which is clearly identical to Eq. (11) and implies that nodes will be induced at the attachment locations. Eq. (16) reveals that there is an infinite number of spring–mass combinations that will induce nodes at x_a^r, as long as Eq. (16) is satisfied. In this case, the tolerable vibration amplitudes of the oscillator masses will dictate the actual spring–mass parameters that are needed to impose nodes at x_n^r, for $r = 1, \ldots, S$. Thus, instead of solving the $2S$ equations for the $2S$ unknowns k_r and m_r, only $|\bar{\underline{z}}| = \bar{\underline{z}}_{max}$ (which consists of S equations) needs to be solved for either the k_r or m_r, because Eq. (16) immediately satisfies the node location constraints.

2.3. One oscillator and one node

When only one node location at x_n is specified, and it does not coincide with the attachment location at x_a (for this case, the attachment and node locations are said to be *not collocated* or *non-collocated*), the desired spring–mass parameters can be readily obtained. For $S = 1$, Eq. (13) simplifies to

$$\underline{\phi}^{\mathrm{T}}(x_n)\left\{[K^d] + \frac{km\omega^2}{m\omega^2 - k}\underline{\phi}(x_a)\underline{\phi}^{\mathrm{T}}(x_a) - \omega^2[M^d]\right\}^{-1} F\underline{\phi}(x_f) = 0. \tag{19}$$

Because the second term of Eq. (19) consists of a matrix modified by a rank one matrix, its inverse can be obtained by applying the Sherman–Morrison formula [11]. Assuming the excitation frequency ω, the oscillator stiffness parameter k, the attachment location x_a, the node location x_n, and the excitation location x_f are all specified, a closed-form expression for the required m in order to impose a node at x_f can be obtained as follows (see Ref. [10] for detailed derivations):

$$m = \frac{c_1 k}{\omega^2(c_1 + c_1 c_3 k - c_2 k)}, \tag{20}$$

where

$$c_1 = \sum_{i=1}^{N} \frac{\phi_i(x_n)\phi_i(x_f)}{K_i - M_i\omega^2}, \tag{21}$$

$$c_2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\phi_i(x_a)\phi_j(x_a)\phi_i(x_n)\phi_j(x_f)}{(K_i - M_i\omega^2)(K_j - M_j\omega^2)} \tag{22}$$

and

$$c_3 = \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{K_i - M_i\omega^2}. \tag{23}$$

Similarly, for $S = 1$, Eq. (15) simplifies to

$$\bar{z} = \frac{k}{k - m\omega^2}\underline{\phi}^{\mathrm{T}}(x_a)\left\{[K^d] + \frac{km\omega^2}{m\omega^2 - k}\underline{\phi}(x_a)\underline{\phi}^{\mathrm{T}}(x_a) - \omega^2[M^d]\right\}^{-1} F\underline{\phi}(x_f). \tag{24}$$

Expanding the triple product of Eq. (24) gives

$$\bar{z} = F\left(c_1' - \frac{\alpha}{1 + c_3\alpha}c_2'\right), \tag{25}$$

where

$$c_1' = \frac{k}{k - m\omega^2} \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i(x_f)}{K_i - M_i\omega^2}, \tag{26}$$

$$c_2' = \frac{k}{k - m\omega^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\phi_i^2(x_a)\phi_j(x_a)\phi_j(x_f)}{(K_i - M_i\omega^2)(K_j - M_j\omega^2)}. \tag{27}$$

After some lengthy algebra, a closed-form expression for $\bar{z}$ is obtained as

$$\bar{z} = \frac{-kF}{m\omega^2 - k + c_3 k m\omega^2} \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i(x_f)}{K_i - M_i\omega^2}. \tag{28}$$

Substituting Eq. (20) into $|\bar{z}| = \bar{z}_{max}$ (the absolute value of Eq. (28)) yields an equation in the unknown parameter k, which can then be solved for the required stiffness value in order to induce a node at x_n. Moreover, for the solution to be physically meaningful, both k and m must be positive. Mathematically, if Eq. (20) and $|\bar{z}| = z_{max}$ lead to a mass value or a stiffness value that is negative, then a node cannot be enforced at the desired location for the given set of ω, x_a and x_f. In this case, the attachment location, x_a, can be varied until physically meaningful, i.e. positive, values of m and k are obtained so that a node can be induced at x_n for the given x_f and ω.

For this one oscillator and one node case, the required mass in order to induce a node and its vibration amplitude are inversely related. Manipulating Eq. (20) for k yields

$$k = \frac{m\omega^2 c_1}{c_1 - m\omega^2 c_1 c_3 + m\omega^2 c_2}. \tag{29}$$

Substituting Eq. (29) into Eq. (28), the product of the required mass and its vibration amplitude is found to be given by

$$\bar{z}m = -\frac{F}{\omega^2}\frac{c_1}{c_2} \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i(x_f)}{K_i - M_i\omega^2} \tag{30}$$

which is independent of k, the oscillator stiffness. For a given x_a, x_n, x_f and ω, the right-hand side of Eq. (30) is a constant. This implies that a small vibration amplitude for the oscillator mass requires a large mass and vice versa, consistent with physical intuition. For this case, once $\bar{z}m$ is evaluated, Eqs. (30) and (29) can be used to solve for the required m and k for any $\bar{z}_{max}$ directly without resorting to *fsolve*.

Consider now the special case where $x_n = x_a$ (collocated). For $S = 1$ and for

$$k = m\omega^2, \tag{31}$$

Eq. (18) shows that a node will be induced at the attachment location, and an infinite number of spring–mass combinations can be used to enforce a node at x_a. In practice, however, the mass amplitude constitutes an additional design constraint that must be met. Interestingly, the result of Eq. (31) can also be obtained by solving Eq. (20) directly. When $x_n = x_a$, then $c_1 c_3 = c_2$ and

Eq. (20) reduces to Eq. (31). Finally, for the same case, Eq. (30) reduces to

$$\bar{z}m = -\frac{F}{\omega^2}\frac{c_1}{c_3} \tag{32}$$

whose right-hand side remains a constant for a specified x_a, x_f and ω.

