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Application of the pothole DAF method to vehicles traversing periodic roadway irregularities

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Abstract

This paper is a sequel to the work discussed in Pesterev et al. (Journal of Sound and Vibration, in press). In that paper, it was suggested that the technique to determine the effect of a local road surface irregularity on the dynamics of a vehicle modelled as a linear multi-degree-of-freedom system relies on the so-called pothole dynamic amplification factor (DAF), which is a complex-valued function specific to the irregularity shape. This paper discusses the companion problem of how to determine the DAF function for an irregularity represented as a superposition of simpler ones. Another purpose of this paper is to demonstrate the application of the pothole DAF functions technique to finding a priori estimates of the effect of irregularities with a repeated structure. Specifically, we solve the problem of finding the conditions under which the dynamic effect of two identical potholes located one after another is greater than that due to the single pothole. We also find the estimate for the number of periods of a periodic irregularity that are sufficient in order to consider the oscillator response as steady state. The discussions are illustrated by numerical examples.

1. Introduction

In the companion paper [1], a technique has been developed to determine the effect of a local road surface irregularity on the dynamics of a moving vehicle modelled as a general linear

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multi-degree-of-freedom (m.d.o.f.) system. The problem has been shown to reduce to solving an uncoupled system of first order complex differential equations in modal co-ordinates. For an irregularity described functionally, solutions of all equations are found analytically and expressed in terms of a function of one complex variable, the so-called *pothole dynamic amplification factor* (DAF), defined in Ref. [1, Eq. (13)]. Thus, to find the contact forces (to be more specific, Fourier coefficients of these forces) arising after passing an irregularity, one needs the DAF of this irregularity. Pothole DAF function $\Phi(\gamma)$ is specific to the pothole shape; mathematically, it is, up to a scalar factor, the complex amplitude of a modal oscillator immediately after passing the pothole. All parameters—oscillator eigenfrequency, vehicle speed, and pothole width—affecting the response are combined in one complex variable γ.

The DAF functions for several typical potholes have been derived in the appendix to Ref. [1]. Clearly, those potholes do not exhaust all possible irregularity configurations that are likely to be met in practice. On the other hand, with any desired degree of accuracy, an arbitrary irregularity can be represented as a superposition of simpler potholes/bumps, for which the DAF functions are available or can easily be found. Thus, the first objective of this paper is to develop the technique for constructing the DAF function of a complex irregularity composed of simpler (“elementary”) ones by the DAF functions of the latter. In this paper, we consider a particular case of complex local roadway irregularities; namely, those represented by periodic functions on finite intervals. Such an irregularity can be represented as a superposition of elementary potholes defined on the interval of length equal to the period of the function. The DAF functions for such irregularities can be obtained in exactly the same way as in Ref. [1]. We, however, will consider another approach, which can be applied not only to periodic irregularities but also to arbitrary non-periodic ones, and establish the equation for the DAF function of a periodic irregularity in terms of the DAF of the corresponding “elementary” pothole. This problem is considered in Section 2.

It is evident that the dynamic effect of several potholes located on the roadway, one after another, can be either greater or less than that of a single pothole, depending on the pothole width, speed, and vehicle parameters. Then, of interest are the conditions under which each succeeding pothole amplifies the dynamic effect of the preceding ones. This is the second objective of the paper, which is discussed in Section 3.

The third objective of the paper is as follows: Due to damping inherent in a vehicle, the response of the vehicle traversing a finite periodic irregularity of sufficient length may be considered as steady state. In many cases, this fact considerably simplifies the analysis. Indeed, let the response of a given vehicle become almost steady state after passing N periods of a periodic irregularity. Then, rather than considering longer finite irregularities, we examine the infinite profile and consider steady state solutions, which are usually much simpler than transient ones. To do this, we need to know how many periods of a given profile result in a steady state response for a given vehicle. This problem is dealt with in Section 4.

The results obtained in Sections 2–4 are applied to profiles described by the sine function, which is discussed in Section 5. It should be noted that, although solutions of all uncoupled equations in the modal state space for a given irregularity are expressed in terms of one DAF function $\Phi(\gamma)$, the complex variable γ takes different values for different modal oscillators. Hence, estimates obtained in Sections 3–5 are valid for the modal oscillators (i.e., separate vehicle eigenvibrations) rather than for the vehicle as a whole. For example, two potholes located one after another can increase the response of one modal oscillator compared to the case of one pothole and reduce the response of another. In view of this remark, we confine our illustrations in Section 6 to those related to a single-degree-of-freedom (s.d.o.f.) oscillator.

2. Dynamic amplification factor functions for periodic irregularities

We will consider irregularities described by continuous periodic functions $r(x)$ with period b defined on an interval of length Nb, where N is the number of periods, and satisfying the conditions $r(x_0) = r(x_0 + Nb) = 0$,[1] where x_0 is the left end of the irregularity. Such an irregularity may be viewed as consisting of N elementary potholes, where, as in Ref. [1], the elementary pothole is described by its width b, depth a, and a shape function $\tilde{r}(x)$. Since, in the examples below, we consider irregularities described by trigonometric functions, it is convenient to assume that the domain of the shape functions is $[0, 2\pi]$. For notational simplicity, we assume that $x_0 = 0$ and extend the definition of the shape function to the domain $[0, 2\pi N]$, so that $r(x) = a\tilde{r}(x/b)$ for $x \in [0, Nb]$.