3. Results

Because the assumed-modes method was used to formulate the equations of motion, the proposed procedures can be easily extended to impose a single node or multiple nodes for any arbitrarily supported elastic structure during harmonic excitations. Without any loss of generality, a simply supported and a fixed–free uniform Euler–Bernoulli beam will be considered.

For a uniform simply supported Euler–Bernoulli beam, its normalized (with respect to the mass per unit length, ρ, of the beam) eigenfunctions are given by

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}}\sin\frac{\mathrm{i}\pi x}{L} \tag{33}$$

such that the generalized masses and stiffnesses of the beam become

$$M_i = 1 \quad \text{and} \quad K_i = (\mathrm{i}\pi)^4 EI/(\rho L^4), \tag{34}$$

where E is the Young's modulus, I is the area moment of inertia of the cross-section of the beam. For a uniform fixed–free Euler–Bernoulli beam, its normalized eigenfunctions are

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cos\beta_i x - \cosh\beta_i x + \frac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}(\sin\beta_i x - \sinh\beta_i x)\right) \tag{35}$$

such that the generalized masses and stiffnesses of the beam are

$$M_i = 1 \quad \text{and} \quad K_i = (\beta_i L)^4 EI/(\rho L^4), \tag{36}$$

where $\beta_i L$ satisfies the following transcendental equation:

$$\cos\beta_i L\cosh\beta_i L = -1. \tag{37}$$

To illustrate the proposed approach of imposing a single node or multiple nodes during harmonic excitations, cases where the node and attachment locations are collocated and cases where they are not collocated will be thoroughly analyzed. In general, the number of modes N used in the expansion depends on the excitation frequency. For the range of excitation frequencies considered in the subsequent numerical examples, $N = 15$ to ensure the convergence of all the numerical results. In addition, the MATLAB routine *fsolve* will be used to find the required mass and stiffness parameters that induce nodes at the desired locations and that satisfy the constraints on the vibration amplitude of the masses.

3.1. Oscillators and node locations are collocated

Consider a uniform simply supported Euler–Bernoulli beam of length L. For a given application, a node is desired at $x_n = 0.35L$, for a concentrated harmonic force of amplitude F, an

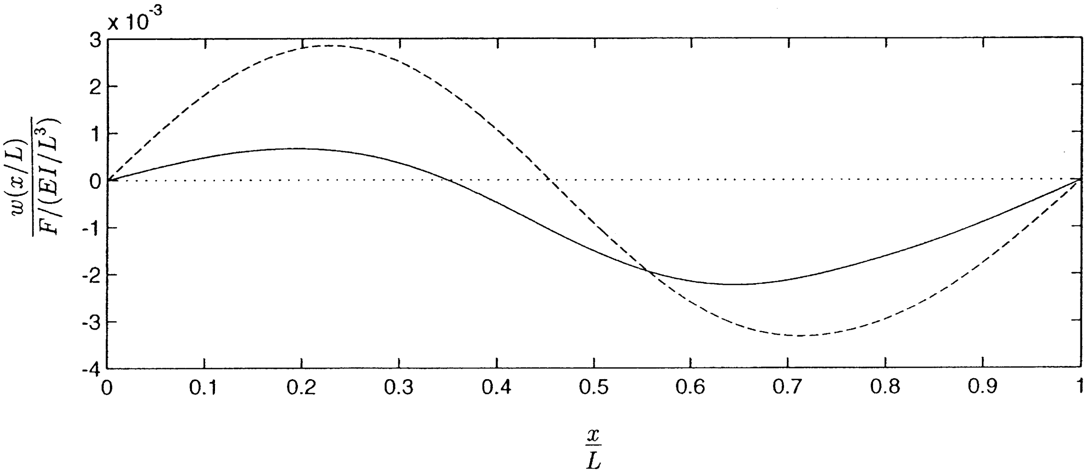

Fig. 2. The steady state deformed shapes of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The horizontal line represents the configuration of the undeformed beam. The system parameters are $\omega = 47\sqrt{EI/(\rho L^4)}$, $x_f = 0.77L$ and $x_a = 0.35L$. The attachment and node locations are collocated.

excitation frequency of $\omega = 47\sqrt{EI/(\rho L^4)}$ and an excitation location of $x_f = 0.77L$. Fig. 2 shows the steady state lateral displacement of the beam. The solid curve corresponds to the deformed shape of the beam with an oscillator attached at $x_a = 0.35L$, whose spring–mass parameters satisfy $k = m\omega^2$. The dotted line corresponds to the deformed shape of the beam with no oscillator, and the horizontal line represents the configuration of the undeformed beam. Note that by attaching an oscillator with a properly chosen set of system parameters, its attachment location, in this case $x_a = 0.35L$, becomes a node, as long as $k = m\omega^2$. Physically, this implies an infinite number of spring–mass combinations can be used to meet the design objective of inducing a node at x_a. In practice, the vibration amplitude of the oscillator mass, $\bar{z}_{max}$, constitutes an important constraint that must be satisfied. Fig. 3 is a design plot that shows $\bar{z}_{max}/(FL^3/(EI))$ as a function of $m/(\rho L)$. Knowing the allowable $\bar{z}_{max}$, the oscillator mass can then be properly chosen, which in turn dictates the required stiffness value, k. For example, suppose the physical constraint of the system requires that $\bar{z}_{max} \leqslant 0.1FL^3/(EI)$. Then from Fig. 3, the oscillator mass must be chosen such that $m \geqslant 3.2452 \times 10^{-3}\rho L$. The results of Fig. 3 are also consistent with Eq. (32). For $x_a = 0.35L$, $x_f = 0.77L$ and $\omega = 47\sqrt{EI/(\rho L^4)}$, Eq. (32) predicts $m\bar{z}_{max} = 3.2452 \times 10^{-4}F\rho L^4/(EI)$; numerically, the product of the required mass m (obtained by using *fsolve*) for any specified $\bar{z}_{max}$ is also found to be $m\bar{z}_{max} = 3.2452 \times 10^{-4}F\rho L^4/(EI)$.