Let $\Phi(\gamma)$ be the dynamic amplification factor for an elementary pothole (see Ref. [1, Eq. (A.2)]), where $\gamma = \lambda b/(2\pi v)$, v is the vehicle speed, $\lambda = \alpha + i\omega$ is a complex parameter, and i is the imaginary unit. Physically, λ is a complex eigenfrequency of a modal oscillator, which implies that $\alpha \leqslant 0$. Hence, we may consider only γ with non-positive real parts, $\mathrm{Re}\,\gamma \leqslant 0$. Denote by $\Phi_N(\gamma)$ the dynamic amplification factor for the periodic irregularity consisting of N elementary potholes. The function $\Phi_N(\gamma)$ is defined similar to $\Phi(\gamma)$ as

$$\Phi_N(\gamma) = \gamma e^{2\pi N\gamma} \int_0^{2\pi N} e^{-\gamma\xi} \tilde{r}(\xi)\, d\xi. \tag{1}$$

Representing the integral on the right-hand side of this equation as the sum of two integrals, we obtain

$$\Phi_N(\gamma) = \gamma e^{2\pi N\gamma} \left(\int_0^{2\pi} e^{-\gamma\xi} \tilde{r}(\xi)\, d\xi + \int_{2\pi}^{2\pi N} e^{-\gamma\xi} \tilde{r}(\xi)\, d\xi \right).$$

Changing the integration variable in the second integral, $\eta = \xi - 2\pi$; taking into account that the function $\tilde{r}(\xi)$ is periodic with period equal to 2π; and simplifying, we get

$$\begin{aligned} \Phi_N(\gamma) &= \gamma e^{2\pi N\gamma} \left(\int_0^{2\pi} e^{-\gamma\xi} \tilde{r}(\xi)\, d\xi + e^{-2\pi\gamma} \int_0^{2\pi(N-1)} e^{-\gamma\eta} \tilde{r}(\eta)\, d\eta \right) \\ &= e^{2\pi(N-1)\gamma} \gamma e^{2\pi\gamma} \int_0^{2\pi} e^{-\gamma\xi} \tilde{r}(\xi)\, d\xi + \gamma e^{2\pi(N-1)\gamma} \int_0^{2\pi(N-1)} e^{-\gamma\eta} \tilde{r}(\eta)\, d\eta. \end{aligned}$$

[1] Note that the latter requirement is not necessary and is assumed to simplify the equations that follow. The case where $r(x_0 + Nb) \neq r(x_0)$ implies that the flat segments of the road before and after the irregularity are at different grades. In this case, the dynamic amplification factor function can be modified to take into account the difference in levels. All resulting equations remain valid if we substitute the modified DAF function for the DAF introduced in Ref. [1].

Comparing this with the definitions of the functions $\Phi(\gamma)$ [1, Eq. (A.2)] and $\Phi_N(\gamma)$ Eq. (1), we get the following recurrence relation for $\Phi_N(\gamma)$:

$$\Phi_N(\gamma) = \mathrm{e}^{2\pi(N-1)\gamma}\Phi(\gamma) + \Phi_{N-1}(\gamma). \quad (2)$$

The first term on the right-hand side of Eq. (2) shows the increment in the dynamic amplification factor due to extending the $(N-1)$-period irregularity by one period. For damped oscillators, $\mathrm{Re}\,\gamma < 0$, and the first term vanishes as N goes to infinity.

It can be verified directly by substituting $N-1,\ N-2, \ldots$ for N into the recurrence relation (2) that $\Phi_N(\gamma)$ can be written as

$$\Phi_N(\gamma) = \frac{1-\mathrm{e}^{2\pi N\gamma}}{1-\mathrm{e}^{2\pi\gamma}}\,\Phi(\gamma). \quad (3)$$

Denoting by $\Phi_\infty(\gamma)$ the dynamic amplification factor for an infinite periodic pothole composed of the specified elementary pothole and letting N in Eq. (3) go to infinity, we obtain

$$\Phi_\infty(\gamma) = \frac{\Phi(\gamma)}{1-\mathrm{e}^{2\pi\gamma}}. \quad (4)$$

As can be seen, the dynamic amplification factor for an infinite periodic profile may cease to exist only when $\mathrm{e}^{2\pi\gamma} = 1$, which holds when $\gamma = n\mathrm{i}$, i.e., when γ is a purely imaginary number. If we confine our consideration to damped modal oscillators, the DAF function exists for any infinite periodic profile.

The next two sections demonstrate the application of the DAF functions technique to finding some a priori estimates, which are valid for potholes of arbitrary shape. First, we consider an irregularity consisting of two elementary potholes located one after another and study when such a configuration results in greater contact forces compared to the case of the single pothole.

3. Two successive potholes

As follows from Eq. (30) [1] governing the amplitude of the harmonic components of the contact forces, for a fixed vehicle model, the greater the magnitude of the DAF function for a given γ, the greater the value of the force. Thus, in order to learn whether two successive elementary potholes result in a greater force, we need to compare two DAF functions, $\Phi(\gamma)$ and $\Phi_2(\gamma)$. From Eq. (3), we have $|\Phi_2(\gamma)| = |\Phi(\gamma)|\sigma(\gamma)$, where $\sigma(\gamma) = |1+\mathrm{e}^{2\pi\gamma}|$. Thus, for a given γ, the contact force due to the two potholes is greater (less) than that due to one pothole when $\sigma(\gamma)$ is greater (less) than one.