As long as $k = m\omega^2$, a node will be induced at the attachment location. However, the actual choice of the oscillator parameters will affect the natural frequencies of the system, and the vibration amplitude of the oscillator mass. Table 1 shows the first six natural frequencies of two combined structures, each of which consists of a simply supported beam carrying an oscillator at $x_a = 0.35L$. For the first system, the oscillator parameters are $m = 0.01\rho L$, $k = 2.2090 \times 10^1 EI/L^3$, and for the second system they are $m = 0.002\rho L$, $k = 4.4180EI/L^3$. Note that both sets of oscillator parameters satisfy $k = m\omega^2$, where $\omega = 47\sqrt{EI/(\rho L^4)}$. Thus, while both systems will exhibit the same deflection shape of Fig. 2 when forced harmonically at $0.77L$ with an excitation

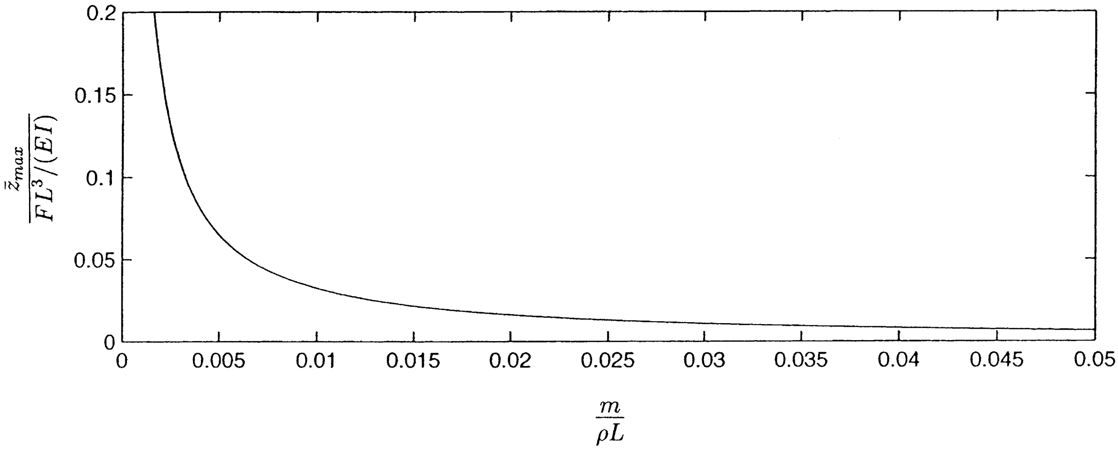

Fig. 3. The design plot of $\bar{z}_{max}/(FL^3/(EI))$ as a function of $m/(\rho L)$, for the system parameters of Fig. 2.

Table 1
The first six natural frequencies of a uniform simply supported beam carrying one oscillator at $x_a = 0.35L$

Natural frequency	System 1	System 2
ω_1	0.97886E+01	0.98533E+01
ω_2	0.38731E+02	0.39309E+01
ω_3	0.48238E+02	0.47268E+02
ω_4	0.88835E+02	0.88828E+02
ω_5	0.15805E+03	0.15794E+03
ω_6	0.24679E+03	0.24675E+03

For the first system, $(m,k) = (0.01\rho L, 22.090EI/L^3)$, and for the second system, $(m,k) = (0.002\rho L, 4.418EI/L^3)$. The natural frequencies are all non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

frequency of ω, they will have different natural frequencies (see Table 1). Moreover, for the first system, $(\bar{z}_{max})_1 = 3.2452 \times 10^{-2} FL^3/(EI)$, and for the second system, $(\bar{z}_{max})_2 = 1.6226 \times 10^{-1} FL^3/(EI)$.

Consider now a uniform cantilever beam. It is desired that two nodes be imposed, at $x_n^1 = 0.3L$ and $x_n^2 = 0.6L$, for a localized harmonic force applied at $x_f = 0.87L$, with a forcing amplitude of F, and an excitation frequency of $\omega = 57\sqrt{EI/(\rho L^4)}$. Fig. 4 illustrates the steady state deformed shape of the beam. Note that by attaching two properly tuned oscillators, with parameters $k_r = m_r\omega^2$, at $x_a^1 = 0.3L$ and $x_a^2 = 0.6L$, nodes are induced at these locations, which consequently leads to very little vibration in the region between 0 and $0.6L$. The selection of k_r and m_r is not unique, and the eventual choice is generally governed by the tolerable vibration amplitudes of the masses. Fig. 5 shows the design plots of $(\bar{z}_{max})_i/(FL^3/(EI))$ as a function of $m_i/(\rho L)$, where the solid and dotted lines correspond to $(\bar{z}_{max})_1$ and $(\bar{z}_{max})_2$, respectively. Interestingly, numerical experiments show that like the collocated, one oscillator and one node case, the product of m_i and $(\bar{z}_{max})_i$ is also a constant. For the given x_a^1, x_a^2, x_f and ω, $(m\bar{z}_{max})_1 = 2.6855 \times 10^{-5} F\rho L^4/(EI)$ and

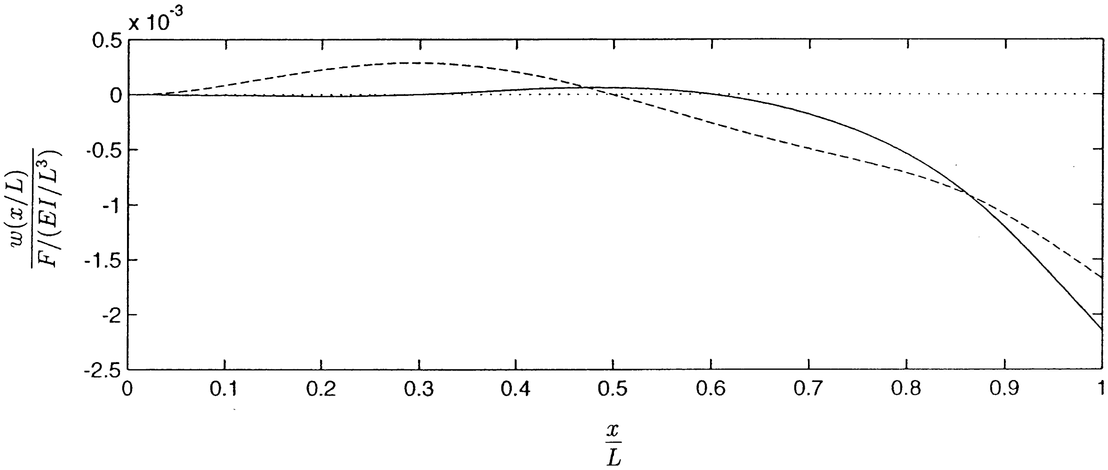

Fig. 4. The steady state deformed shapes of a uniform cantilever Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 57\sqrt{EI/(\rho L^4)}$, $x_f = 0.87L$, $x_a^1 = 0.3L$ and $x_a^2 = 0.6L$. The oscillator parameters, m and k, satisfy $k = m\omega^2$. The attachment and node locations are collocated.

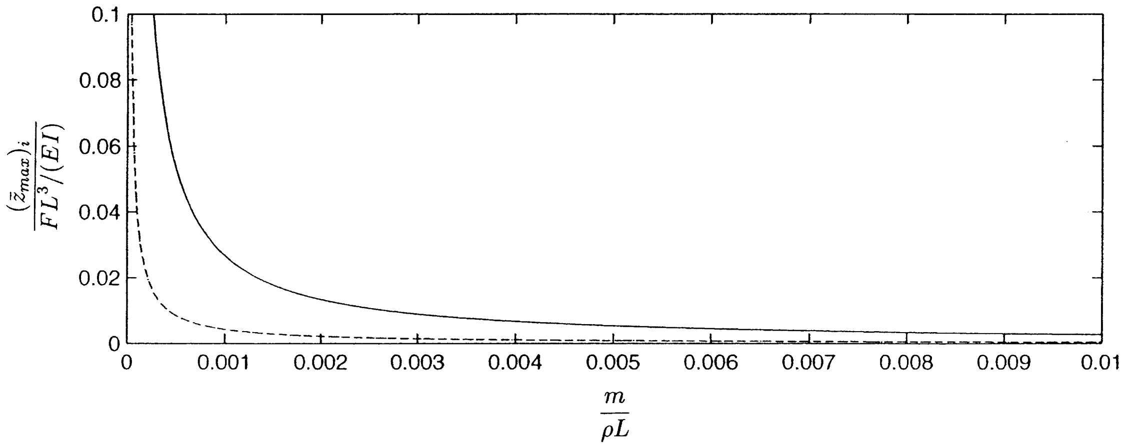

Fig. 5. The design plot of $(\bar{z}_{max})_i/(FL^3/(EI))$ as a function of $m_i/(\rho L)$, for the system parameters of Fig. 4. The solid and dotted lines correspond to $(\bar{z}_{max})_1/(FL^3/(EI))$ and $(\bar{z}_{max})_2/(FL^3/(EI))$, respectively.

$(m\bar{z}_{max})_2 = 4.3011 \times 10^{-6} F\rho L^4/(EI)$. Depending on the allowable vibration amplitudes, a unique set of masses can be selected, which in turn dictates the required stiffnesses, since $k_r = m_r\omega^2$.

3.2. Oscillators and node locations are not collocated

Consider a uniform cantilever beam, with a concentrated harmonic force of excitation frequency $\omega = 3.9\sqrt{EI/(\rho L^4)}$ applied at $x_f = 1.0L$. For a given application, it is desired to have a node at $x_n = 0.68L$. However, due to space constraint, an oscillator cannot be attached at that location but at some other point, say $x_a = 0.44L$. In this case, Eqs. (20)–(23) can be used, in conjunction with the tolerable vibration amplitude of the oscillator mass, to obtain the required spring–mass parameters in order to induce a node. Numerical experiments showed that the deformed shape depends only on x_a, x_n, x_f and ω, as long the k and m satisfy Eq. (20). Alternatively, the tolerable vibration amplitude, $\bar{z}_{max}$, affects only m and k, but does not alter the deformation of the beam. Fig. 6 shows the deformed shape of the uniform cantilever beam carrying an oscillator whose parameters satisfy Eq. (20), for the above set of x_a, x_n, x_f and ω. The excitation frequency is near the first natural frequency of a uniform cantilever beam, and the deformed shape of the combined structure resembles the first mode shape of a cantilever beam and is given by the dotted line. Note that deformed shape of the beam carrying the oscillator has a node at exactly $0.68L$ as desired, and the rest of the beam remains practically stationary compared to the cantilever beam with no oscillator. Fig. 7 is the design plot that shows $\bar{z}_{max}/(FL^3/(EI))$ versus $m/(\rho L)$. Knowing the tolerable vibration amplitude of the oscillator mass, its mass can be readily selected from Fig. 7. Like the collocated case, the product of m and $\bar{z}_{max}$ can be shown analytically (see Eq. (30)) and verified numerically to be a constant. For the chosen set of system parameters, $m\bar{z}_{max} = 2.4517 \times 10^{-1} F\rho L^4/(EI)$. The solid line of Fig. 8 shows the required mass as a function of a given spring stiffness, and the dotted line corresponds to $m = k/\omega^2$. Note that for small k's, the required mass and the corresponding stiffness is related approximately by $m \approx k/\omega^2$. This result will be used later to provide the initial guesses when calling *fsolve*.

Consider now a simply supported beam, with a concentrated harmonic force applied at $x_f = 0.77L$ and a forcing frequency of $\omega = 47\sqrt{EI/(\rho L^4)}$. A node is desired at $x_n = 0.28L$, and the attachment location is at $x_a = 0.64L$. Again, Eqs. (20)–(23) are used, in conjunction with the vibration amplitude constraint of the oscillator mass, to obtain the required spring–mass parameters in order to induce a node. Fig. 9 shows the deformed shape of the uniform simply supported beam carrying an oscillator whose parameters satisfy Eq. (20), for the specified x_a, x_n, x_f and ω. Note that deformed shape of the beam carrying the oscillator has a node at exactly $0.28L$, and the region between 0 and $0.30L$ experiences substantially less vibration comparing to