Noting that

$$|\gamma| = \frac{\sqrt{\alpha^2+\omega^2}b}{2\pi v} \equiv \frac{\omega_0 b}{2\pi v}, \quad (5)$$

where ω_0 is the undamped oscillator eigenfrequency, we can write γ as

$$\gamma = -|\gamma|\zeta + \mathrm{i}|\gamma|\sqrt{1-\zeta^2}, \quad (6)$$

where $\zeta = -\alpha/\alpha_{cr}$ is the modal damping coefficient. In subsequent analyses, we will use two real variables $|\gamma|$ and ζ, which have clear physical meaning, instead of one complex variable γ.

Substituting Eq. (6) into the equation for $\sigma(\gamma)$, we obtain

$$\sigma(\gamma) = |1 + R\mathrm{e}^{\mathrm{i}\varphi}| \equiv \sqrt{1 + R^2 + 2R\cos\varphi}, \quad (7)$$

where $R = \mathrm{e}^{-2\pi|\gamma|\zeta}$ and $\varphi = 2\pi|\gamma|\sqrt{1-\zeta^2}$. As can be seen from Eq. (7), $\sigma(\gamma)$ is the distance from the origin to a complex number lying on the circle of radius R centered at the point $x = 1$ of the real axis. For small $|\gamma| \ll 1$, $R(\gamma) \approx 1$ ($R(0) = 1$), φ is close to zero, and, thus, $\sigma(\gamma) \approx 1 + R \approx 2$, i.e., the second pothole doubles the dynamic effect of the first pothole. However, the range of small $|\gamma|$ is not very interesting since, for any local pothole, $\Phi(0) = 0$ (which follows from physical considerations) such that both $\Phi(\gamma)$ and $\Phi_2(\gamma)$ are small in this range. As $|\gamma|$ grows, φ increases and R decreases, i.e., the point $1 + \mathrm{e}^{2\pi\gamma}$ spirals counterclockwise to the center $x = 1$ on the real axis. At a certain $|\gamma|$, $\sigma(\gamma)$ becomes less than one (which occurs when $\varphi = \pi/2 + \arcsin(R/2)$), takes its minimum value, grows again, becomes greater than one (when $\varphi = 3\pi/2 - \arcsin(R/2)$), and so on. Taking into account that $0 < R \leqslant 1$, it is easy to show that $\sigma(\gamma)$ is, for sure, greater than one when $|\gamma|\sqrt{1-\zeta^2} \leqslant 1/4$ or $3/4 \leqslant |\gamma|\sqrt{1-\zeta^2} \leqslant 5/4$. When $1/3 \leqslant |\gamma|\sqrt{1-\zeta^2} \leqslant 2/3$ or $4/3 \leqslant |\gamma|\sqrt{1-\zeta^2} \leqslant 5/3$, $\sigma(\gamma)$ is certainly less than one. Noting that $\omega_0\sqrt{1-\zeta^2} = \omega$ (damped oscillator eigenfrequency), denoting $\omega_p = 2\pi v/b$, and plotting the ratio ω/ω_p on the real axis, we can summarize the above results as follows:

$$\text{if } \omega/\omega_p \in \left(0, \tfrac{1}{4}\right], \left[\tfrac{3}{4}, 1\tfrac{1}{4}\right], \left[1\tfrac{3}{4}, 2\tfrac{1}{4}\right] \ldots, \text{then } \sigma > 1,$$

$$\text{if } \omega/\omega_p \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right], \left[1\tfrac{1}{3}, 1\tfrac{2}{3}\right], \left[2\tfrac{1}{3}, 2\tfrac{2}{3}\right] \ldots, \text{then } \sigma < 1.$$

It does not make sense to consider greater values of ω/ω_p since R exponentially vanishes, and we can set $\sigma = 1$ without any loss of accuracy. If the value of ω/ω_p falls into a gap between any two adjacent intervals, we can assume that the force after passing two potholes is approximately equal to that after passing the single pothole (or, if needed, to find an exact value of σ from Eq. (7)).

One can find better estimates, as well as determine approximate values of $|\gamma|$ for which σ takes its maximum and minimum values, by taking into account the known dependence of R on $|\gamma|$ and ζ. Here, we note only that, for large damping, we may set $\sigma = 1$ for all $|\gamma|$ on the strength of the fact that R is very small (although, for $|\gamma| \ll 1$, $R \approx 1$, the absolute error of this approximation is negligible by virtue of the fact that $|\Phi(\gamma)| \ll 1$). It also can be shown that, for small damping, $\sigma(\gamma)$ takes its minimum and maximum values, $\sigma_{min} \approx 1 - \mathrm{e}^{-\pi\zeta}$ and $\sigma_{max} \approx 1 + \mathrm{e}^{-2\pi\zeta}$, when $|\gamma| \approx 1/2 - \zeta/(2\pi)$ and $|\gamma| \approx 1 - \zeta/(2\pi)$, respectively. The above analysis implies that an *increase* (*reduction*) *in the contact force due to the second pothole depends on the oscillator eigenfrequency*, *speed*, *and pothole width*, *but does not depend on the pothole shape*.