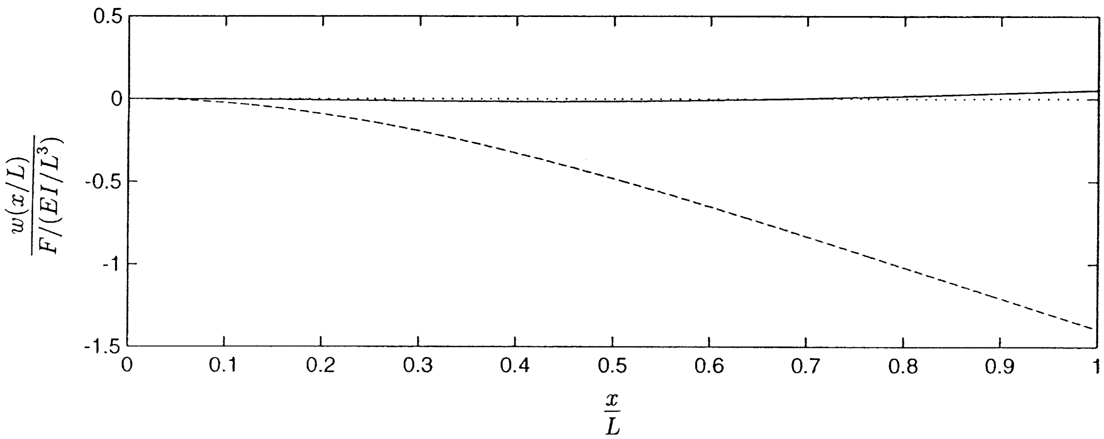

Fig. 6. The steady state deformed shapes of a uniform cantilever Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The system parameters are $\omega = 3.9\sqrt{EI/(\rho L^4)}$, $x_f = 1.0L$, $x_a = 0.44L$, and $x_n = 0.68L$. The oscillator parameters, m and k, are arbitrary but they satisfy Eq. (20). The attachment and node locations are not collocated.

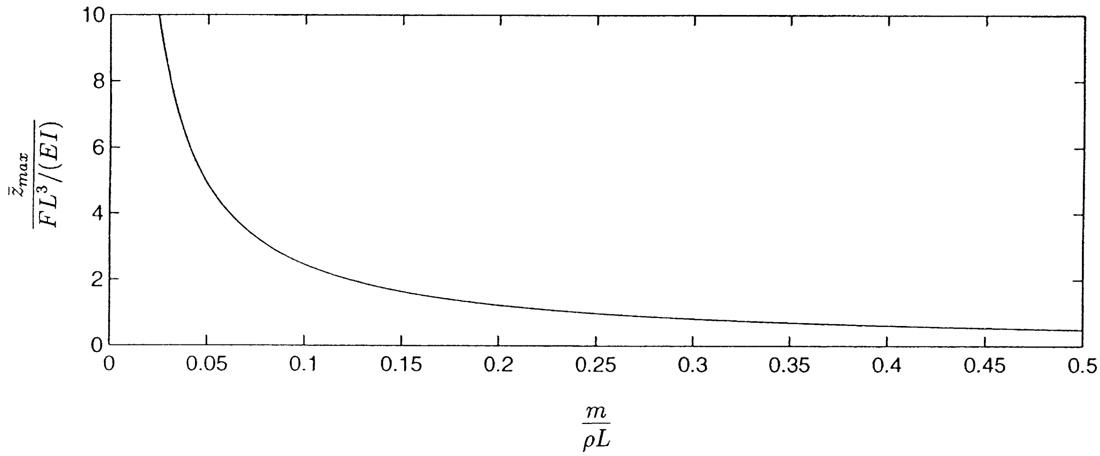

Fig. 7. The design plot of $\bar{z}_{max}/(FL^3/(EI))$ as a function of $m/(\rho L)$, for the system parameters of Fig. 6.

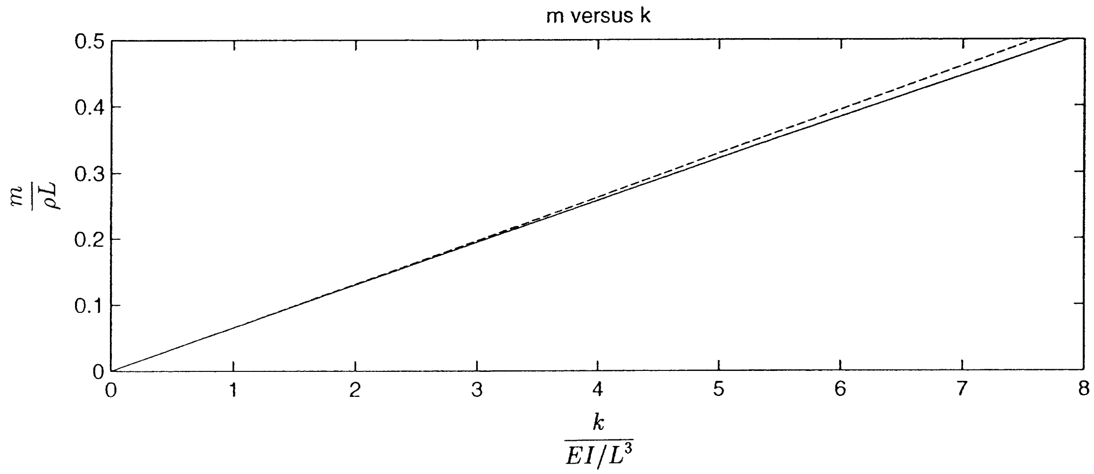

Fig. 8. The required mass, $m/(\rho L)$, as a function of a given spring stiffness, $k/(EI/L^3)$, for the system parameters of Fig. 6.

the beam with no oscillator. Fig. 10 is the design plot that shows $\bar{z}_{max}/(FL^3/(EI))$ versus $m/(\rho L)$, which can be used to select the required mass in order to meet the constraint of a specified oscillator vibration amplitude. For the given set of system parameters, $m\bar{z}_{max} = 7.4739 \times 10^{-4} F\rho L^4/(EI)$. The solid line of Fig. 11 shows the required mass as a function of a given spring stiffness, and the dotted line corresponds to $m = k/\omega^2$. Note again that for small k's, the required mass and the corresponding stiffness is related approximately by $m \approx k/\omega^2$.

For the multiple nodes, non-collocated case, the required oscillator parameters, the m_i and k_i, can be obtained by solving a total of $2S$ non-linear algebraic equations simultaneously, given by Eq. (13) and $|\underline{z}| = \bar{z}_{max}$ (see Eq. (15)). Numerically, the MATLAB routine *fsolve* can be used to

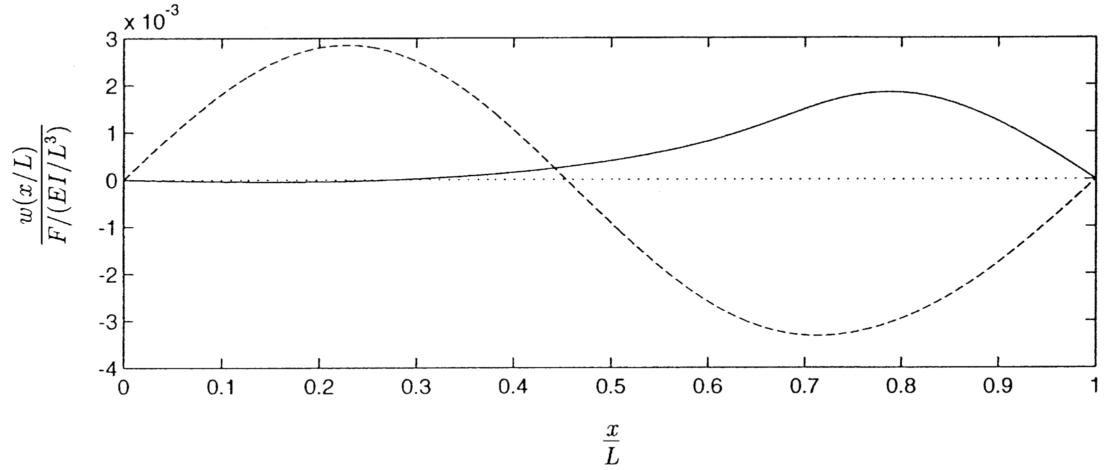

Fig. 9. The steady state deformed shapes of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The system parameters are $\omega = 47\sqrt{EI/(\rho L^4)}$, $x_f = 0.77L$, $x_a = 0.64L$, and $x_n = 0.28L$. The oscillator parameters, m and k, are arbitrary but they satisfy Eq. (20). The attachment and node locations are not collocated.

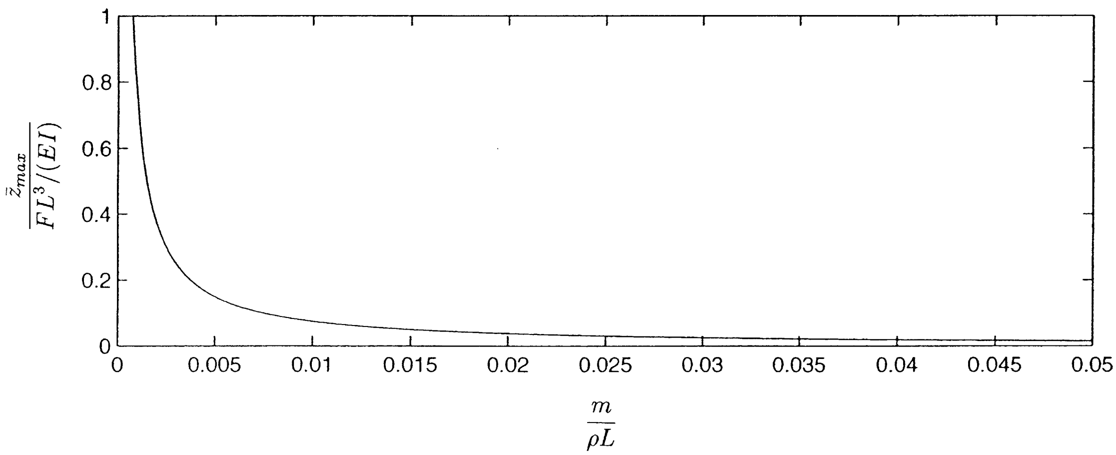

Fig. 10. The design plot of $\bar{z}_{max}/(FL^3/(EI))$ as a function of $m/(\rho L)$, for the system parameters of Fig. 9.

find the roots of these equations. However, numerical experiments showed that *fsolve* is highly sensitive to the set of initial guesses. Thus, unless an estimation of the solutions can be first established, convergence to the actual solution can be quite slow, and this direct approach of finding the required m_i and k_i can be computationally taxing. Fortunately, numerical experiments also showed that the deformed shape of the beam depends only on x_a^i, x_n^i, x_f and ω, as long as the m_i and k_i satisfy Eq. (13). Moreover, like the previous cases, the product $m_i(\bar{z}_{max})_i$ is also a function of x_a^i, x_n^i, x_f and ω only, and is independent of the k_i. Based on the above observations,

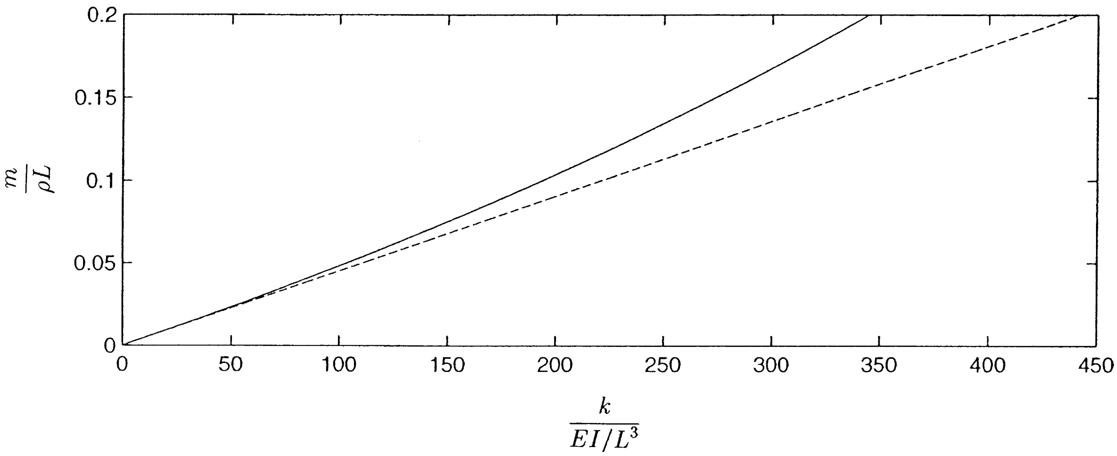

Fig. 11. The required mass, $m/(\rho L)$, as a function of a given spring stiffness, $k/(EI/L^3)$, for the system parameters of Fig. 9.