Also of interest is the overall magnification factor of the second pothole,

$$\Sigma(\zeta) = \frac{\max_{|\gamma|} |\Phi_2(\gamma)|}{\max_{|\gamma|} |\Phi(\gamma)|} \equiv \frac{\max_{|\gamma|} |\sigma(\gamma)\Phi(\gamma)|}{\max_{|\gamma|} |\Phi(\gamma)|},$$

which shows how much the maximum contact force due to two potholes is greater than that due to one pothole. Generally, the maxima of $\sigma(\gamma)$ and $|\Phi(\gamma)|$ may occur at different values of $|\gamma|$; therefore, the analysis should rely on the particular form of $\Phi(\gamma)$. In Section 5, we give a more detailed analysis for the case of a "sine" pothole.

It is evident that, if the number of periods in a finite periodic irregularity is sufficiently large, the response of a vehicle traversing such an irregularity approaches the steady state response owing to damping inherent in the vehicle. In this case, the exact number of periods in a particular irregularity makes no difference, and we may use one pothole amplification factor function, e.g., $\Phi_\infty(\gamma)$ (which corresponds to the steady state vehicle response), for all lengthy irregularities. Thus, we arrive at the following question.

4. How many periods of a periodic irregularity make the oscillator response steady state?

For a given modal oscillator, we set the problem of finding the number N_{ss} of periods such that, for $N \geqslant N_{ss}$, the irregularity consisting of N periods results in an almost steady state response of the oscillator. To be more specific, we want to make sure that the replacement of the DAF function $\Phi_N(\gamma)$ by $\Phi_\infty(\gamma)$ when calculating the contact force results in a relative error less than a prescribed small number ε_0.

The amplitudes of the contact forces arising after passing the N-period and infinite irregularities are obtained by substituting the functions $\Phi_N(\gamma)$ and $\Phi_\infty(\gamma)$, respectively, for $\Phi(\gamma)$ into Eq. (30) in Ref. [1]. Then, it follows from this equation that the relative error in the force due to the replacement of one of these functions by another is equal to

$$\varepsilon = \frac{|\,|\Phi_\infty(\gamma)| - |\Phi_N(\gamma)|\,|}{|\Phi_\infty(\gamma)|}.$$

Substituting Eqs. (3) and (4) into the last equation and applying the triangle inequality, we obtain

$$\varepsilon = \frac{|\Phi(\gamma)|\,|1 - |1 - \mathrm{e}^{2\pi N\gamma}|\,|}{|\Phi(\gamma)|} \leqslant |1 - 1 + \mathrm{e}^{2\pi N\gamma}| = |\mathrm{e}^{2\pi N\gamma}|.$$

Similar to the factor σ (Section 3), the relative error of the approximation of the steady state response by that due to N periods of the irregularity *depends on the oscillator eigenfrequency, speed, and pothole width, but does not depend on the pothole shape.* Taking into account Eq. (6), we get the upper bound for the relative error

$$\varepsilon \leqslant \mathrm{e}^{-2\pi N|\gamma|\zeta}. \quad (8)$$

As can be seen, the relative error monotonically decreases as $|\gamma|$ and ζ increase. For $|\gamma| \ll 1$, the relative error may be sufficiently large; however, as has already been noted, the range of small $|\gamma|$ is of little interest (the absolute error in this range is small by virtue of the fact that $\Phi(\gamma)$ is negligibly small). Then, it seems natural to estimate the relative error in the range where $|\Phi(\gamma)|$ takes its maximum value. Let $|\gamma|_m$ be the maximizer of $|\Phi(\gamma)|$. Substituting it into Eq. (8), equating the right-hand side to ε_0, and taking the logarithm of both sides of the equality, we get the estimate for N_{ss},

$$N_{ss} = \left[-\frac{\ln \varepsilon_0}{2\pi|\gamma|_m\zeta} \right], \quad (9)$$

where the brackets denote rounding up to the nearest integer.

5. Harmonic irregularity

For an example of the above analysis, we consider a harmonic irregularity. For the "elementary pothole", we take the irregularity defined on the interval $[0, b]$ by the equation

$$r(x) = \tfrac{1}{2} a \sin \frac{2\pi x}{b}, \tag{10}$$

which is the combination of a "half-sine bump" and a "half-sine pothole". The DAF function for this pothole is derived similar to those in the appendix of the companion paper [1] and is given by

$$\Phi(\gamma) = -\frac{\gamma(1 - e^{2\pi\gamma})}{2(\gamma^2 + 1)}. \tag{11}$$

Substituting Eq. (11) into Eqs. (3) and (4), we obtain the DAF functions for the irregularity consisting of N sine waves and for the infinite harmonic profile

$$\Phi_N(\gamma) = -\frac{\gamma(1 - e^{2\pi N\gamma})}{2(\gamma^2 + 1)}, \qquad \Phi_\infty(\gamma) = -\frac{\gamma}{2(\gamma^2 + 1)}. \tag{12}$$

It is apparent that $\Phi_\infty(\gamma)$ exists for all γ except $\gamma = \mathrm{i}$. This condition occurs when the modal oscillator is undamped and its eigenfrequency ω_0 is equal to the frequency $\omega_p = 2\pi v/b$ of vertical oscillations of the attachment point moving along the harmonic profile at the speed v. The functions $\Phi(\gamma)$ and $\Phi_N(\gamma)$ exist for any γ, since the singularities on the right-hand sides of the corresponding equations at $\gamma = \mathrm{i}$ are resolvable, which can be checked directly.

Numerical experiments show that, for any fixed $0 \leqslant \zeta < 1$, the maximum of $|\Phi(\gamma)|$ is attained in the neighborhood of $|\gamma| = 1$. Substituting $|\gamma|_m = 1$ into Eq. (9), we get the improved estimate[2] for N_{ss},

$$N_{ss} = \left[-\frac{\ln \varepsilon_0}{2\pi\zeta} \right]. \tag{13}$$

Fig. 1 shows the dependence of N_{ss} on the modal damping coefficient for three values of ε_0: 0.05, 0.1, and 0.2. As can be seen, when $\zeta \to 0$, N_{ss} tends to infinity, which is expected since no steady state response exists in the undamped case. As ζ increases, N_{ss} rapidly decreases; the relative difference of the steady state response (infinite irregularity) and that due to 2 successive sine waves is less than 10% for $\zeta > 0.18$; for $\zeta > 0.25$, this difference is less than 5%. If the damping coefficient is higher ($\zeta > 0.35$ for $\varepsilon_0 = 10\%$ or $\zeta > 0.5$ for $\varepsilon_0 = 5\%$), we may consider the response due to just one "sine" pothole as steady state.

Let us derive the formula for the overall magnification factor of the second pothole $\Sigma(\zeta)$. Consider first the case of small (moderate) damping. As noted in Section 3, σ takes its maximum at $|\gamma| \approx 1 - \zeta/(2\pi)$, i.e., both $\sigma(\gamma)$ and $|\Phi(\gamma)|$ take their maxima in the neighborhood of $|\gamma| = 1$. Hence, the maximum of $|\Phi_2(\gamma)|$ is always greater than that of $|\Phi(\gamma)|$ by the factor of $1 + e^{-2\pi\zeta}$.

Now, consider the case of high damping. Substituting $|\gamma| = 1$ into Eq. (7), we obtain $\sigma = |1 + e^{-2\pi\zeta} e^{i\varphi}|$, where $\varphi = 2\pi\sqrt{1 - \zeta^2}$. Taking into account that, for large ζ, $e^{-2\pi\zeta} \ll 1$, we find that $\sigma \leqslant 1$ when, approximately, $\pi/2 \leqslant \varphi \leqslant 3\pi/2$, i.e., when $1/4 \leqslant \sqrt{1 - \zeta^2} \leqslant 3/4$, or $7/16 \leqslant \zeta \leqslant 15/16$.

[2] Note that Eq. (13) is essentially identical to Eq. (3.116) in Ref. [2], which gives an estimate of the number of cycles of an s.d.o.f. damped oscillator required to reach 95% of final resonant amplitude.

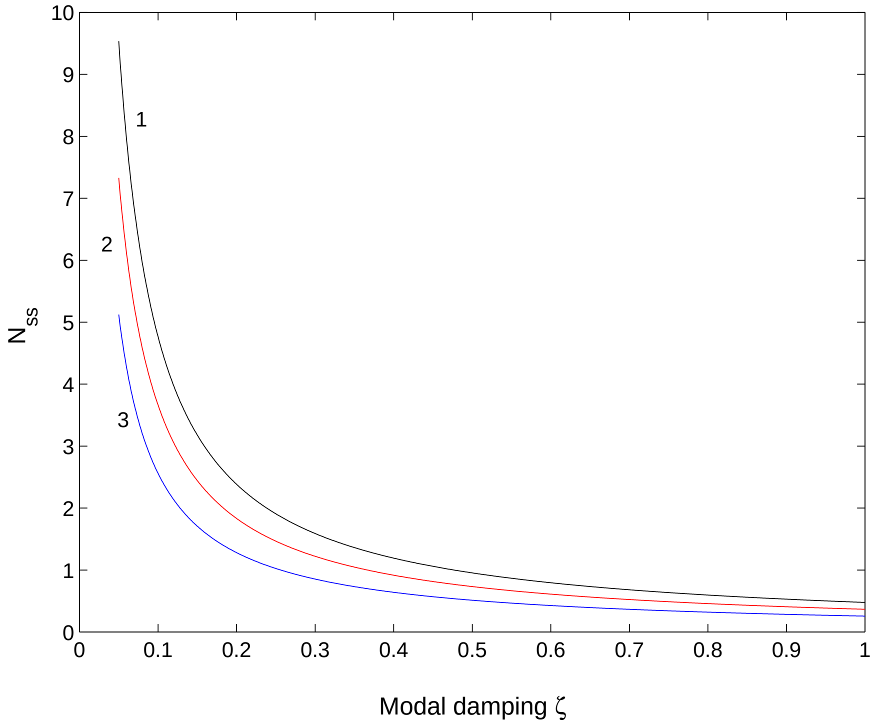

Fig. 1. Dependence of the number of waves of the sine irregularity that result in steady state response on the modal damping for different values of the desired accuracy: $\varepsilon_0 = 0.05$ (1), $\varepsilon_0 = 0.1$ (2), and $\varepsilon_0 = 0.2$ (3).