for a non-collocated case, the following procedure can be used to induce nodes at x_n^i for a given set of ω, x_a^i and x_f, subjected to the constraints of $|\bar{z}| = \bar{z}_{max}$:

1. Assume x_a^i, x_n^i, x_f and ω are specified. For any set of arbitrarily chosen k_i^*, solve Eq. (13) using *fsolve* for the required m_i^* such that x_n^i are nodes. To expedite convergence, use k_i^*/ω^2 as the initial guesses for the required masses.
2. For the given x_a^i, x_n^i, x_f, ω, k_i^* and m_i^*, compute the mass deflections, $\bar{z}_i^*$, using Eq. (15).
3. Determine the product of $|m_i^* \bar{z}_i^*| = m_i^*(\bar{z}_{max}^*)_i$, where $(\bar{z}_{max}^*)_i$ denotes the maximum vibration amplitude of the ith oscillator.
4. Because $m_i^*(\bar{z}_{max}^*)_i = m_i(\bar{z}_{max})_i =$ constant, knowing the desired $(\bar{z}_{max})_i$ (based on physical constraints), the required m_i can be readily determined.
5. Having found the required m_i in order to satisfy the maximum deflection constraints, the required k_i can be found by solving Eq. (13) using *fsolve* to satisfy the location of nodes constraints. To expedite convergence, use $m_i\omega^2$ as the initial guesses for the required stiffnesses.
6. Finally, for the solution to be physically meaningful, both m_i and k_i must be positive. If any of these physical parameters turn out to be negative, this implies that nodes cannot be induced at x_n^i for the specified x_a^i, x_f and ω. In this case, the attachment locations, x_a^i, can be varied, and the proposed procedure repeated until physically realizable quantities of m_i and k_i are obtained.

For a given set of x_a^i, x_n^i, x_f, ω and specified $(\bar{z}_{max})_i$, the required masses m_i are found algebraically once the product of the ith mass and its maximum vibration amplitude is known. Thus, the proposed procedure requires only solving Eq. (13) for the unknown k_i instead of the solving Eqs. (13) and (15) ($|\bar{z}| = \bar{z}_{max}$) simultaneously for the unknown m_i and k_i, effectively reducing the number of non-linear algebraic equations that need to be solved by half. Moreover, once the m_i are known, the initial guesses for the k_i (required for *fsolve*) are simply $m_i\omega^2$. Hence, the proposed procedure of determining the m_i and k_i offers substantial computational savings over the direct approach of solving a set of $2S$ non-linear algebraic equations.

Fig. 12 shows the steady state deformed shape of a uniform cantilever beam excited harmonically at $x_f = 1.0L$ with $\omega = 65\sqrt{EI/(\rho L^4)}$. Nodes are desired at $0.8L$ and $1.0L$, and oscillators are attached at $x_a^1 = 0.55L$ and $x_a^2 = 0.83L$. Arbitrarily choosing $k_1^* = 40EI/L^3$ and $k_2^* = 25EI/L^3$, the initial guesses for the m_i^* are then k_1^*/ω^2 and k_2^*/ω^2. Solving Eq. (13) using *fsolve* (with the aforementioned initial guesses) returns $m_1^* = 9.4213 \times 10^{-3}\rho L$ and $m_2^* = 5.9026 \times 10^{-3}\rho L$. For this set of oscillator parameters, $m_1^*(\bar{z}_{max}^*)_1 = 1.1008 \times 10^{-3}F\rho L^4/(EI)$, and $m_2^*(\bar{z}_{max}^*)_2 = 6.6870 \times 10^{-4}F\rho L^4/(EI)$. Knowing the product of $m_i^*(\bar{z}_{max}^*)_i$, the previously mentioned algorithm can be used to find the desired m_i and k_i that satisfy the tolerable vibration amplitude of the oscillator masses. Table 2 shows the maximum vibration amplitudes of the oscillator masses, along with the required m_i and k_i. In all the cases considered, the beam deflection shape is identical to that of Fig. 12. In addition, note that $m_i(\bar{z}_{max})_i$ remains unchanged for any choice of k_i as long as m_i and k_i satisfy Eq. (13). Moreover, observe that the k_i and m_i are related approximately by $k_i \approx m_i\omega^2$, justifying the earlier statement that $m_i\omega^2$ can be used as initial

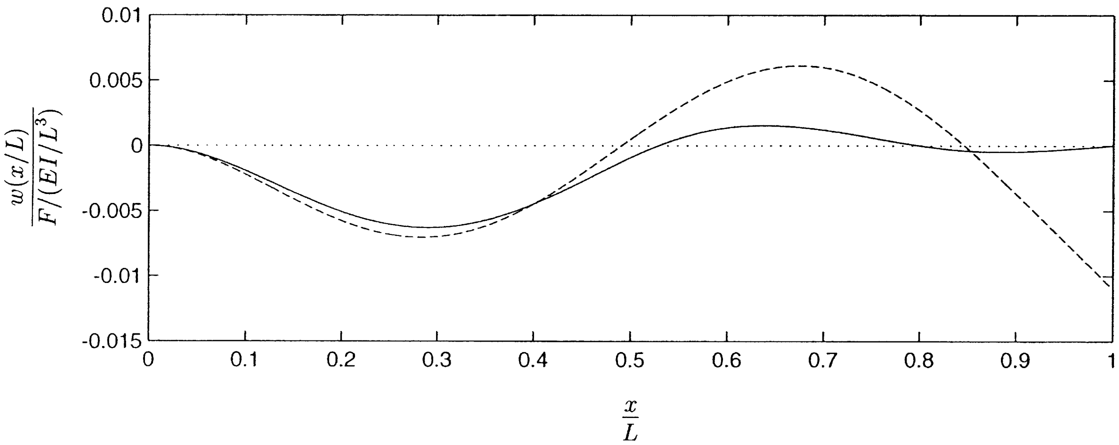

Fig. 12. The steady state deformed shapes of a uniform cantilever Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 65\sqrt{EI/(\rho L^4)}$, $x_f = 1.0L$, $x_a^1 = 0.55L$, $x_a^2 = 0.83L$, $x_n^1 = 0.8L$, $x_n^2 = 1.0L$. The oscillator parameters, m and k, are arbitrary but they satisfy Eq. (13). The attachment and node locations are not collocated.