Hence, it follows that the magnification factor of the second pothole is less than one (reduction of the maximum contact force) when $0.66 \leqslant \zeta \leqslant 0.97$. Now, noting that, for such large damping, the factor $\sigma < 1$ is very close to one, we may neglect this reduction in the maximum contact force and consider that the second pothole does not change the force. Moreover, for the same reason, we may use the same formula for σ as in the case of light damping for the sake of simplicity. Then, the magnification factor of the second "sine" pothole is described by the following simple formula for all $0 \leqslant \zeta \leqslant 1$:

$$\Sigma(\zeta) \approx 1 + e^{-2\pi\zeta}. \tag{14}$$

Fig. 2 depicts the function on the right-hand side of Eq. (14) (solid line). For a given ζ, the ordinate of the curve shows by how much the maximum contact force arising after passage of two "sine" potholes by an oscillator with damping coefficient ζ is greater than that after passage of one pothole. For comparison, the dashed line depicts a more accurate dependence of the magnification factor on ζ given by the function $\sigma(\zeta) = |1 + e^{-2\pi\zeta} e^{i2\pi\sqrt{1-\zeta^2}}|$. As can be seen, the difference in the two curves is negligibly small, and, thus, the use of the simple approximate function (14) is quite justified.

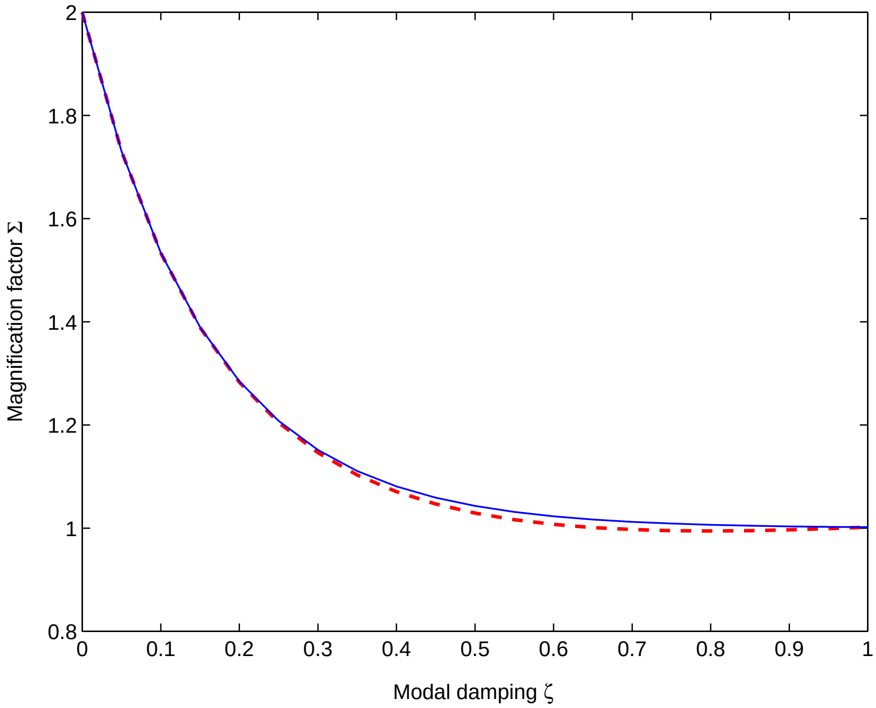

Fig. 2. Magnification factor of the second "sine" pothole vs. modal damping.

6. Examples

To illustrate the above discussions, we apply the technique discussed in the companion paper [1] to the evaluation of the contact forces acting on the road from an SDOF damped oscillator due to passing different irregularities composed of the "sine" pothole (10). Namely, we consider one pothole (10) located on the interval $[-b, 0]$ ($N = 1$), irregularities containing N waves of function (10) on the interval $[-Nb, 0]$, and an irregularity containing an infinite number of sine waves located on the interval $(-\infty, 0]$. In all cases, we are interested in the contact force arising immediately after the oscillator leaves the irregularity (assuming that it moves further along an even horizontal surface). After passing an irregularity, the oscillator freely vibrates, and the contact force can be represented as $F_c(t) = f_0 e^{\alpha t}\cos(\omega t + \psi)$. As discussed in Ref. [1], two quantities of major interest are "amplitude" f_0 and maximum of the contact force, which are calculated by Eqs. (33) and (34) in Ref. [1], respectively. Not to make the following figures messy, we show only plots of the amplitudes f_0. The y-axis in Figs. 3–5 show the dimensionless amplitudes f_0 of the contact forces, which are normalized by the static force ka, where k is the stiffness of the spring connecting the mass to the ground.

Fig. 3 shows results related to the oscillator with light damping, $\zeta = 10\%$. The dashed line depicts the dependence of the amplitude of the contact force after passing one pothole (10) on $|\gamma|$ (i.e., on the undamped eigenfrequency ω_0, speed v, and the pothole width b). The thin solid line corresponds to three potholes (10) located one after another, and the bold solid line corresponds

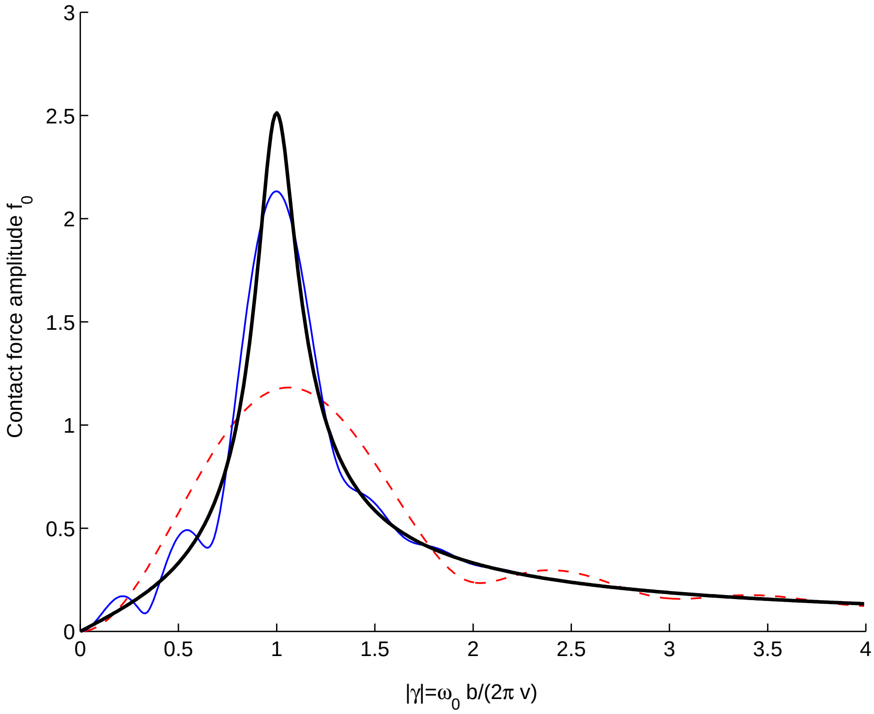

Fig. 3. Dependence of the amplitude of the normalized contact force for an oscillator with damping coefficient $\zeta = 10\%$ on the undamped eigenfrequency, speed, and b after passing (1) the sine pothole (dashed line), (2) three sine potholes (solid line), and (3) infinite harmonic profile (bold line).

to an infinite harmonic profile. As can be seen, the response grows considerably as N increases. The maximum response due to three successive potholes is still far from that due to an infinite harmonic profile (the steady state response). Only after passing five potholes can the response be well approximated by the steady state response (not shown in the figure).

The results shown in Fig. 4 correspond to the case of higher damping, $\zeta = 20\%$. The dashed line shows the amplitude of the contact force after passing one pothole (10), the thin solid line corresponds to two potholes (10), and the bold solid line corresponds to an infinite harmonic profile. Here, the response due to one pothole is again far from the steady state response, and the peak value of the latter is again considerably greater than that due to one pothole. However, the response due to two potholes is now pretty close to the steady state response, and that due to three potholes (not shown) can perfectly be approximated by the latter.

Finally, Fig. 5 corresponds to a heavily damped oscillator with $\zeta = 50\%$. The thin solid line shows the amplitude of the contact force due to one pothole (10), and the bold solid line, that due to an infinite harmonic profile. As can be seen, the dynamic effect of one pothole is almost the same as that of an infinite number of potholes, i.e., for this damping, the response almost achieves steady state after passing as little as one wave of the sine profile. Further increase in the damping leads to virtually no difference in the contact force due to one pothole and that due to infinite number of potholes.