Table 2
Combinations of m_i and k_i that satisfy the specified dimensionless vibration amplitudes of the oscillator masses, $(\bar{z}'_{max})_i = (\bar{z}_{max})_i/(FL^3/(EI))$, for the case of $\omega = 65\sqrt{EI/(\rho L^4)}$, $x_f = 1.0L$, $x_a^1 = 0.55L$, $x_a^2 = 0.83L$, $x_n^1 = 0.8L$ and $x_n^2 = 1.0L$

$(\bar{z}'_{max})_1$	$(\bar{z}'_{max})_2$	$m_1/(\rho L)$	$m_2/(\rho L)$	$k_1/(EI/L^3)$	$k_2/(EI/L^3)$
0.01	0.01	1.1008×10^{-1}	6.6870×10^{-2}	4.9321×10^{2}	2.9064×10^{2}
0.15	0.08	7.3389×10^{-3}	8.3587×10^{-3}	3.1125×10^{1}	3.5439×10^{1}
0.23	0.19	4.7862×10^{-3}	3.5195×10^{-3}	2.0272×10^{1}	1.4892×10^{1}
0.09	0.31	1.2231×10^{-2}	2.1571×10^{-3}	5.2007×10^{1}	9.1219×10^{0}
0.18	0.46	6.1157×10^{-3}	1.4537×10^{-3}	2.5921×10^{1}	6.1456×10^{0}

Table 3
The first six natural frequencies of a uniform cantilever beam (system 1) and a uniform cantilever beam carrying two oscillators at $x_a^1 = 0.55L$ and $x_a^2 = 0.83L$ (system 2)

Natural frequency	System 1	System 2
ω_1	0.35160E+01	0.31730E+01
ω_2	0.22035E+02	0.19898E+02
ω_3	0.61697E+02	0.59184E+02
ω_4	0.12090E+03	0.67382E+02
ω_5	0.19986E+03	0.77932E+03
ω_6	0.29856E+03	0.12698E+03

The sprung masses attached at x_a^1 and x_a^2 have the following parameters respectively: $(m_1, k_1) = (1.1008 \times 10^{-1}\rho L, 4.9321 \times 10^2 EI/L^3)$ and $(m_2, k_2) = (6.6870 \times 10^{-2}\rho L, 2.9064 \times 10^2 EI/L^3)$. The natural frequencies are all non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

guesses when solving for k_i using *fsolve*. Note that for all the combinations of system parameters, the displacements at $0.8L$ and $1.0L$ indeed become zero, leading to very little vibration of the beam between those two points, even though the localized force is applied at $x_f = 1.0L$.

The number of nodes that are eventually induced depends on the excitation frequency. For the example of Fig. 12, while only two node locations are specified, a third node also appears at approximately $0.52L$. Dowell [12] noted that if a spring–mass system is attached to another system, the natural frequencies that were originally higher than the spring–mass natural frequency are increased, those that were originally lower are decreased, and a new natural frequency appears between the original pair of frequencies nearest the oscillator natural frequency. Table 3 shows the first six natural frequencies of a uniform cantilever beam and a uniform cantilever beam carrying two oscillators, attached at $x_a^1 = 0.55L$ and $x_a^2 = 0.83L$, with parameters $(m_1, k_1) = (1.1008 \times 10^{-1}\rho L, 4.9321 \times 10^2 EI/L^3)$ and $(m_2, k_2) = (6.6870 \times 10^{-2}\rho L, 2.9064 \times 10^2 EI/L^3)$ respectively. For the set of spring–mass parameters chosen, two natural frequencies appear between the third and fourth natural frequencies of a uniform cantilever beam. Note that the excitation frequency $\omega = 65\sqrt{EI/(\rho L^4)}$ is near the fourth natural frequency of the combined system, and consequently, a total of three nodes will be induced, but only the nodes at $x_n^1 = 0.8L$ and $x_n^2 = 1.0L$ are imposed as design constraints.

A simple approach has been developed to solve the inverse problem of imposing nodes at multiple locations along any arbitrarily supported elastic structure that is subjected to a localized harmonic excitation, under the constraints of maximum allowable absorber amplitudes. The desired node locations and the vibration amplitudes of the oscillator masses give rise to a total of $2S$ equations that must be solved simultaneously. Using the proposed procedure, only S non-linear algebraic equations need to be solved. In addition, a guideline is provided for the selection of the initial guesses that are required when using *fsolve*. The ability to induce nodes has practical benefits because it allows certain points along the structure to remain stationary without using any rigid supports, and it enables certain regions of the structure to undergo very small deflections, thereby suppressing vibration in those sections.

4. Conclusions

Elastically mounted masses are used as a means to impose a single or multiple nodes on any elastic structure during harmonic excitations. The vibration amplitudes of these masses are imposed as constraints to make the solution scheme more practical. When the parameters of the sprung masses are properly chosen, nodes can always be induced at the attachment locations for any excitation frequency and excitation location. When the attachment and the node locations are not collocated, it is only possible to induce a node or multiple nodes at certain locations along the structure. In addition, if the node locations are properly selected, a region of nearly zero amplitudes can be imposed along the elastic structure for a given localized harmonic force without using any rigid supports, effectively quenching vibration in that segment of the structure. A detailed procedure to assist in the selection of the attached spring–mass systems, subjected to the constraints of tolerable vibration amplitudes of the oscillators, was outlined, and numerical experiments were performed to validate the utility of the proposed scheme of imposing a single or multiple nodes during harmonic excitations for the collocated and non-collocated cases.

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