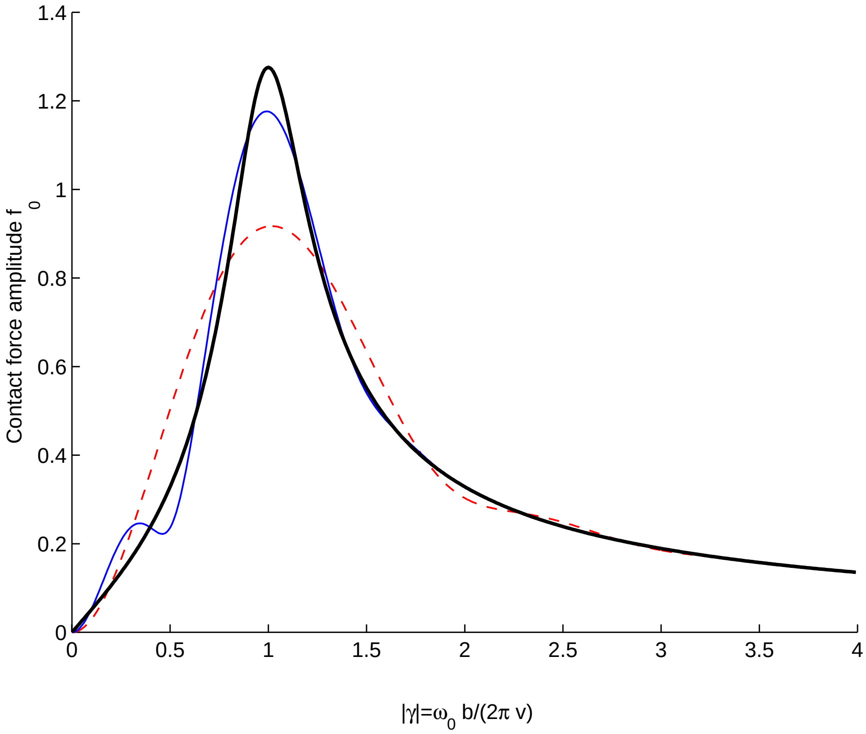

Fig. 4. Dependence of the amplitude of the normalized contact force for an oscillator with damping coefficient $\zeta = 20\%$ on the undamped eigenfrequency, speed, and b after passing (1) the sine pothole (dashed line), (2) two sine potholes (solid line), and (3) infinite harmonic profile (bold line).

Analysis of the plots depicted in Figs. 3–5 shows that they are in a good agreement with the theoretical estimates obtained in Sections 3–5 (see Figs. 1 and 2).

7. Conclusions

In the companion paper [1], an efficient method for calculating amplitudes (maxima) of the Fourier components of the tire forces arising after passing a local road surface irregularity by a general linear vehicle has been developed, which does not require numerical integration of the governing differential equations. All desired characteristics are calculated in terms of a pothole DAF, which is a function of one complex variable specific to the irregularity shape. The DAF functions can be calculated in advance for a number of typical road surface irregularities and then used with any vehicle model.

If an irregularity can be represented as a superposition of simpler ones, its DAF function can be obtained in terms of the DAF functions of the latter. This gives us an opportunity to easily find DAF functions of arbitrary irregularities of complicated shape, thus reducing the problem to construction of a library of DAF functions representing the simplest typical potholes. In this

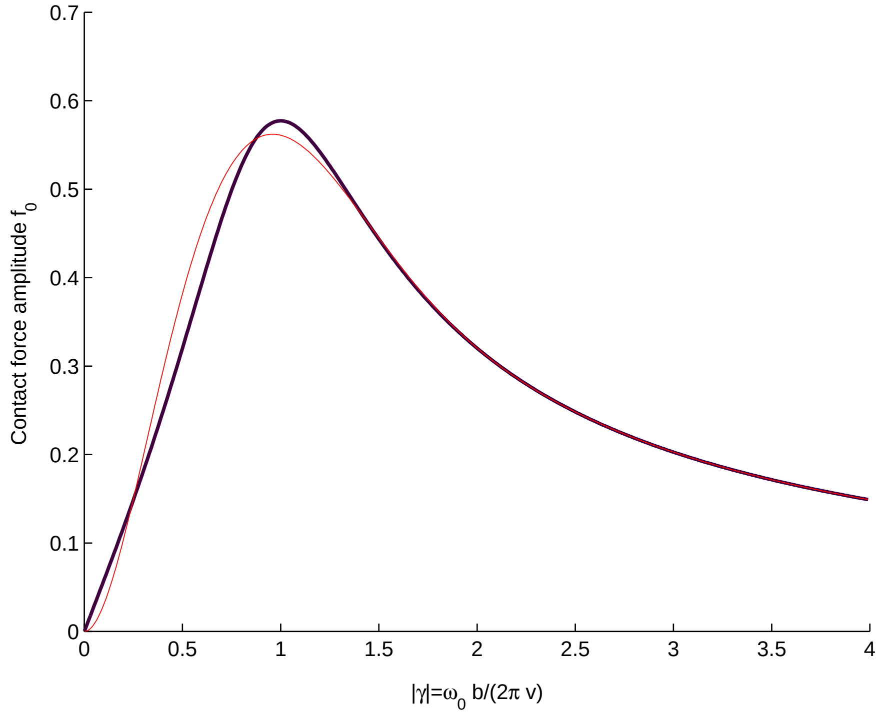

Fig. 5. Dependence of the amplitude of the normalized contact force for an oscillator with damping coefficient $\zeta = 50\%$ on the undamped eigenfrequency, speed, and b after passing (1) the sine pothole (thin line) and (2) infinite harmonic profile (bold line).

paper, the equation relating the DAF function of a periodic, infinite or finite, irregularity to the DAF function of the pothole defined on the interval equal to the period of the irregularity has been derived. The approach used is easily extended to complex non-periodic irregularities, thus giving an effective way of calculating DAF functions for arbitrary irregularity configurations.

The efficiency of the DAF functions technique has also been demonstrated by applying it to two following problems: (1) How to learn without modeling whether two successive identical potholes amplify a given vehicle eigenvibration at a given speed compared to the case of one pothole? (2) How many periods of a periodic irregularity are sufficient in order to consider the response of a modal oscillator as steady state?

Eq. (7), valid for an arbitrary pothole shape, gives us an easy-to-use answer to the first question. The value of γ corresponding to the specified eigenvibration, given vehicle speed and pothole width, is found by Eqs. (5) and (6). Substituting it into Eq. (7), we immediately find how much the contact force due to two potholes is greater/less than that due to one pothole. The answer to the second question is given by Eq. (9).

The discussion has been illustrated by numerical examples demonstrating the application of the technique to finding contact forces for a single-degree-of-freedom vehicle model traversing a harmonic irregularity.

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